U.P.Rajarshi Tandon Open University, Prayagraj

Master of Science/ Master of Arts

PGMM-109N
MAMM-109N
Topology

Block: 1 Metric Spaces-I

Block: 2 Metric Spaces-II

Block: 3 Introduction to Topological Spaces

Block: 4 Separation Axioms on Topological Spaces

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Syllabus

PGMM-109N/MAMM-109N: Topology

Block-1: Metric Spaces-I

Unit-1: Elements of Set Theory
Sets, subset, index set, power set, operations on set, relations, functions, finite and infinite sets, Countable and uncountable sets.

Unit-2: Introduction to Metric Spaces
Metricspace,Pseudo Metric Space, Discrete Metric Space, Usual and Quasi Metric Space, inequalities.

Unit-3: Bounded and Unbounded Metric Spaces
Bounded and Unbounded Metric Space, Usual and Quasi Metric Space, inequalities.

Block-2: Metric Spaces-II

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Unit-6: Complete Metric Space
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Unit-9: Base and Sub-base

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Unit-12: Separation Axioms-II

Separation axioms: normal space, completely normal space, T_4 and T_5, their characterizationsandbasicproperties. Urysohn's lemma and Teitze Extension Theorem,Urysohn'sMetrizationTheorem.

Unit-13: Connectedness

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Unit-14: Compactness

Cover, Open Cover, Compact Space, Compact Set, Finite Intersection Property, Locally Compact Space, Lindelof Space, Bolzano Weierstrass Property, Sequentially Compact, Uniformly Continuous, Lebesgue Covering Lemma, Heine-Borel Theorem, Product Topology, Projection Mappings.

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Block

1 Metric Spaces-I

Block-1

Metric Spaces-I

Set theory has a great importance in the study of mathematics and computer sciences. A German mathematician Georg Cantor (1845-1918) introduce the idea of set theory. The concept of set theory has a great contribution in analysis. In this unit we shall discuss some basic concepts of Sets such as subsets, multi set, empty set, singleton set, finite and infinite set, universal set, comparable and non-comparable sets, set of sets, subset, power set, venn diagrams, operations on sets, cardinality of a set, ordered pairs, cartesian product of sets and some algebraic properties. In our daily life we usually use the word 'set' as set of natural numbers, set of real numbers, set of integers, tea set, set of books of an author, set of an examination papers, set of authors of this book, etc. In all of these, the meaning of the word 'set' is a collection of well-defined objects.In the set theory of real numbers, R can be geometrically demonstrated through the points on a straight line. Set theory and real number system are the fundamental of the Mathematics. The key concept of analysis must be based on an exactly defined on the concept of number.

Metric spaces are essential in mathematics, especially in analysis and its practical applications. Metric spaces have numerous applications in various fields, including physics, computer science, and engineering. For example, they are used in algorithms for data analysis, optimization, and machine learning.Metric spaces are crucial in the study of analysis, particularly in real and complex analysis. They provide a framework for defining limits, continuity, and convergence, which are central concepts in analysis. Metric spaces are the foundation of topology, a branch of mathematics that deals with the properties of spaces that are preserved under continuous transformations.

In the first unit, we shall discussed the Sets, subset, index set, power set, operations on set, relations, functions, finite and infinite sets, Countable and uncountable sets.In the second unit we shall discuss the Metric space, Pseudo Metric Space, Discrete Metric Space, Bounded and Unbounded Metric Space, Usual and Quasi Metric Space, inequalities. Bounded and unbounded metric space, Usual and Quasi Metric Space, inequalities with details are discussed in unit third.

UNIT-1: Elementary of Set Theory

Structure

1.1 Introduction

In modern mathematics, the words set and element are very common and appear in most texts. They are even overused. There are instances when it is not appropriate to use them. For example, it is not good to use the word element as a replacement for other, more meaningful words. When you call something an element, then the set whose element is this one should be clear. The word element makes sense only in combination with the word set, unless we deal with a nonmathematical term (like chemical element), or a rare old-fashioned exception from the common mathematical terminology (sometimes the expression under the sign of integral is called an infinitesimal element; lines, planes, and other geometric images are also called elements). In dictionary the word set is defined as a collection, a group, a class or an assemblage etc.

Finite sets are very important for the study of combinatory theory of counting. George Cantor (1874) discussed the term of countable set. Countable sets have great importance in real and discrete mathematics.

1.2 Objectives

After reading this unit the learner should be able to understand about the:

- the introductory concepts about sets
- the subsets, superset, proper subsets and improper subsets
- the index set and power set
- the operations on set
- the relations and types of relations
- the functions and types of functions
- the finite and infinite sets
- the countable and uncountable sets

1.3 Set

A *set* is a well defined collection of objects. The objects in a set are known as members or elements or points. Suppose A is a set and a is an element of A, then we write $a \in A$ (a belongs to A). If a is not an element of A, then we write $a \notin A$ (a does not belongs to A).

Let A be the set

$$A = \{1, 3, 5, 7, 9, 11\} \qquad \text{….. (i)}$$

Here $1 \in A$, $3 \in A$, $5 \in A$, $7 \in A$, $9 \in A$, $11 \in A$ but $2 \notin A$. The form of presentation of set A in (i) is known as *tabular method* or *roster method*. Also the equation (i) can be written as

$$A = \{x \mid x \text{ is an odd positive integer and } x < 13\} \qquad \text{…. (ii)}$$

It means that A is the set of all odd positive integers which are less than 11. The form of presentation of set A in (ii) is known as *set-builder method* or *rule method.*

For example: The set consisting of all the letters in the word "DELHI" can be written as

$\{D, E, L, H, I\}$ or

$\{x \mid x \text{ is a letter in the word "DELHI"}\}$.

Sets are denoted by capital letters and their element by lower case letter. Some notations o standard sets are given below:

- C: the set of complex numbers.
- R: the set of real number.
- Q: the set of rational numbers.
- I: the set of integers.
- N: the set of natural numbers.

For example:. The set consisting of all even positive integers is denoted by {2, 4, 6, 8, ………} or $\{x \mid x$ is an even positive integer}.

For example: The set consisting of fourth roots of unity is denoted by $\{1, -1, i, -i\}$ or $\{x \mid x^4 = 1\}$.

Empty Set

A set is said to be *empty set* or *null set* or *void set* if it contains no element. It is denoted by ϕ or {}. Let A $=\{x \mid x$ is a real number and $x^2 = -1\}$, B $= \{x \mid x < x\}$ and C $= \{x \mid x \in \mathrm{I}$ and $1<x<2\}$. Here A, B and C are empty set.

Singleton Set

A set is said to be *singleton set* or *unit set* if it contains only one element. Let A $=\{x \mid x$ is a positive integer and $x^2 = 4\}$ and B = {0}. Here A and B are singleton set.

Comparable and Non-comparable Set

Let A and B be any two sets. Then A and B are said to be *comparable* if all the elements of A belongs to B or all the elements of B belongs to A (*i.e.*, $A \subseteq B$ or $B \subseteq A$). But if $A \not\subset B$ or $B \not\subset A$, then A and B are known as *non-comparable set*.

Note: Every set is comparable with itself, *i.e.*, $A \subset A$, therefore A and A (itself) are comparable sets.

For example: Let A = {1, 2, 3, 4, 5, 6, 7, 8}, B = {1, 3, 5, 7} and C = {2, 5, 6, 7, 9}.

Here we see that all the elements of B belongs to A, *i.e.*, $B \subset A$, therefore A and B are comparable sets. Also $C \subset A$, therefore A and C are comparable sets. But $C \not\subset B$ or $B \not\subset C$, therefore B and C are non-comparable sets.

For example: Let A = $\{a, m, a, r, j, e, e, t\}$ and B = $\{a, j, e, e, t\}$.

Here we see that all the elements of B belongs to A, *i.e.*, $B \subset A$, therefore A and B both are comparable sets.

Equality of Sets

Let A and B be any two sets. If all the elements of A belongs to B and all the elements of B belongs

to A, (*i.e.*, A $\subseteq$ B and B $\subseteq$ A) then A and B are said to be equal set and written as A = B.
Consider two sets A = {N, I, R, A, N, J, A, N} and B = {N, I, R, A, J}. Here A and B are equal set, *i.e.*, A = B.

Multi Set

A *multi set* is an unordered collection of objects in which an object can appear more than once. Let A = {*a*, *a*, *b*, *b*, *b*, *c*}. Here *a* appears two times, *b* appears three times and *c* appear one time.

Universal Set

A set under consideration in the problem is a fixed set in which includes each given set known as *universal set*.
For example: For the sets of numbers, the set of complex number (C) will be the universal set. It is denoted by U.

Disjoint Set

Let A and B be any two sets. Then A and B are said to be disjoint sets if they have no common elements.

For example: Let A = {1, 3, 5, 7} and B = {2, 4, 6, 8}. Here A and B have no common elements.

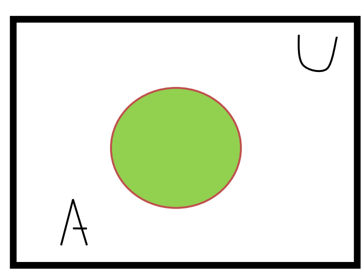
Therefore A and B are disjoint sets.

Venn Diagram

A *venn diagram* is a pictorial representation of sets in which it represented by a rectangle and the sets with by circle.

For example: Let A = {1, 2, 3} and

U = {1, 2, 3, 4, 5, 6, 7, 8}.



1.4 Subset

Let A and B be any two sets. If all the elements of A belong to B, then A is said to be *subset* of B. It is denoted as $A \subset B$, read as "A is a subset of B" or "A is contained in B".

For example: Let A be the set A= {*a*, b, c}. Then ϕ, {*a*}, {b}, {c}, {*a*, b}, {*a*, c}, {b, c} and A are all subsets of A.

For example: If A= {*a*, b, c}, then {b, d} is not a subset of A because $d \notin A$.

Superset

Let A and B be any two sets. $A \subset B$ is also expressed by writing as $B \supset A$ and is read as "B contains A" or B is a super set of A.

For example: Let A = {1, 2, 3} and B = {1, 2, 3, 4}.

Here we see that all elements of set A belong to set B, i.e., $A \subset B$, *i.e.*, B contains A**.** Therefore B is a super set of A.

Proper Subset

Let A and B be any two sets. Then A is said to be proper subset of B if $A \subset B$, $A \neq \phi$ and $A \neq B$.

Improper Subset

Let A be any set then ϕ and itself A are improper subsets of A.

For example: Let A be the set A= {*a*, b, c}. Then the all subsets of A are ϕ, {*a*}, {b}, {c}, {*a*, b}, {*a*, c}, {b, c} and A. Here {*a*}, {b}, {c}, {*a*, b}, {*a*, c} and {b, c} are all subsets of A. ϕ and A are improper subsets of A.

1.5 Index Set

Index set is a set whose elements are used as names. It is usually denoted by Λ. An index set may be finite or infinite.

For example: Let A={*a*, *b*, *c*, ……..}, B={α, β, γ, ………} and C={*i*, *j*, *k*, ……..} be any three sets. Here we see that the all elements of A, B and C are used as names. Therefore A, B and C are index sets.

Cardinality of a set

Let A be any finite sets. The number of distinct elements contained in A is known as *cardinality* of the set A. It is denoted by *n*(A) or $|A|$.

For example: Let A be the set: A= {1, 2, 3, 4, 5}. Then *n*(A) = 5. For a empty set, *n*(ϕ) = 0.

1.6 Power Set

Let A be any set. The power set of A is the set of all subsets of A. It is denoted by P(A).

Let A = {*a*, b, c} be the set. Then the power set of A is

P(A) ={ϕ, {*a*}, {b}, {c}, {*a*, b}, {*a*, c}, {b, c}, {*a*, b, c}}.

The number of elements in a P(A) is 2 raised to the cardinality of A *i.e.*,

Number of P(A) = $2^{n(A)}$.

For example: If A = {*a*, b, c}, then number of P(A) = 2^3 = 8.

Set of Sets

If a set contains a number of sets as its elements then it is known as set of sets or family of sets or class of sets.

For example: Let A = {{*a*}, {*a*, b}, {*a*, b, c}, {*a*, b, c, d}, {*a*, b, c, d, e}} and B = {{0}, {0, 1}, {0, 1, 2}}.

Here A and B are set of sets.

1.7 Operations on Set

Complement of a Set

Let U be the universal set. The complement of a set A with respect to U is the set of elements which belong to U but do not belong to A. It is denoted by U-A or $\overline{A}$ *or* A' or A^c and is defined as $\overline{A} = \{x: x \in U \text{ and } x \notin A\}$.

For example: Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9} and A = {1, 3, 5, 7, 9}. Then $\overline{A}$ = {2, 4, 6, 8}.

For example: Let U= {*x*: *x* is a letter in English alphabet} and A = {*x*: *x* is a vowel}. Then $\overline{A}$ = {*x*: *x* is a consonant}.

Union of Sets

Let A and B be any two sets. The *union* of A and B is the set of all elements which belong to A or to B and is denoted by $A \cup B$. Thus $A \cup B = \{x: x \in A \text{ or } x \in B\}$.

For example: Let A = {1, 2, 3, 4, 5} and B = {2, 4, 6, 8, 10}. Then the union of A and B is

$$A \cup B = \{1, 2, 3, 4, 5, 6, 8, 10\}.$$

Intersection of Sets

Let A and B be any two sets. The *intersection* of A and B is the set of elements which belong to both A and B and is denoted by $A \cap B$. Thus $A \cap B = \{x: x \in A \text{ and } x \in B\}$.

For example: Let A = {1, 2, 3, 4, 5} and B = {2, 4, 6, 8, 10}. Then the intersection of A and B is

$$A \cap B = \{2, 4\}.$$

Difference of Sets

Let A and B be any two sets. The *difference* of A and B is the set of elements which belong to A but do not belong to B. It is denoted by A-B *or* A~B *or* A/B= $\{x: x \in A \text{ and } x \notin B\}$.

For example: Let A = {1, 2, 3, 4, 5, 6} and B = {3, 4, 5, 6, 7, 8}.Then the difference of A and B is

A-B = {1, 2}

and the difference of B and A is

B-A={7, 8}.

Symmetric Difference of Sets

Let A and B be any two sets. The *symmetric difference* of A and B is the set of elements which belong to A or B but do not belong to A and B. It is denoted by A⊕B and defined as

$A \oplus B = \{x: (x \in A \text{ and } x \notin B) \text{ or } (x \notin A \text{ and } x \in B)\}$ or $A \oplus B = (A-B) \cup (B-A)$.

For example: Let A = {1, 2, 3, 4, 5} and B = {1, 3, 5, 7}.Then A⊕B = {2, 4, 7}.

1.8 Relation

In our day to life, a word used relation means something like as marriage and friendship, etc. “Is the mother of”, “is the father of”, “is the sister of”, is the brother of”, “is the friend of”, are all relations over the set of men. Similarly, “is equal to”, is less than”, “is greater than”, “is the divisor of” are relations on the set of numbers. In this book we study binary relations. A binary relation is the relation between two objects.

For example, “is the son of” is a relation between two men *a* and *b*. Therefore the binary relation involves certain ordered pair (*a*, *b*) in which the first element *a* is related to the second element *b*.

Let *A* and *B* be any two sets. A *relation R* from a set *A* to set *B* is a subset of $A \times B$ and defined as

xRy if and only if $(x, y) \in R$, $x \in A$ and $y \in B$

or $\quad xRy \Leftrightarrow (x, y) \in R$

and $\quad x \not R\, y \Leftrightarrow (x, y) \notin R$,

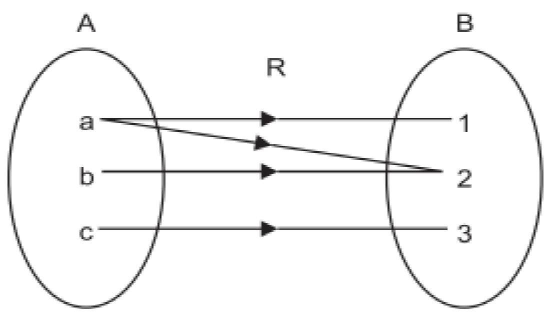
$x\ R\ y$ reads "x is R-related to y".

Note: (i) If R is a relation from A to A then R is known as relation on A.

(ii) A binary relation on a set A is a subset of A × A.

For example: Let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$ be any two sets.

Then $R = \{(a, 1), (a, 2), (b, 2), (c, 3)\}$ is a relation from A to B.



Inverse Relation

Let R be a relation from a set A to a set B. Then R^{-1} from B to A is known as the *inverse relation* of R if and only if

$$R^{-1} = \{(y, x) : (x, y) \in R\}.$$

For example: Let $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$ be any two sets.

Then $R = \{(1, 2), (1, 4), (2, 4), (3, 6)\}$ is a relation from A to B

and $R^{-1} = \{(2, 1), (4, 1), (4, 2), (6, 3)\}$ is an inverse relation from B to A.

Note: (i) Every relation has an inverse relation.

Identity Relation

Let $A = \{a, b, c\}$ be any set. Then a relation R on a set A is known as an *identity relation* if $R = \{(a, a) : a \in A\}$.

For example: Let $A = \{a, b, c, d\}$ be any set. Then the relation $R = \{(a, a), (b, b), (c, c), (d, d)\}$ is an identity relation on A.

Universal Relation

Let $A = \{a, b, c\}$ be any set. Then a relation R on a set A is called as *universal relation* if

$$R = A \times A$$

or $R = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$ is a universal relation on A.

For example: Let $A = \{a, b\}$ be any set. Then the relation $R = \{(a, a), (a, b), (b, a), (b, b)\}$ is a universal relation on A.

Note:(i) If R is a relation from A to A then R is known as relation on A.

(ii) A binary relation on a set A is a subset of $A \times A$.

(iii) Every relation has an inverse relation.

(iv) Let $A=\{1, 2, 3, 4\}$ and R be the relation > (is greater than). Then we have

$$R=\{(2, 1), (3, 2), (3, 1), (4, 3), (4, 2), (4, 1)\}.$$

1.9 Types of Relation

Some important types of relations are as follows:

(i) Reflexive Relation

A relation R on a set A is known as *reflexive* relation if and only if

$$aRa, \forall a \in A.$$

(ii) Symmetric Relation

A relation R on a set A is known as *symmetric* relation if and only if

$$aRb \Rightarrow bRa, \forall(a, b) \in R.$$

(iii) Anti-symmetric Relation

A relation R on a set A is known as *anti-symmetric* relation if and only if

$$aRb, bRa \Rightarrow a = b, \ \forall(a, b) \in R.$$

(iv) Transitive Relation

A relation R on a set A is known as *transitive* relation if and only if $aRb, bRc \Rightarrow aRc$, $(a, b, c \in A)$.

Note: (i) In R, the relation "is equal to" is reflexive, symmetric and transitive.

(ii) In R, the relation "less than" is anti-symmetric and transitive.

(iii) The relation "is the friend of" on the set of all human beings is reflexive.

(iv) The relation "less than", "greater than", "is the father of", "is the wife of" on the set of people are not reflexive.

(v) The relation "a divides b" on set of natural numbers is anti-symmetric for a divides b and b divides a if and only if $a = b$.

(vi) The relation "is the brother of" on any set of men is transitive for a is brother of b, b is brother of c then a is brother of c.

(vii) The relation "is the father of" is not transitive.

Example.1. Write a relation which is reflexive but neither symmetric nor transitive.

Solution: Let $A = \{a, b, c\}$ be any set and the relation R on A defined as

$$R = \{(a, a), (a, c), (b, a), (b, b), (c, b), (c, c)\}.$$

Then (i) Reflexive: We have $(a, a) \in \text{R}, \forall a \in \text{A}$.

Therefore R is reflexive on A, *i.e.*,$(a, a), (b, b), (c, c) \in$ R.

(ii) Symmetric: We have $(a, c) \in$ R but $(c, a) \notin$ R.

Therefore R is not symmetric on A, i.e., $(a, c), (b, a), (c, b) \in$ R but $(c, a), (a, b), (b, c) \notin$ R.

(iii) Transitive: We have $(a, c), (c, b) \in$ R but $(a, b) \notin$ R.

Therefore R is not transitive on A. Hence R is reflexive but neither symmetric nor transitive.

Equivalence Relation

A relation R on a set A is known as an *equivalence relation* if and only if it is reflexive, symmetric and transitive. Equivalence relation is denoted by ~.

Note: A universal relation R on any set A always satisfied the properties of equivalence relation.

For example: Let $A = \{a, b, c\}$ and the relation R on a set A is defined as

$R = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$ is an equivalence relation.

Example.2. Let I be an integer set and R is a relation on I defined as

$R = \{(a, b): a<b \text{ and } a, b \in I\}$ is not an equivalence relation.

Solution: Let $R = \{(a, b): a<b \text{ and } a, b \in I\}$.

Then (i) Reflexive: We have $(a, a) \notin$ R, i.e., a is not less than $a, \forall a \in I$.

Therefore R is not reflexive on I.

(ii) Symmetric: Suppose $(a, b) \in$ R i.e., $a<b \Rightarrow (b, a) \notin$ R, i.e., b is not less than a.

Therefore R is not symmetric on I, i.e., $(a, b) \in$ R $\Rightarrow (b, a) \notin$ R (because if a is less than b then b is not less than a)

(iii) Transitive: We have $(a, b), (b, c) \in$ R $\Rightarrow (a, c) \in$ R i.e., $a<b$ and $b<c \Rightarrow a<c$.

Therefore R is transitive on I. Hence R is transitive but neither reflexive nor symmetric.

Example.3. If R is an equivalence relation on a set A then show that R^{-1} is also an equivalence relation on A.

Solution: Let $A = \{a, b, c\}$ be any set and the relation R on a set A. Suppose R is an equivalence relation, i.e., R is reflexive, symmetric and transitive.To show that R^{-1} is an equivalence relation.

Then (i) R is reflexive: We have $(a, a)\in R$, $\forall a\in A$

$$\Rightarrow (a, a)\in R^{-1}, \forall a\in A$$

Therefore R^{-1} is reflexive on A.

(ii) R is symmetric: We have $(a, b)\in R \Rightarrow (b, a)\in R$.

Now we have $(a, b)\in R^{-1}\Rightarrow (b, a)\in R$

$\Rightarrow (a, b)\in R$

$\Rightarrow (b, a)\in R^{-1}$.

Therefore $(a, b)\in R^{-1}\Rightarrow (b, a)\in R^{-1}$, i.e., R^{-1} is symmetric on A.

(iii) R is transitive: We have $(a, b), (b, c)\in R \Rightarrow (a, c)\in R$.

Now we have $(a, b), (b, c)\in R^{-1}\Rightarrow (b, a), (c, b)\in R$

$\Rightarrow (c, b), (b, a)\in R$

$\Rightarrow (c, a)\in R$

$\Rightarrow (a, c)\in R^{-1}$

Therefore $(a, b), (b, c)\in R^{-1}\Rightarrow (a, c)\in R^{-1}$, i.e., R^{-1} is transitive on A.

Hence R^{-1} is reflexive on A i.e., R^{-1} is reflexive, symmetric and transitive.

Equivalence Classes

Let R be an equivalence relation on a set A. Let a be any arbitrary element of A. The set of all element $x \in A$ such that xRa constitute a subset of A (say $[a]$). Thus subset $[a]$ is known as *equivalence class* of a with respect to R, denoted as

$$[a]=\{x : x\in A \text{ and } xRa\}.$$

Order Relation

A relation which is transitive but not an equivalence relation is known as an *order relation*.

If R is an order relation on a set X, then

$$x\mathrm{R}y \text{ and } y\mathrm{R}z \Rightarrow x\mathrm{R}z,\ \forall x, y, z\in \mathrm{X}.$$

Partial Order Relation

A relation R on a set X is said to be a *partial order relation* if it is at the same time

(i) Reflexive

(ii) Anti-symmetric and

(ii) Transitive.

It is denoted by the symbol $\leq$. A set X together with a partial order relation defined on it, *i.e.*,(X, $\leq$) is known as a *partial ordered set*.

For example: The relation "x divides y" on the set of natural numbers is a partial order relation. The relation "sub-set of" on the set of all sub-sets of a set is a partial order relation.

1.10 Function

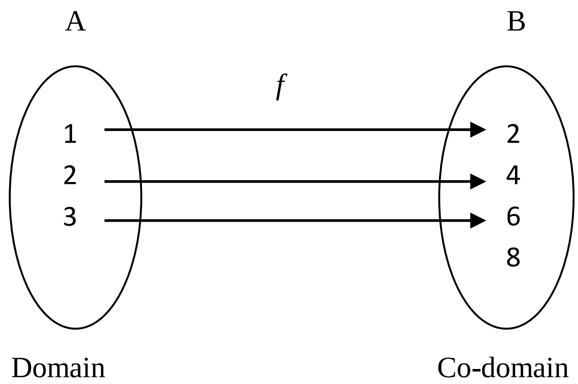
Let A and B be any two non-empty sets. If there exists a rule or a correspondence f which associate each element of A has a unique image in B then f is a function or mapping from A to B. This mapping is denoted by

$$f : A\rightarrow B$$

or $\quad A \xrightarrow{f} B.$

Here the set A is known as domain and the set B is known as co-domain of the function f.

For example: Let $A = \{1, 2, 3\}$, $B = \{2, 4, 6, 8\}$ and $f : A \rightarrow B$ is defined as



Here range is {2, 4, 6}. We know that the range is a subset of co-domain.

Example.4. If $A = \{1, 2, 3\}$ and $B = \{a, b, c, d\}$ then does

(i) $\{(1, a), (2, c), (3, d)\}$

(ii) $\{(1, a), (2, b), (2, c), (3, d)\}$

(iii) $\{(1, a), (2, b)\}$

(iv) $\{(1, a), (2, b), (3, a)\}$ represent a function from $f : A \rightarrow B$.

Solution: (i) Here we see that $f(1)=a$, $f(2)=c$ and $f(3)=d$. Therefore f is a function from A to B because every element of A has a unique image in B.

(ii) Here we see that $f(1)=a$, $f(2)=b$, $f(2)=c$ and $f(3)=d$. Therefore f is not a function from A to B because every element of A has not a unique image in B, i.e., one element (2) of A has two images (b, c) in B.

(iii) Here we see that $f(1)=a$ and $f(2)=b$. Therefore f is not a function from A to B because every element of A has not a unique image in B, i.e., one element (3) of A has not any image in B.

(iv) Here we see that $f(1)=a$, $f(2)=b$ and $f(3)=a$. Therefore f is a function from A to B because every element of A has a unique image in B.

Example.5. If $A = \{1, 2, 3, 4\}$ and $f(1)=2$, $f(2)=3$, $f(3)=4$ and $f(4)=2$, then does f represent a function.

Solution: (i) We have $A = \{1, 2, 3, 4\}$ and $B = \{2, 3, 4\}$.

Here we see that every element of A has a unique image in B. Therefore f is a function from A to B.

1.11 Types of Function

Here we discuss some types of functions which as follows:

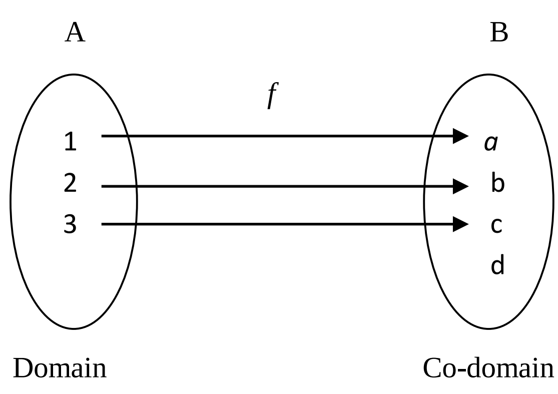
(i) One-One Function

A function $f: A \to B$ is called *one-one* if $x_1, x_2 \in A$, we have

$$x_1 = x_2 \quad \Rightarrow \quad f(x_1) = f(x_2)$$

or $$x_1 \neq x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2).$$

For example: Let $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$ and $f : A \to B$ is defined as



Here f is known as one-one function and range of f is $\{a, b, c\}$.

Example.6. If $A = \{1, 2, 3, 4, 5\}$, $B = \{a, b, c\}$ and $f$is defined as $f(1)=a$, $f(2)=b$, $f(3)=a$,$f(4)=a$ and $f(5)=c$, then state whether f is a function from A to B or not, if yes write its type.

Solution: Here we see that $f(1)=a$, $f(2)=b$, $f(3)=a$,$f(4)=a$ and $f(5)=c$,

therefore f is a function from A to B because every element of A has a unique image in B.

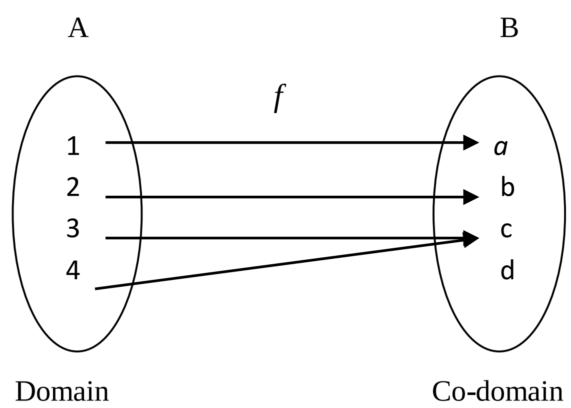
Hence f is a function from A to B.

Also we see that three elements (1, 3, 4) of A has same image (a) in B. Hence f is not one-one function from A to B, i.e., one element (2) of A has two images (b, c) in B.

(ii) Many-One Function

A function $f : A \rightarrow B$ is said to be *many-one* if at least one element of B has two or more than two pre-image in A.

For example: Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$ and $f : A \rightarrow B$ is defined as

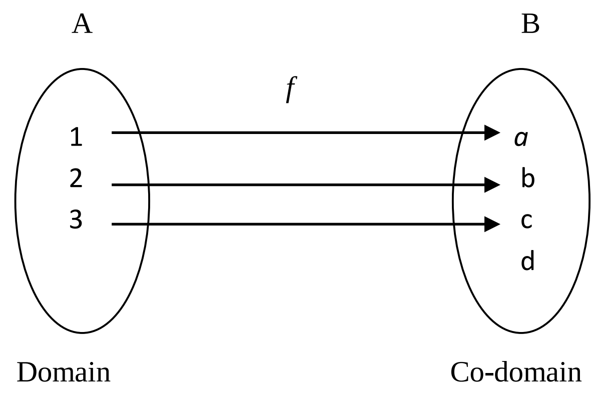


Here f is known as many-one function and range of f is $\{a, b, c\}$.

(iii) Into Function

A function $f : A \rightarrow$ B is said to be *into* if there is at least one element of B, has no pre-image in A.

For example: Let $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$ and $f : A \rightarrow B$ is defined as

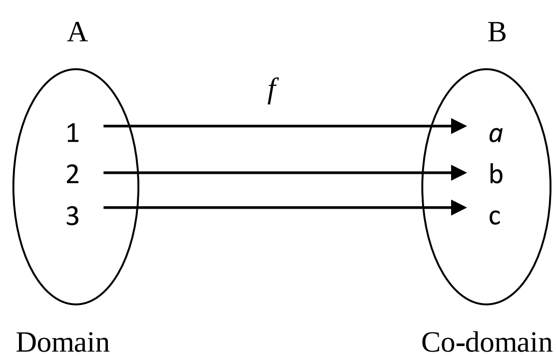


Here one element *d* of the set *B* has no pre-image in the set *A*. Then *f* is known as into function and range of *f* is {*a, b, c*}.

(iv) Onto Function

A function $f : A \rightarrow B$ is said to be *onto* if there is no element of *B*, which is not an image of some element of *A*.

For example: Let *A* {1, 2, 3}, *B* = {*a, b, c*} and $f : A \rightarrow B$ is defined as



Here *f* is known as onto function and range of *f* is {*a, b, c*}.

Inverse of a Mapping

Let $f : X \rightarrow Y$ be a *one-one ontomapping* and $f(x)=y$, $\forall x \in X$, $\forall y \in Y$.

Now we define a mapping $f^{-1}: y \rightarrow X$ such that $f^{-1}(y)=x$, $\forall x \in X$, $\forall y \in Y$, where f^{-1} is called the inverse of f. Here f is inversible mapping because inverse of f is exists.

Example.7. Let f be a function $f : \mathrm{R} \to \mathrm{R}$ is defined as $f(x)=x^2$, $\forall x\in\mathrm{R}$, where R is the set of real numbers. Find the value of $f^{-1}(9)$.

Solution: It is given that f be a function $f : \mathrm{R} \to \mathrm{R}$ is defined as $f(x)=x^2$, $\forall x\in\mathrm{R}$.

We have $$f^{-1}(9) = \{x\in\mathrm{R}: f(x) = 9\}= \{x\in\mathrm{R}: x^2 = 9\}$$

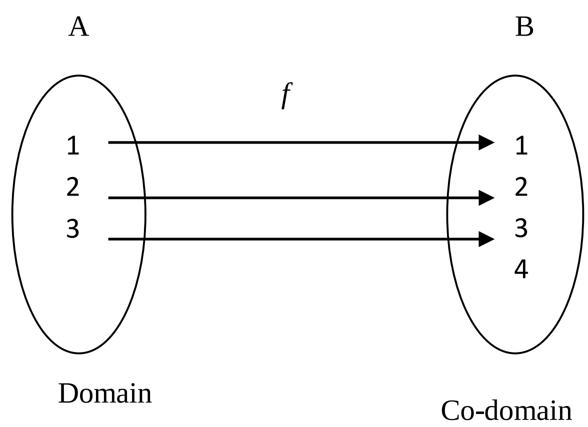
$$= \{x\in\mathrm{R}: x =3, -3\}= \{3, -3\}.$$

Inclusion Mapping

Let X be any subset of Y. Then the mapping $f{:}X \to Y$ is said to be *inclusion mapping* if

$$f(x) = x,\ \forall x\in X.$$

For example: Let $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4\}$ and $f : A \to B$ is defined as

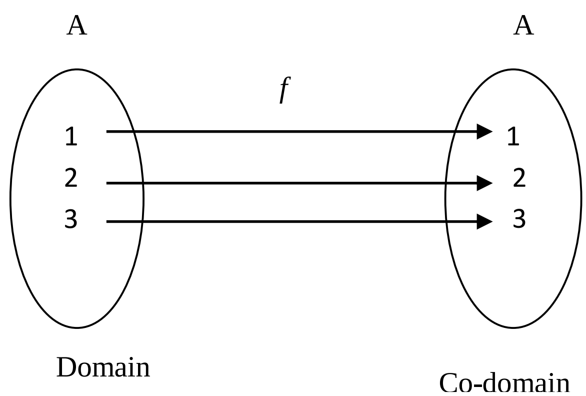


Here f is known as inclusion mapping.

Identity Mapping

Let $f : X \to X$ be a mapping. Then f is said to be *identity mapping* if $f(x) = x$, $\forall x\in X$.

For example: Let $A = \{1, 2, 3\}$ and $f : A \to A$ is defined as



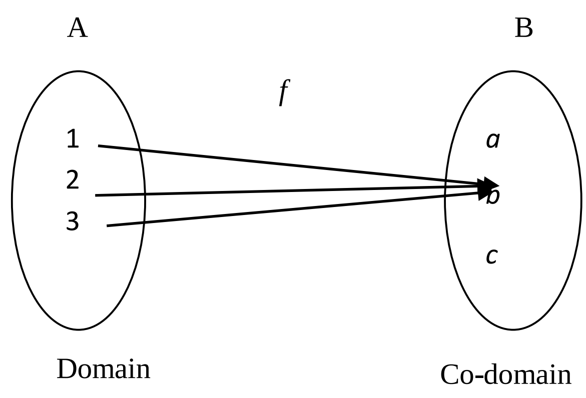
Here f is known as identity mapping.

Constant Function

Let $f{:}X \to Y$ be a function. Then f is said to be *constant function* if $f(x) = a$, $\forall x \in X$

i.e., a function $f : X \to Y$ is known as *constant function* if each element of X is mapped onto a single element of Y.

For example: Let $A = \{1, 2, 3\}$, $B=\{a, b, c\}$ and $f : A \to B$ is defined as



Here f is known as constant function, *i.e.*, $f(1) = b$, $f(2) = b$, $f(3) = b$.

Real Valued Mapping

A mapping $f : X \rightarrow R$, where R is the set of real numbers, is known as *real valued mapping*.

Characteristic Function

Let U be the universal set and A be a subset of U. Then the real valued function f: U→ {0, 1} such that $f_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$ is known as characteristic function of A.

Zero Function

The function $f : X \rightarrow Y$ is known as *zero function* if the image of each element of X under f is zero *i.e.*,$f(x) = 0$.

Injective (or Injection) Mapping

A mapping f is said to be *injective* (or *injection*) which is either one-one into or one-one onto.

Bijective (or Bijection) Mapping

A mapping f is said to be *bijective* (or *bijection*) which is both one-one and onto.

Equality of Mapping

Let $f : X \rightarrow$ and $g : X \rightarrow Y$ be two mapping. Then the mapping f and g are said to be *equal mapping* if and only if $f(x) = g(x)\ \forall x \in X$.

In case of equal mappings, the domains of mappings must be the same.

Composition of Functions or Product of Functions

Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be any two mapping. Then a function $gof : X \rightarrow Z$ is defined as

$$gof = g[\,f\,(x)],\ \forall x \in X \text{ is known as } \textit{composition of functions}.$$

Example.8. Let $f(x)=x^2$, $g(x)=x+3$, $\forall x \in R$. Find gof and fog.

Solution: Here $gof=g[f(x)]= g(x^2)= x^2 + 3$

and $\quad fog = f[g(x)]= f(x + 3)= (x + 3)^2= x^2 + 6x + 9.$

Example.9. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be any two mapping such that $f(x)=\log(1+x)$, $g(x)=e^x$, then find the value of $gof\,(x)$ and $fog\,(x)$.

Solution: Here we have $gof : X \rightarrow Z$ is a mapping such that

$$gof\,(x) = g[f(x)] = g[\log(1+x)] = e^{\log(1+x)} = (1+x).$$

Now we have $\quad fog\,(x)= f[g(x)] = f(e^x) = \log(1+ e^x).$

Example.10. Let $f : R \rightarrow R$ and $g : R \rightarrow R$ be any two mapping such that $f(x)=x^2$, $g(x)= x^3$, $\forall x \in R$. Find the values of $gof\,(x)$ and $fog\,(x)$.

Solution: Here we have $gof : R \rightarrow R$ is a mapping such that

$$gof\,(x) = g[f(x)] = g(x^2) = (x^2)^3 = x^6.$$

and $\quad fog = f[g(x)]= f(x^3)= (x^3)^2 = x^6.$

1.12 Finite and Infinite Sets

A set is said to be *finite set* if it contains finite number of elements, otherwise it is infinite. Let A be the set of all students of an engineering college, B is the set of vowels and N is the set of natural numbers. Here A and B are finite set and N is infinite set.

1.13 Contable and Uncountable Sets

A set which is either finite of denumerable is called a *countable set*. An infinite set is said to be denumerable or enumerable if it equivalent to the set N, the set of all natural number. *For example*, Let $A = \{1,2,3,4,5,6\}$. Then A is finite so that by definition A is countable. A set A is called on *uncountable set* if A is an infinite set and A is not cardinally equivalent to N. Here we state the following theorem without proof:

1. Every infinite set contains an enumerable set.

2. The open interval (0, 1) is not enumerable.

3. The set of all irrational numbers is uncountable.

Note: 1. R and C are uncountable sets.

1.14 Summary

A set is a well-defined collection of objects. The objects in a set are known as members or elements or points. A multi set is an unordered collection of objects in which an object can appear more than once. A set is said to be empty set or null set or void set if it contains no element. It is denoted by ϕ or {}. Let A and B be any two sets. If all the element of A belongs to B, then A is said to be subset of B. If a set contains a number of sets as its elements then it is known as set of sets or family of sets or class of sets. Two sets A and B are said to be disjoint sets if they have no common elements.

A set is said to be finite s*et* if it contains finite number of elements, otherwise it is infinite.Let A be any set. The power set of A is the set of all subsets of A.Index set is a set whose elements are used as names. The difference of A and B is the set of elements which belong to A but do not belong to B. The symmetric difference of A and B is the set of elements which belong to A or B but do not belong to A and B.

The Cartesian products of *A* and *B* is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$

i.e.,$A\times B = \{(a, b) : a\in A, b\in B\}$ and $B \times A = \{(b, a) : b\in B, a\in A\}$. Let R be a relation from a set A to a set B. Then R^{-1} from B to A is known as the inverse relation of R if and only if $R^{-1} = \{(y, x) : (x, y)\in R\}$. Let $A = \{a, b, c\}$ be any set. Then a relation R on a set A is known as an identity relation if $R = \{(a, a) : a\in A\}$.

A relation R on a set A is known as reflexive relation if and only if aRa, $\forall a\in A$. A relation R on a set A is known as symmetric relation if and only if $aRb \Rightarrow bRa\ \forall(a, b)\in R$. A relation R on a set A is known as anti-symmetric relation if and only if aRb, $bRa \Rightarrow a = b\ \forall(a, b)\in R$. A relation R on a set A is known as transitive relation if and only if aRb, $bRc \Rightarrow aRc$, $(a, b, c\in A)$. A relation R on a set A is known as an equivalence relation if and only if it is reflexive, symmetric and transitive.

A relation which is transitive but not an equivalence relation is known as an order relation. If R is an order relation on a set X, then $x\mathrm{R}y$ and $y\mathrm{R}z\Rightarrow x\mathrm{R}z$, $\forall x, y, z\in \mathrm{X}$. A relation R on a set X is said to be a partial order relation if it is at the same time (i) Reflexive (ii) Anti-symmetric and (ii) Transitive. A set X together with a partial order relation defined on it, *i.e.*,$(X, \leq)$ is known as a partial ordered set.

Let A and B be any two non-empty sets. If there exists a rule or a correspondence f which associate each element of A has a unique image in B then f is a function or mapping from A to B. A function $f: A\rightarrow B$ is called one-one if $x_1, x_2\in A$, we have $x_1 = x_2\Rightarrow f(x_1) = f(x_2)$ or $x_1\neq x_2\Rightarrow f(x_1) \neq f(x_2)$.

A function $f : A \rightarrow B$ is said to be many-one if at least one element of B has two or more than two pre-image in A. A function $f : A \rightarrow B$ is said to be many-one if at least one element of B has two or more than two pre-image in A. A function $f : A \rightarrow \mathrm{B}$ is said to be into if there is at least one element of B, has no pre-image in A.

A function $f : A \rightarrow B$ is said to be onto if there is no element of B, which is not an image of some element of A. Let $f{:}X\rightarrow Y$ be a one-one onto mapping and $f(x)=y$, $\forall x\in X$, $\forall y\in Y$. Now we define a mapping $f^{-1}{:}y\rightarrow X$ such that $f^{-1}(y)=x$, $\forall x\in X$, $\forall y\in Y$, where f^{-1} is called the inverse of f.

1.15 Terminal Questions

Q.1. List of elements of the following sets:

(a) $\{x : x \in I , x^2 < 11\}$

(b) $\{x : x \in N, x$ is even and $x < 17\}$

(c) $\{x : x \in N, x$ is prime and $x < 21\}$

(d) $\{x : x$ is a solution of $x^2 + 3x + 2 = 0\}$

Q.2 Let $U = \{1, 2, 3, \ldots., 9, 10\}$ be the universal set and $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 7. 9\}$, $C = \{2, 5, 6, 8\}$. Find

(a) A', B', C'

(b) $A \cup B$, $B \cup C$, and $A \cup C$

(c) $A \cap B$, $B \cap C$, $A \cap C$

(d) $A - B$, $B - A$, $B - C$, $C - B$, $A - C$ and $C - A$.

(e) $A \oplus B$, $B \oplus C$, and $A \oplus C$

Q.3 Which of the sets are equal?

(a) $\{x : x$ is a letter in the word ‘wolf ’$\}$

(b) $\{x : x$ is a letter in the word ‘follow’$\}$

(c) The lettters *f, l, o, w*.

(d) The letters which appear in the word ‘flow’.

Q.4 Is a set *A* comparable with itself?

Q.5. Find the power set of $\{1, 2\}$

Q.6. Let $A = \{a, b, c\}$ and $B = \{c, d, e, f\}$. Find the $A - B$, $B - A$ and $A \oplus B$.

Q.7. Prove that $A \cap (B - C) = A \cap B) - (A \cap C)$

Q.8. If $A = \{a, b, c\}$. find all the subsets of *A*.

Q.9. Let $A = \{1, 2,\}$ and $B = \{3, 4\}$. Find $A \times B$ and B × A.

Q.10. Give an example of a relation which is symmetric and transitive but not reflexive.

Q.11. Give an example of a relation that is reflexive but neither symmetric nor transitive.

Q.12. Give an example of a relation which is transitive but not reflexive or symmetric.

Q.13. If the function $f : R \to R$ be defined by $f(x) = x^2$, find $f^{-1}(g)$ and $f^{-1}(-g)$.

Q.14. If the function $f : R \to R$ be defined by $f(x) = x^2 - 1$ then find $f^{-1}(-2)$ and f^{-1} $\{8, 15\}$.

Answers

1. {(a) $\{-3, -2, -1, 0, 1, 2, 3\}$ (b) $\{2, 4, 6, 8, 10, 12, 14, 16\}$

(c) $\{2, 3, 5, 7, 11, 13, 17, 19\}$ (d) $\{-1, -2\}$

2. (a) $A' = \{5, 6, 7, 8, 9, 10\}$, $B' = 1, 2, 5, 6, 8, 10\}$, $C' = \{1, 3, 4, 7, 9, 10\}$

(b) $A \cup B = \{1, 2, 3, 4, 7, 9\}$, $B \cup C = \{2, 3, 4, 5, 6, 7, 8, 9\}$ and $A \cup C = \{1, 2, 3, 4, 5, 6, 8\}$

(c) $A \cap B = \{3, 4\}$, $B \cap C = \varphi$ and $A \cap C = \{2\}$.

(d) $A - B = \{1, 2\}$, $B - A = \{7, 9\}$, $B - C = \{3, 4, 7, 9\}$, $C - B = \{2, 5, 6, 8\}$,

$A - C = \{1, 3, 4\}$ and $C - A = \{5, 6, 8\}$.

(e) $A \oplus B = \{1, 2, 7, 9\}$, $B \oplus C = \{2, 3, 4, 5, 6, 7, 8, 9\}$ and $A \oplus C = \{1, 3, 4, 5, 6, 8\}$.

3. All the given sets are equal.

4. Yes

5. ϕ, {1}, {2}, {1, 2}.

6. $\{a, b\}$, $(d, e, f\}$ and $\{a, b, d, e, f\}$

8. A, ϕ, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{b, c\}$, $\{a, c\}$, $\{a, b, c\}$

9. $A \times B = \{(1, 3\}, (1, 4), (2, 3), (2, 4)\}$ and $B \times A = \{(3, 1), \{3, 2\}, (4, 1), (4, 2)\}$

10. $A = \{a, b, c\}$ and $R = \{(a, a), \{b, b), (a, b), \{b, a)\}$.

11. $A = \{a, b, c\}$ and $R = \{(a, a), (b, b), (c, c), (a, b), (b, c)\}$.

12. $A = \{a, b, c, d\}$ and $R = \{(a, b), (b, c), (a, c)\}$

13 {3, –3}, ϕ

14. ϕ, {3, –3, 4, – 4}.

UNIT- 2: Introduction to Metric Spaces

Structure

2.1 Introduction

Metric spaces play a crucial role in the realms of topology and analysis. In mathematics, a metric space consists of a set endowed with a metric, a function that establishes the distance between each pair of elements within the set. These spaces serve as a foundation for defining key concepts like convergence, continuity, and completeness, which are essential in both analysis and topology. By providing a framework for studying properties of spaces broader than Euclidean realms, metric spaces allow mathematicians to extend classical geometric ideas to more abstract settings. While the most familiar example is Euclidean space, where distance is measured by the Euclidean metric, numerous other examples exist. These include spaces of real numbers, where the metric is the absolute difference, as well as more abstract spaces like function spaces, where the metric is defined through integrals or other methods.

2.2 Objectives

After studying this unit, the learner will be able to understand the :

- Metric spaces and Pseudo Metric Space
- Discrete Metric Space or Trivial Metric Space
- Metrizable and Usual Metric and Norm
- Inequality, Triangular Inequality and An Auxilary Inequality
- Holder Inequality and Cauchy Schwarz-Inequality
- Minkowski’s Inequality and Minkowski’s Inequality in terms of norms

2.3 Metric Spaces

Metric spaces are fundamental in the study of topology and analysis. In mathematics, a metric space is a set equipped with a metric, which is a function that defines a distance between each pair of elements in the set. Let X be a non-empty set and $d(x,y), \forall x, y \in X$ is a distance function. A real valued function $d : X \times X \to R^0$ which satisfies the following axioms:

(i) $d(x,y) \geq 0, \forall x, y \in X$.

(ii) $d(x,y) = d(y,x), \forall x, y \in X$ (symmetric property)

(iii) $d(x,y) \leq d(x,z) + d(z,y), \quad \forall x, y, z \in X$ (Triangular in equality)

(iv) If $x = y \quad \Rightarrow \quad d(x,y) = 0$

(v) If $d(x,y) = 0 \quad \Rightarrow \quad x = y$.

Then d is said to be metric on X and the pair (X,d) is called a metric space. The real number $d(x,y)$ is called the distance of x to y .

The first axiom means that the distance between any two points x and y of X is a non-negative real number.The second axiom means that the distance does not depend on the order of the points x and y . The third axiom means that in the triangle, the sum of the length of two sides is greater than the length of the third side and equal sign shows that three points are in a straight line.The fourth axiom means that if two points x and y are the same then the distance between x and y is equal to zero.

The fifth axiom means that if the distance between two points x and y is equal to zero then the points x and y is equal to zero then the points x and y are the same.

Examples

Example.1. Let R be the set of real numbers and let d be the function $d: R \times R \to R$ defined by $d(x,y) = |x-y|, \forall x, y \in R$ **.**

Then show that d is a metric on R.

Solution: It is given that $d: R \times R \to R$ defined by

$$d(x,y) = |x-y|, \quad \forall x, y \in R \qquad \text{....(1)}$$

To show that d is a metric on R if it satisfies all the following five axioms:

(i) $d(x,y) \geq 0, \quad \forall x, y \in R$

$\Rightarrow \quad |x-y| \geq 0$ is always a non-negative real number.

(ii) $d(x,y) = d(y,x), \ \forall x, y \in R$

We have

$$d(x,y) = |x-y|$$

$$= |-(y-x)|$$

$$= |y-x|$$

$$= d(y,x), \ \forall x, y \in R$$

(iii) $d(x,y) \leq d(x,z) + d(z,y), \quad \forall x, y, z \in R$

We have

$$d(x,y) = |x-y|$$

$$= |(x-z)+(z-y)|$$

$$\leq |x-z|+|z-y|$$

$i.e., d(x,y) \leq d(x,z)+d(z,y), \quad \forall x,y,z \in R$

(iv) If $x = y \quad \Rightarrow \quad d(x,y) = 0, \quad \forall x,y \in R$

We have

$$x = y \quad \Rightarrow \quad x-y=0$$

$$\Rightarrow \quad |x-y| = 0$$

$$\Rightarrow \quad d(x,y) = 0, \quad \forall x,y \in R$$

(v) If $d(x,y) = 0 \quad \Rightarrow \quad x = y, \quad \forall x,y \in R$

We have

$$d(x,y) = 0$$

$$\Rightarrow \quad |x-y| = 0$$

$$\Rightarrow \quad x-y=0$$

$$\Rightarrow \quad x = y, \quad \forall x,y \in R$$

Hence d is a metric on R. Also the above metric $d(x,y) = |x-y|, \forall x,y \in R$ is known as usual metric for reals.

2.4 Pseudometric Space

A pseudometric space is a generalization of the concept of a metric space, where the distance function (called a pseudometric) satisfies all the properties of a metric except possibly the requirement that the distance between distinct points must be positive.The key difference between a pseudometric and a metric is that in a pseudometric space, the distance between distinct points can be zero. This means that two different points can be "infinitesimally close" to each other, but not necessarily equal.

Pseudometric spaces are particularly useful in situations where a notion of distance that allows for such "coincidence" is desirable, such as in certain areas of analysis, geometry, and topology. Let us consider X be a non-empty set. If a real valued function $d: X \times X \to R$ satisfies the following axioms:

(i) $d(x, y) \geq 0, \quad \forall x, y \in X$.

(ii) $d(x, y) = d(x, y), \quad \forall x, y \in X$ (symmetric property)

(iii) $d(x, y) \leq d(x, z) + d(z, y), \quad \forall x, y, z \in X$ (Triangular inequality)

(iv) If $x = y \quad \Rightarrow \quad d(x, y) = 0$

then d is known as pseudometric on X and the pair (X, d) is known as a pseudometric space.

Note: Every metric on X is pseudometric on X but a pseudometric on X is not necessarily a metric on X .

Examples

Example.2. Give an example of a pseudo metric which is not metric,

Solution: Consider a mapping $d: R \times R \to R$ defined by

$$d(x,y)=\left|x^2-y^2\right|, \quad \forall x,y\in R \qquad \ldots(1)$$

To show that d is a metric on R if it satisfies all the following five axioms:

(i) $d(x,y)\geq 0, \quad \forall x,y\in R$

$\Rightarrow \quad \left|x^2-y^2\right|\geq 0$ is always a non-negative real number.

(ii) $d(x,y)=d(y,x), \; \forall x,y\in R$

We have

$$d(x,y)=\left|x^2-y^2\right|$$

$$=\left|-\left(y^2-x^2\right)\right|$$

$$=\left|y^2-x^2\right|$$

$$=d(y,x), \; \forall x,y\in R$$

(iii) $d(x,y)\leq d(x,z)+d(z,y), \quad \forall x,y,z\in R$

We have

$$d(x,y)=\left|x^2-y^2\right|$$

$$=\left|\left(x^2-z^2\right)+\left(z^2-y^2\right)\right|$$

$$\leq\left|x^2-z^2\right|+\left|z^2-y^2\right|, \quad \forall x,y,z\in R$$

i.e., $d(x,y)\leq d(x,z)+d(z,y), \quad \forall x,y,z\in R$

(iv) If $x = y \quad \Rightarrow \quad d(x, y) = 0, \quad \forall x, y \in R$

We have

$x = y \quad \Rightarrow \quad x^2 = y^2$

$\Rightarrow \quad x^2 - y^2 = 0$

$\Rightarrow \quad \left|x^2 - y^2\right| = 0$

$\Rightarrow \quad d(x, y) = 0, \ \forall x, y \in R$

(v) If $d(x, y) = 0 \quad \Rightarrow \quad x = y, \quad \forall x, y \in R$

We have

$$d(x, y) = 0$$

$\Rightarrow \quad \left|x^2 - y^2\right| = 0$

$\Rightarrow \quad x^2 - y^2 = 0$

$\Rightarrow \quad x^2 - y^2$

$\Rightarrow \quad x = \pm y$, *i.e.*, property is not hold good.

Therefore $d(x, y) = 0$ does not necessarily imply that $x = y$.

For example, we have

$$d(2, -2) = \left|(2)^2 - (-2)^2\right| = 0 \text{ while } 2 \neq -2.$$

Hence d is not a metric on R .

2.5 Discrete Metric Space or Trivial Metric Space

In mathematics, a discrete metric space is a metric space in which the distance between any two distinct points is either 0 or 1. This metric essentially measures whether two points are the same (distance 0) or different (distance 1). The discrete metric induces the discrete topology on the set, where every subset is open, making the space a particularly simple and well-behaved example in topology. Let X be any non-empty set and d be the function defined by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$$

Then d is said to be metric on X and (X, d) is called discrete metric space or trivial metric space.

Examples

Example.3. Let X be a non-empty set and let $d : X \times X \to R$ be defined by

$$d(x, y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$$

Then show that d is a metric on X.

Solution: Using definition of d, we have

(i) $d(x, y) \geq 0, \ \forall x, y \in X$

$\Rightarrow$ 0 or 1, *i.e.*, always a non-negative real number.

(ii) $d(x, y) = d(y, x) \ \forall x, y \in X$.

When $x = y$

$\Rightarrow \qquad d(x,y)=0=d(y,x)$

and when $x \neq y$

$\Rightarrow \qquad d(x,y)=1=d(y,x), \quad \forall x,y \in X$

(iii) $d(x,y) \leq d(x,z)+d(z,y), \quad \forall x,y,z \in X$

If $z \neq y \neq z$ then we have

$$d(x,y)=d(x,z)=d(z,y)=1$$

i.e., $d(x,y) \leq d(x,z)+d(z,y)$.

But if $x=y=z=0$ then we have

$$d(x,y)=d(x,z)+d(z,y)=0$$

i.e., $d(x,y) \leq d(x,z)+d(z,y)$.

(iv) If $x=y \quad \Rightarrow \quad d(x,y)=0, \quad \forall x,y \in X$

It is given that $d(x,y)=\begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x=y \end{cases}$.

(v) $d(x,y)=0 \quad \Rightarrow \quad x=y \ \ \forall x,y \in X$

It is given that $d(x,y)=\begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x=y \end{cases}$.

Hence, d is a metric on X. The space (X,d) is known as discrete metric space.

2.6 Metrizable and Usual Metric

Metrizable metrics are significant because they allow us to use the tools and concepts of metric spaces in the study of topological spaces, providing a bridge between the more concrete world of distances and the more abstract world of topologies.A set X is said to be metrizable if and only if a metric can be defined on X .

The term "usual metric space" typically refers to a specific metric space that is commonly associated with a particular set.The metric defined on a real line is called usual metric or euclidean metric on R .

For example: 1.The usual metric space on the set of real numbers R is the space where the metric is the absolute difference, given by $d(x, y) = |x - y|, \forall x, y \in R$.

2.The usual metric space on the set of complex numbers C is the space where the metric is the modulus of the difference, given by

$$d(z,w) = |z - w|, \; \forall z, w \in C .$$

2.7 Norm

The size of an element x is a real number denoted by $\|x\|$ and is called as norm (which is distance $d(x,0)$) if satisfies the following properties:

(i) $\|x\| \geq 0$

(ii) $\|x\| = 0$ if and only if $x = 0$

(iii) $\|kx\| = |k| . \|x\| \qquad \{\|-x\| = \|x\|\}$

(iv) $\|x+y\| \le \|x\| + \|y\|$

Now we define a metric d for a set X with the help of norm as follows:

$$d(x,y) = \|x-y\|, \ \forall\, x,y \in X$$

This metric is known as metric induced by the norm. Let f and g be two real bounded functions defined on the closed interval $[0,1]$. Define the norms of f and g by

$$\|f\| = \int_0^1 |f(x)|\,dx \quad \text{and}$$

$$\|g\| = \int_0^1 |g(x)|\,dx \qquad \forall x \in [0,1]$$

The induced metric is defined by

$$d(f,g) = \|f-g\|$$

$$= \int_0^1 |f(x) - g(x)|\,dx\,.$$

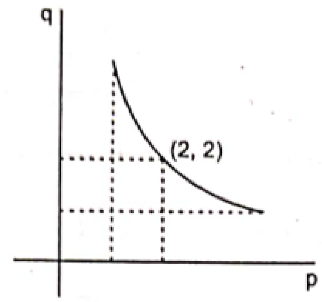
2.8 Inequality

Suppose a number $p>1$ then we say that a number q is known as conjugate index of p if

$$\frac{1}{p} + \frac{1}{q} = 1 \text{ for } 1 < p < \infty$$

$q = \infty$ for $p = 1$

$q = 1$ for $q = \infty$



The graph of

$$\frac{1}{P}+\frac{1}{2}=1 \text{ for } 1< p,q<\infty$$

For $1< p,q<\infty$, the first condition in the above combination can be put in any one of the form

$$\frac{1}{p}+\frac{1}{q}=1$$

$\Rightarrow \qquad (p-1)(q-1)=1$, where $p=\dfrac{q}{q-1}$ and $q=\dfrac{p}{p-1}$

$\Rightarrow \qquad p+q=pq$

Hence only 2 is number which has it own conjugate 1 and ∞ are considered to be conjugate index.

2.9 Triangular Inequality

If x_1 and x_2 are two real numbers, then we have

$$|x_1+x_2|\le|x_1|+|x_2|$$

If z_1 and z_2 are two complex numbers, then we have

$$|z_1+z_2|\le|z_1|+|z_2|$$

In general, we have

$$|z_1+z_2+.....+z_n|\le|z_1|+|z_2|+....+|z_n|$$

where $z_1, z_2, z_3, ..., z_n$ are complex numbers.

Note: If z_1 and z_2 are two complex numbers, then we have

$$\frac{|z_1+z_2|}{1+|z_1+z_2|} \le \frac{|z_1|}{1+|z_1|} + \frac{|z_2|}{1+|z_2|} \qquad \{\because |z_1+z_2| \le |z_1| + |z_2|\}$$

2.10 An Auxilary Inequality

If $1 < p < \infty$ **and** $\frac{1}{p} + \frac{1}{q} = 1$ **then** $ab \le \frac{a^p}{p} + \frac{b^2}{q}$, **where** a, b **are two non-negative real number.**

Proof: Suppose if $a = 0$ or $b = 0$, the result is obvious. Now let us consider a case, when

$$a \ne 0,\ b \ne 0.$$

Suppose $\quad f(t) = 1 - \lambda + \lambda t - t^2$ for $0 < \lambda < 1 \qquad \ldots(1)$

Then $\quad f'(t) = \lambda - \lambda t^{\lambda-1} \qquad \ldots(2)$

For minimum, we have

$$f'(t) = 0$$

$\Rightarrow \quad \lambda - \lambda t^{\lambda-1} = 0$ where $\lambda,\ \lambda - 1 \ne 0$

$\Rightarrow \quad t = 1 \qquad \ldots(3)$

From equation (2), we have

$$f''(t) = -\lambda(\lambda-1)t^{\lambda-2} \qquad \ldots(4)$$

At $t = 1$, we have

$$f''(t) = -\lambda(\lambda-1)(1)^{\lambda-2}$$

$$= -\lambda(\lambda-1)$$

$$= \lambda(1-\lambda)$$

= Positive as $0 < \lambda < 1$

Hence, $f'(t) = 0,\ f''(t) > 0$ at $t = 1$

Thus, $f(t)$ is minimum at $t = 1$,

$\therefore \quad f(1) \le f(t)$

This implies $\quad 1-\lambda+\lambda-1 \le 1-\lambda+\lambda t - t^{\lambda}$

or $\quad 1-\lambda+\lambda t - t^{\lambda} \ge 0$

$$(1-\lambda)+\lambda t \ge t^{\lambda} \qquad \ldots(5)$$

Putting $\lambda = \frac{1}{p}$ in the equation (5), we get

$$\left(1-\frac{1}{p}\right)+\frac{1}{p}t \ge t^{1/p} \qquad \ldots(6)$$

But $\quad \frac{1}{p}+\frac{1}{q} = 1$

$\Rightarrow \quad \frac{1}{p} = 1-\frac{1}{q}$

Substituting these value in equation (6) we get

$$\left(1-1+\frac{1}{q}\right)+\frac{1}{p}t \geq t^{1/p}$$

$$\frac{1}{q}+\frac{1}{p}t \geq t^{1/p} \qquad \ldots(7)$$

Put $t=\frac{a^p}{b^q}$ in the equation (7), we get

$$\frac{1}{q}+\frac{a^p}{b^q}.\frac{1}{p} \geq \left(\frac{a^p}{b^q}\right)^{1/p}$$

or $$\frac{1}{q}+\frac{a^p}{pb^q} \geq \frac{a}{b^{q/p}}$$

or $$\frac{b^q}{q}+\frac{a^p}{p} \geq \frac{ab^q}{b^{q/p}}$$

or $$\frac{b^q}{q}+\frac{a^p}{p} \geq ab^{q-\frac{q}{p}} \qquad \ldots.(8)$$

or $$\frac{b^q}{q}+\frac{a^p}{p} \geq ab$$

$$\left\{\begin{array}{l} \because \frac{1}{p}+\frac{1}{q}=1 \quad \Rightarrow \quad p+q=pq \\ \Rightarrow \quad 1-\frac{q}{p}=q \quad \Rightarrow \quad q-\frac{q}{p}=1 \end{array}\right\}$$

or $$\frac{a^p}{p}+\frac{b^q}{q} \geq ab$$

Where a,b are two non-negative real number.

2.11 Holder Inequality

If a_i, b_i **are non-negative real numbers, then** $\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} b_i^q\right)^{1/q}$

where $\frac{1}{p}+\frac{1}{q}=1$ **and** $p>1$**.**

Proof: To prove this inequality, first we prove an auxilary inequality follows as:

$$\frac{A^p}{p}+\frac{B^q}{q} \geq AB` \qquad \ldots(1)$$

Now put $A=\frac{a_i}{\left(\sum_i a_i^p\right)^{\frac{1}{p}}}$ and $B=\frac{b_i}{\left(\sum_i b_i^q\right)^{\frac{1}{q}}}$ in equation (1), we get,

$$\frac{1}{p}\frac{a_i^p}{\left(\sum_i a_i^p\right)}+\frac{1}{q}\frac{b_i^q}{\sum_i b_i^q} \geq \frac{a_i b_i}{\left(\sum_i a_i^p\right)^{1/p}\left(\sum_i b_i^q\right)^{1/q}}$$

Taking sum of the result from $i=1$ to n.

$$\frac{1}{p}\frac{\sum_{i=1}^{n} a_i^p}{\left(\sum_i a_i^p\right)}+\frac{1}{q}\frac{\sum_{i=1}^{n} b_i^q}{\sum_i b_i^q} \geq \frac{\sum_{i=1}^{n} a_i b_i}{\left(\sum_i a_i^p\right)^{1/p}\left(\sum_i b_i^q\right)^{1/q}}$$

or $$\frac{1}{p}+\frac{1}{q} \geq \frac{\sum_{i=1}^{n} a_i b_i}{\left(\sum_i a_i^p\right)^{1/p}\left(\sum_i b_i^q\right)^{1/q}}$$

$$\Rightarrow \qquad 1 \geq \frac{\sum_{i=1}^{n} a_i b_i}{\left(\sum_i a_i^p\right)^{1/p} \left(\sum_i b_i^q\right)^{1/q}} \qquad \left\{\because \frac{1}{p}+\frac{1}{q}=1\right\}$$

Hence, $\left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} b_i^q\right)^{1/q} \geq \sum_{i=1}^{n} a_i b_i$

Note: 1. Holder's inequality for complex numbers, we have

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |b_i|^q\right)^{1/q}$$

2. Holder inequality for integrals, we have

$$\int_a^b fg dx \leq \left(\int_a^b f^p dx\right)^{1/p} \left(\int_a^b g^q dx\right)^{1/q}$$

3. If we put $p = 2$ ($i.e.,\ q = 2$) in holder inequality, we get

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} \left(\sum_{i=1}^{n} b_i^2\right)^{1/2}$$

Where $a_{i'} b_i$ are non-negative real numbers. This inequality known as cauchy's inequality.

2.12 Cauchy Schwarz-Inequality

If $a_{i'}\ b_i\ (i = 1, 2, 3, \ldots\ldots, n)$ are real or complex numbers, then

$$\sum_{i=1}^{n} |a_i b_i| \leq \left(\sum_{i=1}^{n} |a_i|^2\right)^{1/2} \left(\sum_{i=1}^{n} |b_i|^2\right)^{1/2}$$

or $$\sum_{i=1}^{n}|a_i b_i| \le \|a\|\,\|b\|$$

Proof: Here we have

$$\text{and}\quad \left.\begin{aligned} a &= (a_1, a_2, a_3, \ldots., a_n) \\ b &= (b_1, b_2, b_3, \ldots.., a_n) \end{aligned}\right\} \qquad \ldots.(1)$$

The magnitudes of these above vectors are given as

$$\left.\begin{aligned} \|a\|^2 &= \sum_{i=1}^{n}|a_i|^2 \\ \text{and}\ \ \|b\|^2 &= \sum_{i=1}^{n}|b_i|^2 \end{aligned}\right\} \qquad \ldots.(2)$$

If we take $a = 0$ or $b = 0$ then the inequality reduces to equality. So we let $a \ne 0$ and $b \ne 0$. we know that

Geometric mean $\le$ Arithmetic mean

$$\Rightarrow \qquad \sqrt{xy} \le \frac{x+y}{2}$$

Putting $\sqrt{x} = \dfrac{|a_i|}{\|a\|}$ and $\sqrt{y} = \dfrac{|b_i|}{\|b\|}$, then we get

$$2\,\frac{|a_i||b_i|}{\|a\|\,\|b\|} \le \frac{|a_i|^2}{\|a\|^2} + \frac{|b_i|^2}{\|b\|}$$

or $$2\sum_{i=1}^{n}\ \frac{|a_i||b_i|}{\|a\|\,\|b\|} \le \frac{\sum|a_i|^2}{\|a\|^2} + \frac{\sum|b_i|^2}{\|b\|^2}$$

or $$2\sum_{i=1}^{n}\frac{|a_i||b_i|}{\|a\|\,\|b\|} \le 1+1 \qquad \{\text{using equation (2)}\}$$

or $$\sum_{i=1}^{n} |a_i||b_i| \le \|a\|.\|b\|$$

or $$\sum_{i=1}^{n} |a_i b_i| \le \left(\sum_{i=1}^{n} |a_i|^2 \right)^{1/2} \left(\sum_{i=1}^{n} |b_i|^2 \right)^{1/2}$$

2.13 Minkowski's Inequality

If $p \ge 1$ **and** $a_i, b_i \ (i = 1, 2, 3, \ldots, n)$ **are non-negative real numbers, then**

$$\left(\sum_{i=1}^{n} (a_i + b_i)^p \right)^{1/p} \le \left(\sum_{i=1}^{n} a_i^p \right)^{1/p} + \left(\sum_{i=1}^{n} b_i^p \right)^{1/p}$$

Proof: We know that the holder inequality is

$$\sum_{i=1}^{n} a_i b_i \le \left(\sum_{i=1}^{n} a_i^p \right)^{1/p} \left(\sum_{i=1}^{n} b_i^q \right)^{1/q} \qquad \ldots(1)$$

Where $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$

For $$\frac{1}{p} + \frac{1}{q} = 1$$

$\Rightarrow$ $$q + p = pq$$

$$q(p-1) = p \qquad \ldots.(2)$$

Now, we have

$$\sum_{i=1}^{n} (a_i + b_i)^p = \sum_{i=1}^{n} (a_i + b_i)(a_i + b_i)^{p-1}$$

$$= \sum_{i=1}^{n} a_i \left(a_i + b_i\right)^{p-1} + \sum_{i=1}^{n} b_i \left(a_i + b_i\right)^{p-1}$$

By equation (1), we have

$$\sum_{i=1}^{n} \left(a_i + b_i\right)^p \le \left(\sum_{i=1}^{n} a_i^p \right)^{1/p} \left(\sum_{i=1}^{n} \left(a_i + b_i\right)^{q(p-1)} \right)^{1/q}$$
$$+ \left(\sum_{i=1}^{n} b_i^p \right)^{1/p} \left(\sum_{i=1}^{n} \left(a_i + b_i\right)^{q(p-1)} \right)^{1/q}$$

By equation (2), we have

$$\sum_{i=1}^{n} \left(a_i + b_i\right)^p \le \left(\sum_{i=1}^{n} a_i^p \right)^{1/p} \left(\sum_{i=1}^{n} \left(a_i + b_i\right)^{p} \right)^{1/p}$$
$$+ \left(\sum_{i=1}^{n} b_i^p \right)^{1/p} \left(\sum_{i=1}^{n} \left(a_i + b_i\right)^{p} \right)^{1/q}$$

Dividing by $\left(\sum_{i=1}^{n} \left(a_i + b_i\right)^p \right)^{1/q}$ both sides, we get

$$\left(\sum_{i=1}^{n} \left(a_i + b_i\right)^p \right)^{1-\frac{1}{q}} \le \left(\sum_{i=1}^{n} a_i^p \right)^{1/p} + \left(\sum_{i=1}^{n} b_i^p \right)^{1/p}$$

or $$\left(\sum_{i=1}^{n} \left(a_i + b_i\right)^p \right)^{1/p} \le \left(\sum_{i=1}^{n} a_i^p \right)^{1/p} + \left(\sum_{i=1}^{n} b_i^p \right)^{1/p}$$

Note: Minkowski's inequality for integrals

$$\left(\int_a^b \left(f + g \right)^p dx \right)^{1/p} \le \left(\int_a^b f^p dx \right)^{1/p} + \left(\int_a^b g^p dx \right)^{1/p}$$

Where f and g are non-negative real valued function defined on $\left[a, b\right]$ and $p \ge 1$.

2.14 Minkowski's Inequality in terms of norms

Let $a_i, b_i \left(i = 1, 2, 3, ..., n\right)$ be two $n-$tuples of real or complex numbers, then

$$\left(\sum_{i=1}^{n}|a_i + b_i|^2\right)^{1/2} \leq \left(\sum_{i=1}^{n}|a_i|^2\right)^{1/2} + \left(\sum_{i=1}^{n}|b_i|^2\right)^{1/2}$$

or $$\|a+b\| \leq \|a\| + \|b\|$$

Proof: We know that

$$\|a+b\|^2 = \sum_{i=1}^{n}|a_i + b_i|^2$$

$$= \sum_{i=1}^{n}|a_i + b_i|.|a_i + b_i|$$

$$\leq \sum_{i=1}^{n}|a_i + b_i|.\left(|a_i| + |b_i|\right)$$

$$= \sum_{i=1}^{n}|a_i + b_i|.|a_i| + \sum_{i=1}^{n}|a_i + b_i|.|b_1|$$

Using cauchy-Suchwarz inequality, we have

$$\|a+b\|^2 \leq \left(\sum_{i=1}^{n}|a_i + bi|^2\right)^{1/2}\left(\sum_{i=1}^{n}|a_i|^2\right)^{1/2}$$
$$+\left(\sum_{i=1}^{n}|a_i + b_i|^2\right)^{1/2}\left(\sum_{i=1}^{n}|b_i|^2\right)^{1/2}$$

or $$\|a+b\|^2 \leq \|a+b\|.\|a\| + \|a+b\|.\|b\|$$

or $$\|a+b\| \leq \|a\| + \|a+b\|$$

or $$\left(\sum_{i=1}^{n}|a_i+b_i|^2\right)^{1/2}\le\left(\sum_{i=1}^{n}|a_i|^2\right)^{1/2}\left(\sum_{i=1}^{n}|b_i|^2\right)^{1/2}$$

Examples

Example.4: Let R^2 be the set of all ordered pairs of real numbers and let $d:R^2\times R^2\to R^0$ defined by $d(x,y)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$ then show that d is a metric on R^2.

Solution : Using definition of d, we have

(i) $d(x,y)\ge 0,\ \ \forall x,y\in R^2$, where $x=(x_1,x_2)$ and $y=(y_1,y_2)$

$d(x,y)\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}\ge 0$ is always a non- negative real number.

(ii) $d(x,y)=d(y,x),\ \ \forall x,y\in R^2$

We have

$$d(x,y)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$$

$$=\sqrt{(y_1-x_1)^2+(y_2-x_2)^2}$$

$$=d(y,x),\ \ \forall x,y\in R^2$$

(iii) $d(x,y)\le d(x,z)+d(z,y),\ \ \forall x,y,z\in R^2$

We have

$$d(x,y)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$$

$$= \sqrt{\left\{(x_1 - z_1) + (z_1 - y_1)\right\}^2 + \left\{(x_2 - z_2) + (z_2 - y_2)\right\}^2}$$

$$\leq \sqrt{(x_1 - z_1)^2 + (z_1 - y_1)^2} + \sqrt{(x_2 - z_2)^2 + (z_2 - y_2)^2} \quad \text{(Using Minkowski's inequality)}$$

i.e., $d(x, y) \leq d(x, z) + d(z, y), \quad \forall x, y, z \in R^2$

(iv) If $x = y \quad \Rightarrow \quad d(x, y) = 0, \ \forall x, y \in R^2$

We have

$$x = y \quad \Rightarrow \quad (x_1, x_2) = (y_1, y_2)$$

$$\Rightarrow \quad (x_1 - y_1) = 0 \text{ and } (x_2 - y_2) = 0$$

$$\Rightarrow \quad (x_1 - y_1)^2 = 0 \text{ and } (x_2 - y_2)^2 = 0$$

$$\Rightarrow \quad (x_1 - y_1)^2 + (x_2 - y_2)^2 = 0$$

$$\Rightarrow \quad \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = 0$$

$$\Rightarrow \quad d(x, y) = 0, \ \forall x, y \in R^2$$

(v) If $d(x, y) = 0 \quad \Rightarrow \quad x = y, \ \forall x, y \in R^2$

We have

$$d(x, y) = 0$$

$$\Rightarrow \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = 0$$

$$\Rightarrow \quad (x_1 - y_1)^2 + (x_2 - y_2)^2 = 0$$

$$\Rightarrow \quad (x_1 - y_1)^2 = 0 \text{ and } (x_2 - y_2)^2 = 0$$

$$\Rightarrow \quad x_1 - y_1 = 0 \text{ and } (x_2 - y_2) = 0$$

$$\Rightarrow \quad (x_1, x_2) = (y_1, y_2)$$

$$\Rightarrow \quad x = y.$$

Hence, d is a metric on R^2. The metric space (R^2, d) is known as the Euclidean metric space.

Example.5: The usual metric for R^3 is defined by $d(x,y) = \sqrt{\left\{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2\right\}}$ **or** $d(x,y) = \sqrt{\left\{\sum_{r=1}^{3}(x_r - y_r)^2\right\}}$.

Where $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in R^3$. **To show that** d **is metric on** R^3.

Solution: Using definition of d, we have

(i) $d(x,y) \geq 0, \ \forall x, y \in R^3$

$\Rightarrow \sqrt{\left\{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2\right\}} \geq 0$ is always a non-negative real number.

(ii) $d(x,y) = d(y,x), \ \forall x, y \in R^3$

We have

$$d(x,y) = \sqrt{\left\{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2\right\}}$$

$$= \sqrt{\left\{(y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_3 - x_3)^2\right\}}$$

$$= d(y, x), \quad \forall x, y \in R^3$$

(iii) $d(x, y) \le d(x, z) + d(z, y), \ \forall x, y, z \in R^3$

We have

$$d(x, y) = \sqrt{\left\{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2\right\}}$$

$$= \sqrt{\left\{(x_1 - z_1) + (z_1 - y_1)\right\}^2 + \left\{x_2 - z_2) + z_2 - y_2\right\}^2 + \left\{(x_3 - z_3) + (z_3 - y_3)^2\right\}}$$

$$\le \sqrt{(x_1 - z_1)^2 + (x_2 - z_2)^2 + (x_3 - z_3)^2} + \sqrt{(z_1 - y_1)^2 + (z_2 - y_2)^2 + (z_3 - y_3)^2}$$

$\forall x, y, z \in R^3$ (Using minkowski's enequality)

i.e., $d(x, y) \le d(x, z) + d(z, y), \quad \forall x, y, z \in R^3$

(iv) If $x = y \ \Rightarrow \ d(x, y) = 0, \ \forall x, y \in R^3$

We have

$$x = y \quad \Rightarrow \quad (x_1, x_2, x_3) = (y_1, y_2, y_3)$$

$$\Rightarrow \quad x_1 - y_1 = 0, \quad x_2 - y_2 = 0 \text{ and } x_3 - y_3 = 0$$

$$\Rightarrow \quad (x_1 - y_1)^2 = 0, (x_2 - y_2)^2 = 0, (x_3 - y_3)^2 = 0$$

$$\Rightarrow \quad (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 = 0$$

$$\Rightarrow \quad \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2} = 0$$

$$\Rightarrow \quad d(x, y) = 0, \ \forall x, y \in R^3$$

(v) If $d(x,y)=0 \quad \Rightarrow \quad x=y, \quad \forall x,y \in R^3$

We have

$$d(x,y)=0$$

$$\Rightarrow \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}=0$$

$$\Rightarrow \quad (x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2=0$$

$$\Rightarrow \quad (x_1-y_1)^2=0, (x_2-y_2)^2=0 \text{ and } (x_3-y_3)^2=0$$

$$\Rightarrow \quad x_1-y_1=0, \quad x_2-y_2=0 \text{ and } x_3-y_3=0$$

$$\Rightarrow \quad (x_1,x_2,x_3)=(y_1,y_2,y_3)$$

$$\Rightarrow \quad x=y.$$

Hence, d is ba metric on R^3. The metric space (R^3,d) is known as the Euclidean plane space.

2.15 Summary

Let X be a non-empty set and $d(x,y), \forall x,y \in X$ is a distance function. A real valued function $d: X \times X \to R$ which satisfies the following axioms:

(1) $d(x,y) \geq 0, \forall x,y \in X$.

(2) $d(x,y)=d(y,x), \forall x,y \in X$. (symmetric property)

(3) $d(x,y) \leq d(x,z)+d(z,y), \forall x,y,z \in X$.(Triangular in equality)

(4) If $x = y \Rightarrow d(x, y) = 0$.

(5) If $d(x, y) = 0 \Rightarrow x = y$.

Every metric on X is pseudo metric on X but a pseudo metric on X is not necessarily a metric on X.

Let X be any non-empty set and d be the function defined by $d(x, y) = \begin{cases} 0, \text{ if } x = y \\ 1, \text{ if } x \neq y \end{cases}$ then d is said to be metric on X and (X, d) is called discrete metric space or trivial metric space. A set X is said to be metrizable if and only if a metric can be defined on X. The metric defined on a real line is called usual metric or Euclidean metric on R.

The size of an element x is a real number denoted by $\|x\|$ and is called norm (which is distance $d(x,0)$) if satisfies the following properties.

(1) $\|x\| \geq 0$

(2) $\|x\| = 0$ if and only if $x = 0$

(3) $\|kx\| = |k|.\|x\| \quad \{\|-x\| = \|x\|\}$

(4) $\|x + y\| \leq \|x\| + \|y\|$

Now we define a metric d for a set X with the help of norm as follows:

$$d(x, y) = \|x - y\|, \ \forall x, y \in X$$

This metric is known as metric induced by the norm.

If z_1 and z_2 are two complex numbers, then we have

$$\frac{|z_1+z_2|}{1+|z_1+z_2|} \leq \frac{|z_1|}{1+|z_1|} + \frac{|z_2|}{1+|z_2|} \quad \{\because |z_1+z_2| \leq |z_1|+|z_2|\}$$

If $1 < p < \infty$ and $\frac{1}{p}+\frac{1}{q}=1$ then $ab \leq \frac{a^p}{p}+\frac{b^q}{q}$, where a, b are two non-negative real number.

If a_i, b_i are non-negative real numbers, then $\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} b_i^q\right)^{1/q}$

where $\frac{1}{p}+\frac{1}{q}=1$ and $p>1$.

If $a_i, b_i \ (i=1,2,3,.....,n)$ are real or complex numbers, then

$$\sum_{i=1}^{n} |a_i b_i| \leq \left(\sum_{i=1}^{n} |a_i|^2\right)^{1/2} \left(\sum_{i=1}^{n} |b_i|^2\right)^{1/2} \quad \text{or} \quad \sum_{i=1}^{n} |a_i b_i| \leq \|a\| \|b\|$$

If $p \geq 1$ and $a_i, b_i \ (i=1,2,3,...,n)$ are non-negative real numbers, then

$$\left(\sum_{i=1}^{n} (a_i+b_i)^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} a_i^p\right)^{1/p} + \left(\sum_{i=1}^{n} b_i^p\right)^{1/p}$$

2.16 Terminal Questions

Q.1.Explain the metric spaces.

Q.2. What do you mean by Pseudo metric space.

Q.3. Give an example of a pseudometric which is not a metric. Is every metric a pseudo-metric?

Q.4.State and prove Holder’s ineuality.

Q.5. State and prove Minkowski's inequality.

Q.6. State and prove Cauchy-schwarz inequality.

Q.7.Does $d(x,y)=(x-y)^2$ define a metric on the set of the real numbers? Give reason for your answer.

Q.8. Show that $d(x,y)=\sqrt{|x-y|}$ defines a metric on the set of all real numbers.

Q.9. Let $R[0,1]$ denotes the class of all Reimann integrable function f from $f\,[0,1]$ into R. Let a mapping $d:R[0,1]\times R[0,1]\to R$ defined by $d(f,g)=\int_0^1 |f-g|(x)\,dx$ $=\int_0^1 |f(x)-g(x)|\,dx$. Then to show that d is pseudometric but not metric on R.

Answers

3. A function $d:R\times R\to R$ defined by $d(x,y)=\left|x^2-y^2\right|\forall x,y\in R$ is a pseudometric on R but not metric on R. Yes, every metric is a pseudometric but converse is not true.

12. $d(x,y)=(x-y)^2$ not define a matric on the set of all real number because triangular inequality is not satisfied as:

$$d(x,y)=(x-y)^2$$

$$=(x-z+z-y)^2$$

$$=(x-z)^2+(z-y)^2+2(x-z)(z-y)$$

$$\geq(x-z)^2+(z-y)^2$$

Hence $d(x,y)\geq d(x,z)+d(z,y)$, which is not true.

UNIT- 3: Bounded and Unbounded Metric Spaces

Structure

3.1 Introduction

In mathematics, the concepts of bounded and unbounded metric spaces are related to the behavior of distances within the space.A metric space is said to be bounded if there exists a real number M such that the distance between any two points in the space is less than or equal to M. Formally, a metric space (X,d) is bounded if there exists a real number M such that for all $d(x,y) \le k, \forall x, y \in X$. Conversely, a metric space is said to be unbounded if it is not bounded, meaning that there is no such real number M that satisfies the above condition for all pairs of points in the space.

For example, the real line R with the usual metric is an unbounded metric space, as there is no finite value of M that bounds the distances between all pairs of points on the real line. On the other hand, the closed interval [0,1] in R with the usual metric is a bounded metric space, as the distances between any two points in the interval are always less than or equal to 1.

In this unit we shall discuss about the bounded and unbounded metric spaces, and Quasi metric with their applications in details.

3.2 Objectives

After studying this unit the learner will be able to understand the:

- Bounded and unbounded metric spaces
- Quasi Metric and their applications

3.3 Bounded and Unbounded Metric Space

The concepts of boundedness and unboundedness in metric spaces relate directly to the distances between points in the space. In a bounded metric space, there is a finite upper bound on the distances between any two points, meaning that no two points are "too far apart." In contrast, an unbounded metric space lacks such a finite bound, allowing for the possibility of arbitrarily large distances between points.

Let (X,d) be a metric space and let k be a positive real number. If there exists a number such that

$$d(x,y) \le k, \forall x, y \in X$$

Then (X,d) is known as bounded metric space.A metric space which is not bounded known as unbounded metric space.

Examples

Example.1. Let (X,d) be a metric space and consider $d^*(x,y) = \dfrac{d(x,y)}{1+d(x,y)} \quad \forall x, y \in X$.

Show that d^* is a bounded metric on X.

Solution: Consider (X,d) a metric space and $d^*(x,y) = \dfrac{d(x,y)}{1+d(x,y)} \quad \forall x, y \in X$.

Using definition of d, we have

(i) $d^*(x,y) \ge 0, \ \forall x, y \in X$

$\Rightarrow \quad \dfrac{d(x,y)}{1+d(x,y)} \ge 0$ is always a non-negative real number.

(ii) $d^*(x,y)=d^*(y,x) \;\; \forall x,y\in X$

We have $d^*(x,y)=\dfrac{d(x,y)}{1+d(x,y)}$

$$=\frac{d(y,x)}{1+d(y,x)}$$

$$=d^*(y,x) \;\; \forall x,y\in X$$

(iii) $d^*(x,y)\le d^*(x,z)+d^*(z,y), \;\; \forall x,y,z\in X$

We have $\dfrac{d(x,y)}{1+d(x,y)}\le\dfrac{d(x,z)+d(z,y)}{1+d(x,z)+d(z,y)}$

$$\le\frac{d(x,z)}{1+d(x,z)+d(z,y)}+\frac{d(z,y)}{1+d(x,z)+d(z,y)}$$

$$\le\frac{d(x,z)}{1+d(x,z)}+\frac{d(z,y)}{1+d(z,y)} \;\; \forall x,y,z\in X$$

i.e., $d^*(x,y)\le d^*(x,z)+d^*(z,y), \quad \forall x,y,z\in X$

(iv) If $x=y \quad \Rightarrow \quad d^*(x,y)=0, \quad \forall x,y\in X$

We have $\quad x=y \quad \Rightarrow \quad d(x,y)=0$

$$\Rightarrow \qquad \frac{d(x,y)}{1+d(x,y)}=0, \;\; \forall x,y\in X$$

(v) If $d^*(x,y)=0 \quad \Rightarrow \quad x=y$

We have $\quad \dfrac{d(x,y)}{1+d(x,y)}=0$

$$\Rightarrow \quad d(x,y)=0$$

$$\Rightarrow \quad x=y \ \ \forall x,y\in X$$

Hence d^* is a metric on X.

Now we have

$$d^*(x,y)=\frac{d(x,y)}{1+d(x,y)}$$

$$=\frac{1}{1+\frac{1}{d(x,y)}}$$

$$\leq 1, \ \ \forall x,y\in X$$

Hence the given metric d^* is bounded metric on X.

3.4 Quasi Metric

A quasi-metric (or semimetric) is a generalization of the concept of a metric that relaxes the requirement that the distance between distinct points must be positive.Consider X be a non-empty set and suppose $x,y,z\in X$ be arbitrary. A mapping $d:X\times X\to[0,\infty)$ satisfies the following axioms:

(i) $d(x,x)=0$

(ii) $d(x,y)=d(y,x)$

(iii) $d(x,y)\leq d(x,z)+d(z,y)$

is known as quasi-metric on X. The set X together with the quasi-metric d is called auasi-metric space denoted by (X,d). The quasi-metric is said to be finite if and only if $d(x,y)<\infty, \forall x,y\in X$.

Hence the key difference between a quasi-metric and a metric is that in a quasi-metric space, the distance between distinct points can be zero, meaning that two different points can be "infinitesimally close" to each other but not necessarily equal. Quasi-metrics are used in situations where a notion of distance that allows for such "coincidence" is desirable, such as in certain areas of analysis, geometry, and topology.

Examples

Example.2. Let (X,d) be a metric space and let x, y, z be any three points of X . Then show that

$$d(x,y)\geq |d(x,z)-d(z,y)|.$$

Solution: Using definition of d, we have

$$d(x,z)\leq d(x,y)+d(y,z) \qquad \text{(Using triangular inequality)}$$

$$=d(x,y)+d(z,y) \qquad \text{(Using symmetric property)}$$

Thus we have $d(x,y)\geq d(x,z)-d(z,y)$ ….(1)

Now we have

$$d(z,y)\leq d(z,x)+d(x,y) \qquad \text{(Using triangular inequality)}$$

$$=d(x,z)+d(x,y) \qquad \text{(Using symmetric property)}$$

$$d(x,y) \geq d(z,y) - d(x,z)$$

i.e. $$d(x,y) \geq -\left[d(x,z) - d(z,y)\right] \qquad \ldots.(2)$$

From theequations (1) and (2), we have

$$d(x,y) \geq \left|d(x,z) - d(z,y)\right|$$

Example.3. Let (X,d) be a metric space and let d_1 be defined by

$$d_1(x,y) = \min\{d(x,y),1\}, \forall x,y \in X$$

Show that (X,d_1) is a metric space.

Solution: Using definition of d, we have

1. $d_1(x,y) \geq 0, \ \forall x,y \in X$

   $\Rightarrow \quad \min\{d(x,y),1\} \geq 0$ is always a non-negative real number.

2. $d_1(x,y) = d_1(y,x), \ \forall x,y \in X$

   $\Rightarrow \quad \min\{d(x,y),1\} = \min\{d(y,x),1\}, \quad \forall x,y,z \in X$

3. $d_1(x,y) \leq d_1(x,z) + d_1(z,y), \ \forall x,y,z \in X$

Suppose if $d_1(x,y) = \min\{d(x,y),1\} = 1$ then we get

$$1 \leq 1+1$$

If $d_1(x,y) = \min\{d(x,y),1\} = d(x,y)$ then we get

$$d(x,y) \le d(x,z) + d(z,y), \ \forall x,y,z \in X$$

4. If $x = y \quad \Rightarrow \quad d_1(x,y) = 0 \ \forall x,y \in X$

We have $\quad x = y \quad \Rightarrow \quad d(x,y) = 0$

$\Rightarrow \quad \min\{d(x,y),1\} = \min\{0,1\} = 0$

$\Rightarrow \quad d_1(x,y) = 0, \ \forall x,y \in X$.

5. If $d_1(x,y) = 0$ then $x = y$

$\Rightarrow \quad \min\{d(x,y),1\} = 0$

$\Rightarrow \quad d(x,y) = 0 \quad \Rightarrow \quad x = y \quad \forall x,y \in X$

Hence d_1 is a metric on X and (X,d_1) is a metric space.

Example.4. Suppose d metric on X. Determine the all constant k such that:

(i) $kd(x,y)$ (ii) $x(x,y)+k$ is a metric on X .

Solution: Given that (X,d) is a metric space *i.e.*, it satisfies all the axioms of metric. To determine all constants k, we have

$$d_1(x,y) = kd(x,y)$$

1. Using first axiom of metric, we have

$d_1(x,y) \ge 0$

$\Rightarrow \quad kd(x,y) \ge 0$

$\Rightarrow \quad k > 0.$

Now we let $d^*(x,y) = d(x,y) + k$

2. Using fifth axiom of metric, we have

$d^*(x,y) = 0$ if $x = y$

$\Rightarrow \quad d(x,y) + k = 0$ if $x = y$

$\Rightarrow \quad k = 0.$

Example.5. Let R^2 **denote the set of all ordered pairs of real numbers. Show that the mapping (function)** $d : R^2 \times R^2 \to R^0$ **defined by**

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2|, \textbf{ where } x = (x_1, x_2), y = (y_1, y_2) \in R^2$$

Is a metric on R^2.

Solution: Using definition of d, we have

1. $d(x,y) \geq 0 \ \ \forall x, y \in R^2 \Rightarrow |x_1 - y_1| + |x_2 - y_2| \geq 0$ is always a non-negative real number.

2. $d(x,y) = d(y,x), \ \ \forall x, y \in R^2 \Rightarrow |x_1 - y_1| + |x_2 - y_2| = |y_1 - x_1| + |y_2 - x_2| \ \ \forall x, y \in R^2$

3. $d(x,y) \leq d(x,z) + d(z,y), \ \forall x, y, z \in R^2$

$$|x_1 - y_1| + |x_2 - y_2| = |(x_1 - z_1) + (z_1 - y_1)| + |(x_2 - z_2) + (z_2 - y_2)|$$

$$\leq \{|x_1 - z_1| + |z_1 - y_1|\} + \{|x_2 - z_2| + |z_2 - y_2|\}$$

(Using triangular inequality)

$$= \{|x_1 - z_1| + |x_2 - z_2|\} + \{|z_1 - y_1| + |z_2 - y_2|\}$$

i.e., $d(x, y) \le d(x, z) + d(z, y) \;\; \forall x, y, z \in R^2$

4. If $x = y \Rightarrow d(x, y) = 0 \;\forall x, y \in R^2$

$$\Rightarrow (x_1, x_2) = (y_1, y_2) \Rightarrow x_1 - y_1 = 0 \text{ and } x_2 - y_2 = 0$$

$$\Rightarrow |x_1 - y_1| = 0 \text{ and } |x_2 - y_2| = 0 |x_1 - y_1| + |x_2 - y_2| = 0, \;\forall x, y \in R^2$$

5. If $d(x, y) = 0 \Rightarrow x = y \,\forall x, y \in R^2 \Rightarrow |x_1 - y_1| + |x_2 - y_2| = 0 \Rightarrow |x_1 - y_1| = 0$ and $|x_2 - y_2| = 0$

$$\Rightarrow x_1 - y_1 = 0 \text{ and } x_2 - y_2 = 0$$

$$\Rightarrow x_1 = y_1 \text{ and } x_2 = y_2$$

$$\Rightarrow (x_1, y_2) = (y_1 = y_2) \forall x, y \in R^2$$

Hence, d is a metric on R^2.

Example.6. Let R^2 **be the set of all ordered pairs of real numbers and let** $d : R^2 \times R^2 \to R^0$ **be defined by** $d(x, y) = \max.\{|x_1 - y_1|.|x_2 - y_2|\}$

Where $x = (x_1, x_2), y = (y_1, y_2) \in R^2$

Is a metric on R^2**.**

Solution: Using definition of d, we have

1. $d(x,y) \geq 0, \quad \forall x, y \in R^2 \Rightarrow \max.\{|x_1 - y_1|.|x_2 - y_2|\} \geq 0$ is always a non-negative real number.

2. $d(x,y) = d(y,x), \forall x, y \in R^2$

   $\Rightarrow \max.\{|x_1 - y_1|.|x_2 - y_2|\} = \max.\{|y_1 - x_1|.|y_2 - x_2|\}, \quad \forall x, y \in R^2$

3. $d(x,y) \leq d(x,z) + d(z,y) \forall x, y, z \in R^2$

   $\Rightarrow \max.\{|x_1 - y_1|.|x_2 - y_2|\} = \max.\{[x_1 - z_1) + (z_1 - y_1)|].[1(x_2 - z_2) + (z_2 - y_2)|]\}$

   $\leq \max.\{|x_1 - z_1|.|z_1 - y_1|\} + \max.\{|x_2 - z_2| + |z_2 - y_2|\}$ {since $|x + y| \leq |x| + |y|$ therefore

   $|x + y| \leq \max.\{|x_1|.|x_2| + \max\{|y_1|.|y_2|\}$

   i.e, $d(x,y) \leq d(x,z) + d(z,y), \quad \forall x, y, z \in R^2$

4. If $x = y \Rightarrow d(x,y) = 0, \forall x, y \in R^2$

   $\Rightarrow (x_1, x_2) = (y_1, y_2) \Rightarrow x_1 - y_1 = 0$ and $x_2 - y_2 = 0$

   $\Rightarrow |x_1 - y_1| = 0$ and $|x_2 - y_2| = 0$

   $\Rightarrow \{|x_1 - y_1|.|x_2 - y_2| = 0$

   $\Rightarrow \max.\{|x_1 - y_1|.|x_2 - y_2| = 0, \quad \forall x, y \in R^2$

5. If $d(x,y) = 0 \Rightarrow x = y, \quad \forall x, y \in R^2$

$\Rightarrow \quad \max.\{|x_1 - y_1|.|x_2 - y_2|\} = 0$

$\Rightarrow \quad \{|x_1 - y_1|.|x_2 - y_2|\} = 0$

$\Rightarrow \qquad |x_1 - y_1| = 0$ and $|x_2 - y_2| = 0$

$\Rightarrow \qquad x_1 - y_1$ and $x_2 - y_2$

$\Rightarrow \qquad (x_1 - x_2) = (y_2, y_2), \ \forall x, y \in R^2$

Hence, d is a metric on R^2.

Example.7: The usual metric for R^n **is defined by** $d : R^n \times R^n \to R^0$ **such that**

$$d(x, y) = \sqrt{\left(\sum_{r=1}^{n} (x_r - y_r)^2\right)}$$

Where $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in R^n$

To show that d **is metric on** R^n **.**

Solution: Using definition of d, we have

(1) $\quad d(x, y) \geq 0 \ \forall x, y \in R^n$

$\Rightarrow \quad \sqrt{\left(\sum_{r=1}^{n} (x_r - y_r)^2\right)} \geq 0$ is always a non-negative real number.

(2) $\quad d(x, y) = d(y, x), \ \forall x, y, z \in R^n$

$$\Rightarrow \quad \sqrt{\left[\sum_{r=1}^{n} (x_r - y_r)^2\right]} = \sqrt{\left[\sum_{r=1}^{n} (y_r - x_r)^2\right]}, \quad \forall x, y \in R^n$$

(3) $\quad d(x, y) \leq d(x, z) + d(z, y), \ \forall x, y, z \in R^n$

$$\Rightarrow \quad \sqrt{\left[\sum_{r=1}^{n}\left(x_r - y_r\right)^2\right]} = \sqrt{\left[\sum_{r=1}^{n}\left\{\left(x_r - z_r\right) + \left(z_r - y_r\right)\right\}^2\right]}$$

$$\leq \sqrt{\left[\sum_{r=1}^{n}\left(x_r - z_r\right)^2\right]} + \sqrt{\left[\sum_{r=1}^{n}\left(z_r - y_r\right)^2\right]} \quad \forall x, y, z \in R^n$$

(Using Minkowski's inequality)

(4) If $x = y \quad \Rightarrow \quad d\left(x, y\right) = 0, \quad \forall x, y \in R^n$

$$\Rightarrow \quad \left(x_1, x_2, ..., x_n\right) = \left(y_1, y_2, ..., y_n\right)$$

$$\Rightarrow \quad x_1 - y_1 = 0, x_2 - y_2 = 0, ..., x_n - y_n = 0$$

$$\Rightarrow \quad \left(x_1 - y_1\right)^2 = 0, \left(x_2 - y_2\right)^2 = 0, ..., \left(x_n - y_n\right)^2 = 0$$

$$\Rightarrow \quad \left(x_1 - y_1\right)^2 + \left(x_2 - y_2\right)^2 + ... + \left(x_n - y_n\right)^2 = 0$$

$$\Rightarrow \quad \sqrt{\left(x_1 - y_1\right)^2 + \left(x_2 - y_2\right)^2 + ... + \left(x_n - y_n\right)^2} = 0$$

$$\Rightarrow \quad \sqrt{\left[\sum_{r=1}^{n}\left(x_r - y_r\right)^2\right.} = 0$$

(5) If $d\left(x, y\right) = 0 \Rightarrow x = y \quad \forall x, y \in R^n$

$$\Rightarrow \quad \sqrt{\left[\sum_{r=1}^{n}\left(x_r - y_r\right)^2\right.} = 0$$

$$\Rightarrow \quad \sum_{r=0}^{n}\left(x_1 - y_1\right)^2 = 0$$

$$\Rightarrow \quad \left(x_1 - y_1\right)^2 = 0, \left(x_2 - y_2\right)^2 = 0, ..., \left(x_n - y_n\right)^2 = 0$$

$$\Rightarrow \quad x_1 - y_1 = 0, x_2 - y_2 = 0, ..., x_n - y_n = 0$$

$\Rightarrow \qquad (x_1, x_2, ..., x_n) = (y_1, y_2, ..., y_n) = 0$

Hence, d is a metric on R^n. The metric space (R^n, d) is called the real Euclidean space.

Example.8. Let $X = X_1 \times X_2$, **where** X_1 **and** X_2 **are metric spaces with metrics** d_1 **and** d_2 **respectively. Show that a metric** d **is defined by** $d(x, y) = d_1(x_1, y_1) + d_2(x_2, y_2)$

Where $x = (x_1, x_2)$ **and** $y = (y_1, y_2)$ **is a product metric space.**

Solution: Using definition of d, we have

1. $\qquad d(x, y) \geq 0, \ \forall x, y \in X \Rightarrow d_1(x_1, y_1) + d_2(x_2 y_2) \geq 0$ is always a non-negative real number.

2. $\qquad d(x, y) = d(y, x), \forall x, y \in X$

$\Rightarrow d_1(x_1, y_1) + d_2(x_2, y_2) = d_1(y_1, x_1) + d_2(y_2, x_2), \ \forall x, y \in X$

3. $\qquad d(x, y) \leq d(x, z) + d(z, y), \ \forall x, y, z \in X$

$d(x, y) = d_1(x_1, y_1) + d_2(x_2, y_2) \leq d_1(x_1, y_2) + d_1(z_1, y_1) + d_2(x_2, z_2) + d_2(z_2, y_2)$

i.e., $d(x, y) \leq d(x, z) + d(z, y) \ \forall x, y, z \in X$

4. $\qquad$ If $x = y \Rightarrow d(x, y) = 0, \forall x, y \in z$

$\Rightarrow x = y \Rightarrow (x_1, x_2) = (y_1, y_2) \Rightarrow d_1(x_1, y_1) = 0$ and $d_2(x_2, y_2) = 0$

$\Rightarrow d_1(x_1, y_1) + d_2(x_2, y_2) = 0 \Rightarrow d(x, y) = 0, \ \forall x, y \in X$

5. $\qquad$ If $d(x, y) = 0 \Rightarrow x = y, \ \forall x, y \in X$

$\Rightarrow d_1(x_1, y_1) + d_2(x_2, y_2) = 0$

$\Rightarrow d_1(x_1, y_1) = 0$ and $d_2(x_2, y_2) = 0$

$\Rightarrow x_1 = y_1$ and $x_2 = y_2$

$$\Rightarrow (x_1, x_2) = (y_1, y_2)$$

$$\Rightarrow x = y,\ \forall x, y \in X\ .$$

Hence, d is a metric on X and (X, d) is a product metric space.

3.7 Summary

Let (X, d) be a metric space and let k be a positive real number. If there exists a number such that $d(x, y) \le k, \forall x, y \in X$ then (X, d) is said to be a bounded metric space. A metric said which is not bounded known as unbounded metric space.

Consider X be a non-empty set and suppose $x, y, z \in X$ be arbitrary. A mapping $d : X \times X \to [0, \infty)$ satisfies the following axioms:

(i) $d(x, x) = 0$

(ii) $d(x, y) = d(y, x)$

(iii) $d(x, y) \le d(x, z) + d(z, y)$

is known as quasi-metric on X. The set X together with the quasi-metric d is called quasi-metric space denoted by (X, d). The quasi-metric is said to be finite if and only if

$d(x,y)<\infty, \forall x,y \in X$.

3.8 Terminal Questions

Q.1. What do you mean by Bounded and Unbounded Metric Spaces.

Q.2. Explain the Quasi metric space.

Q.3. Let (X,d) be a metric space and let $x_1, x_2, y_1, y_2 \in X$. Then show that $|d(x_1,y_1)-d(x_2,y_2)| \le d(x_1,x_2)+d(y_1,y_2)$.

Q.4. Give two different matrices for the set R of real numbers.

Q.5. Let b be a metric for a non-empty set X . Show that d_1 defined as $d_1(x,y)=2d(x,y)$ is also a metric for X .

Q.6. Let X be a non-empty set and let d be a real valued function of ordered pairs of elements of X which satisfies the following conditions:

(i) $d(x,y)$ if and only if $x=y$

(ii) $d(x,y) \le d(x,z)+d(y,z), \forall x,y,z \in X$. Show that d is a metric on X .

Q.7. Let (X,d) be any metric space and let k be a positive number, then there exists a metric d_1 for X such that the metric space (X,d_1) is bounded with $\delta(X) \le k$.

Q.8. If d is a metric for a non-empty set X , then show that the function d_1 defined by $d_1(x,y)=\min\{2,|x-y|\}$ is a metric for X .

Q.9. Let (X,d) be a metric space and let x, y, z, w be any point of X. Then $|d(x,y)-d(z,w)| \leq d(x,z)+d(y,w)$.

Answer

4. (i) Consider a function $d: R\times R \to R$ defined by $d(x,y)=|x-y|, \quad \forall\, x,y \in R.$

(ii) Consider a function $d: R\times R \to R$ defined by $d(x,y)=\begin{cases} 0, \text{ if } x=y \\ 1, \text{ if } x\neq y \end{cases}.$

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Master of Science/ Master of Arts

PGMM-109N
MAMM-109N
Topology

Block

2 Metric Spaces-II

Block-2

Metric Spaces-II

Metric spaces are used in engineering for tasks such as optimization, control theory, and signal processing. They provide a mathematical framework for analyzing and solving problems in these areas. Many important concepts in topology, such as convergence, compactness, and connectedness, are defined using the notion of a metric. Metric spaces play a crucial role in modern mathematics, providing a framework for studying the properties of spaces and their relationships in a precise and rigorous manner.

Metric spaces play a foundational role in mathematics, especially in analysis and its applications. They provide a generalization of the concept of distance beyond Euclidean spaces, enabling mathematicians to explore properties of spaces in more abstract settings. Metric spaces are valuable in geometry, offering a framework to study geometric properties of spaces that cannot be described by traditional Euclidean geometry.

In the fourth unit, we shall discussed Sequence spaces l^∞, Function space, sequence space l^p, Hilbert sequence space l^2, Open and closed ball, sphere, neighbourhood of a point, limit point, equivalent Metrics.

In the fifth unit we shall discuss the Sequence in a Metric Space, Convergent Sequence in a Metric Space, Bounded Set, Cauchy Sequence, Continuity and Homeomorphism of metric spaces, Homeomorphic Spaces.

Complete Metric Space, Incomplete Metric Space, Contor' Intersection theorem, Completeness of Care discussed in details in the unit sixth.

UNIT-4: Spaces in Metric

Structure

4.1 **Introduction**

4.2 **Objectives**

4.3 **Sequence Spaces** l^{∞}

4.4 **Function Spaces** $C[A,B]$

4.5 **Sequence Spaces** $[S \ or \ F(\text{Frechet Space})]$

4.6 **Space B(A) or Bounded Function**

4.7 **Sequence space** l^p

4.8 **Hilbert sequence space** l^2

4.9 **Open Ball, closed ball and sphere**

4.10 **Neighbourhood of a point**

4.11 **Limit point of A Set**

4.12 **Equivalent Metrics**

4.13 **Summary**

4.14 **Terminal Questions**

4.1 Introduction

Sequence spaces are a specific type of metric space that is particularly useful in analysis and functional analysis. In sequence spaces, the elements are sequences of real or complex numbers, and the metric is often defined in terms of a norm. Sequence spaces and their properties are important in the study of functional analysis, especially in the context of studying the convergence and properties of sequences of functions.

Function spaces in the context of metric spaces refer to spaces where the elements are functions and the metric is used to define distances between these functions. These spaces are fundamental in various areas of mathematics, especially in analysis and functional analysis.

4.2 Objectives

After reading this unit the learner should be able to understand about:

- Sequence spaces l^∞ and Function Spaces $C[A, B]$
- Sequence Spaces $[S \text{ or } F (\text{Frechet Space})]$
- Space B(A) or Bounded Function
- Sequence space l^p
- Hilbert sequence space l^2
- Open Ball, closed ball and sphere
- Neighbourhood of a point, limit point of A Set
- Equivalent matrices

4.3 Sequence Spaces l^{∞}

Sequence spaces are a significant concept in functional analysis and various branches of mathematics. Let X be a non-empty set. Then a sequence in a set X is any mapping from the set of natural numbers into X. Here we shall discuss some examples of metric sequence spaces which are following:

Sequence Spaces ℓ^{∞}

Let X be the set of all bounded sequences of complex number *i.e.*, every element of X is a complex sequence

$$x=\left(x_1, x_2, ...\right) \qquad\qquad i.e., x=\left(x_i\right)$$

For all $i=1,2,3,...,$ we have

$$\left|x_i\right| \leq C_x$$

Where C_x is a real number which may depend on x but does not depend on i then the metric dfined by

$$d\left(x, y\right)=\sup_{i\in N}\left|x_i-y_i\right|$$

Where $y=\left(y_i\right)\in X$ and $N=\{1,2,3,,...\}$

Now we shall show that the function d satisfies all the five axioms of a metric on X.

1. $d\left(x, y\right)\geq 0, \quad \forall x, y \in X$

$\Rightarrow \sup\limits_{i\in N}\left|x_i-y_i\right|\geq 0$ is always a non-negative real number.

2. $d\left(x, y\right)=d\left(y, x\right), \quad \forall x, y\in X$

$$\Rightarrow \sup_{i\in N}|x_i - y_i| = \sup_{i\in N}|y_i - x_i|, \quad \forall x, y \in X$$

3. $d(x,y) \le d(x,z) + d(z,y), \ \forall x, y, z \in X$

$$\Rightarrow \sup_{i\in N}|x_i - y_i| = \sup_{i\in N}|(x_i - z_i) + (z_i - y_i)|$$

$$\le \sup_{i\in N}|x_i - z_i| + \sup_{i\in N}|z_i - y_i|, \quad \forall x, y, z \in X$$

4. If $x = y \Rightarrow d(x,y) = 0, \quad \forall x, y \in X$

$$\Rightarrow x = y \Rightarrow x_i = y_i \Rightarrow x_i - y_i = 0$$

$$\Rightarrow |x_i - y_i| = 0 \Rightarrow \sup_{i\in N}|x_i - y_i| = 0, \quad \forall x, y \in X$$

5. If $d(x,y) = 0 \Rightarrow x = y, \quad \forall x, y \in X$

$$\Rightarrow \sup_{i\in N}|x_i - y_i| = 0 \Rightarrow |x_i - y_i| = 0 \Rightarrow x_i - y_i = 0$$

$$\Rightarrow x_i = y_i \Rightarrow x = y, \ \forall x, y \in X$$

Hence, d is a metric on X and (X, d) is called metric space. This metric space is denoted by ℓ^∞. Thus, ℓ^∞ is a sequence space because each element of X is a sequence.

4.4 Function Spaces $C[A, B]$

Sequence spaces are fundamental in the study of linear operators and functional spaces. Let X be the set of all real valued functions $x, y, z, \ldots,$ which are functions of independent real variable t and are defined and continuous on a given closed interval $I = [a, b]$. Then the metric defined by

$$d(x,y) = \max_{t\in I} |x(t) - y(t)|$$

Now we shall show that the function d satisfies all the five axioms of a metric on X .

1. $d(x,y) \geq 0, \quad \forall x, y \in X$

$\Rightarrow \qquad \max_{t\in I} |x(t) - y(t)| \geq 0$ is always a non-negative real number.

2. $d(x,y) = d(y,x) \ \forall x, y \in X$

$$\max_{t\in I} |x(t) - y(t)| = \max_{t\in I} |y(t) - x(t)| \ \ \forall x, y \in X.$$

3. $d(x,y) \leq d(x,z) + d(z,y), \quad \forall x, y, z \in X$

$$\Rightarrow \max_{t\in I} |x(t) - y(t)| = \max_{t\in I} \left|\{y(t) - z(t)\} + \{z(t) - y(t)\}\right|$$

$$\leq \max_{t\in I} |x(t) - z(t)| + \max_{t\in I} |z(t) - y(t), \quad \forall x, y \in X \ .$$

4. If $x = y \Rightarrow d(x,y) = 0, \ \forall x, y \in X$

$$\Rightarrow x = y \Rightarrow x(t) = y(t) \Rightarrow x(t) - y(t) = 0$$

$$|x(t) - y(t)| = 0 \Rightarrow \max_{t\in I} |x(t) - y(t)| = 0 \quad \forall x, y \in X$$

5. If $d(x,y) = 0 \Rightarrow x = y \ \ \forall x, y \in X$

$$\Rightarrow \max_{t\in I} |x(t) - y(t)| = 0 \Rightarrow |x(t) - y(t)| = 0$$

$$\Rightarrow x(t) - y(t) = 0 \Rightarrow x - y = 0$$

$$\Rightarrow x = y \quad \forall x, y \in X$$

Hence, d is metric on X and (X,d) is called metric space. This metric space is denoted by $C[a,b]$.

Note: $C[a,b]$ is also known as function space because every point of $C[a,b]$ is a function.

4.5 Sequence Spaces $[S \text{ or } F\,(\text{Frechet Space})]$

Let X be the set of all (bounded or unbounded) sequences of complex numbers and the metric d is defined by

$$d(x,y)=\sum_{i=1}^{\infty}\frac{1}{2^i}\frac{|x_i-y_i|}{1+|x_i-y_i|}$$

Where $x=(x_i)$ and $y=(y_i)$

Now we shall show that the function d satisfies all the five axioms of a metric on X.

1. $d(x,y)\geq 0, \quad \forall\, x,y\in X$

   $\Rightarrow \sum_{i=1}^{\infty}\frac{1}{2^i}\frac{|x_i-y_i|}{1+|x_i-y_i|}\geq 0$ is always a non-negative real number.

2. $d(x,y)=d(y,x), \ \forall\, x,y\in X$

$$\Rightarrow \sum_{i=1}^{\infty}\frac{1}{2^i}\frac{|x_i-y_i|}{1+|x_i-y_i|}=\sum_{i=1}^{\infty}\frac{1}{2^i}\frac{|y_i-x_i|}{1+|y_i-x_i|} \quad \forall x,y\in X$$

3. $d(x,y)\leq d(x,z)+d(z,y), \quad \forall\, x,y,z\in X$

$$\Rightarrow \sum_{i=1}^{\infty}\frac{1}{2^i}\frac{|x_i-y_i|}{1+|x_i-y_i|}=\sum_{i=1}^{\infty}\frac{1}{2^i}\frac{|(x_i-z_i)+(z_i-y_i)|}{1+|(x_i-z_i)+(z_i-y_i)|}$$

$$\le \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\left|\left(x_i - z_i\right)\right)|}{1+\left|\left(x_i - z_i\right)+\left(z_i - y_i\right)\right|} + \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\left|z_i - y_i\right|}{1+\left|x_i - z_i\right|+\left|z_i - y_i\right|}$$

$$\le \sum_{i=1}^{\infty} \frac{1}{z^i} \frac{\left|x_i - z_i\right|}{1+\left|\left(x_i - z_i\right)\right.} + \sum_{i=1}^{\infty} \frac{1}{z^i} \frac{\left|z_i - y_i\right|}{1+z_i - y_i} \quad \forall x, y, z \in X \ .$$

4. If $x = y \Rightarrow d\left(x, y\right) = 0, \quad \forall x, y \in X$

$$\Rightarrow x = y \Rightarrow x_i - y_i = 0 \Rightarrow \left|x_i - y_i\right| = 0$$

$$\Rightarrow \frac{\left|x_i - y_i\right|}{1+\left|x_i - y_i\right|} = 0 \Rightarrow \frac{1}{2^i} \frac{\left|x_i - y_i\right|}{1+\left|x_i - y_i\right|} = 0$$

$$\Rightarrow \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\left|x_i - y_i\right|}{1+\left|x_i - y_i\right|} = 0 \quad \forall x, y, z \in X \ .$$

5. If $d\left(x, y\right) = 0 \Rightarrow x = y \quad \forall x, y \in X$

$$\Rightarrow \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\left|\left(x_i - y_i\right)\right|}{1+\left|\left(x_i - y_i\right)\right|} = 0 \Rightarrow \frac{1}{2^i} \frac{\left|x_i - y_i\right|}{1+\left|x_i - y_i\right|} = 0 \Rightarrow \frac{\left|x_i - y_i\right|}{1+\left|x_i - y_i\right|} = 0$$

$$\Rightarrow \left|x_i - y_i\right| = 0 \Rightarrow x_i - y_i = 0 \Rightarrow x_i = y_i \Rightarrow x = y \quad \forall x, y \in X \ .$$

Hence, d is metric on X and $\left(X, d\right)$ is called metric space. This metric space is denoted by S or F (Frechet space).

4.6 Space B(A) or Bounded Function

The space B(A) or the space of bounded functions on a set A is a fundamental concept in functional analysis. Let X be a non-empty set and A be a subset of R. Each element $x \in B\left(A\right)$

is a function defined and bounded on a given set A and the metric defined by

$$d(x,y)=\sup_{t\in A}\left|x(t)-y(t)\right|$$

If in a case of an interval $A[a,b]\subset R,$ we write $B[a,b]$ in place of $B(A)$.

Now we shall show that the function d satisfies all the five axioms of a metric on X .

1. $d(x,y)\geq 0, \quad \forall x,y\in X$

   $\Rightarrow \sup\limits_{t\in A}\left|x(t)-y(t)\right|\geq 0$ is always a non-negative real number

2. $d(x,y)=d(x,y), \quad \forall x,y\in X$

   $d(x,y)=d(x,y), \quad \forall x,y\in X$

3. $d(x,y)\leq d(x,z)+d(z,y) \quad \forall x,y,z\in X$

   $\Rightarrow \sup\limits_{t\in A}\left|x(t)-y(t)\right|=\sup\limits_{t\in A}\left|\{x(t)-z(t)\}+\{z(t)-y(t)\}\right|$

   $\Rightarrow\leq \sup\limits_{t\in A}\left|x(t)-z(t)\right|=\sup\limits_{t\in A}\left|\{z(t)-y(t)\}\right| \quad \forall x,y,z\in X$.

4. If $x=y\Rightarrow d(x,y)=0, \quad \forall x,y\in X$

   $\Rightarrow x=y\Rightarrow x(t)=y(t)\Rightarrow (x(t)-y(t)=0$

   $\Rightarrow \left|x(t)-y(t)\right|=0\Rightarrow \sup\limits_{t\in A}\left|x(t)-y(t)\right|=0, \quad \forall x,y\in X$.

5. If $d(x,y)=0\Rightarrow x=y, \ \forall x,y\in X$

   $\Rightarrow \sup\limits_{t\in A}\left|x(t)-y(t)\right|=0\Rightarrow \left|x(t)-y(t)\right|=0$

$$\Rightarrow x(t)-y(t)=0 \Rightarrow x, y=y(t)$$

$$\Rightarrow x=y \ \forall x, y \in X \ .$$

This show that $x-y$ is bounded on A. Hence d is metric on X and (X,d) is called metric space. This metric space is denoted by $B(A)$.

4.7 Sequence Space l^p

l^p Space is also Banach space. Suppose $p \geq 1$ is a fixed real number and each element in the pace l^p is a sequence $x=(x_1, x_2, \ldots)$ of numbers such that $|x_1|^p + |x_2|^p + \ldots\ldots..$ converges.

Thus,

$$\sum_{i=1}^{\infty} |x_i|^p < \infty \qquad\qquad (p \geq 1, \text{fixed})$$

The metric defined by

$$d(x,y)=\left(\sum_{i=1}^{\infty} |x_i - y_i|^p\right)^{1/p}$$

Where $y=(y_i), \sum |y_i|^p < \infty$ and $\forall x, y \in X$

Now we shall show that the function d satisfies all the five axioms:

(i) $\quad d(x,y) \geq 0, \ \forall x, y \in X$

$$\Rightarrow \left(\sum_{i=1}^{\infty} |x_i - y_i|^p\right)^{1/p} \geq 1 \text{ is always a non-negative real number for } p \geq 1.$$

(ii) $d(x,y) = d(y,x)\ \forall x, y \in X$

$$\Rightarrow \left(\sum_{i=1}^{\infty} |x_i - y_i|^p\right)^{1/p} = \left(\sum_{i=1}^{\infty} |x_i - y_i|^p\right)^{1/p} \ \forall x, y \in X .$$

(iii) $d(x,y) \le d(x,z) + d(z,y)\ \forall x, y, z \in X$

$$\Rightarrow \left(\sum_{i=1}^{\infty} |x_i - y_i|^p\right)^{1/p} = \left(\sum_{i=1}^{\infty} |(x_i - z_i) + (z_i - y_i)|^p\right)^{1/p}$$

(Using Minkowski inequality)

$$\le \left(\sum_{i=1}^{\infty} |x_i - z_i|^p\right)^{1/p} + \left(\sum_{i=1}^{\infty} |(z_i - y_i)|^p\right)^{1/p} \ \forall x, y, z \in X .$$

(iv) If $x = y \Rightarrow d(x,y) = 0, \forall x, y \in X$

$$\Rightarrow x = y \Rightarrow x_i, y_i \Rightarrow x_i - y_i = 0$$

$$\Rightarrow |x_i - y_i| = 0 \Rightarrow |x_i - y_i|^p = 0$$

$$\Rightarrow \sum_{i=1}^{\infty} |x_i - y_i|^p = 0 \Rightarrow \left(\sum_{i=1}^{\infty} |x_i - y_i|^p\right)^{1/p} = 0 \ (p \ge 1) \ \ \forall x, y \in X .$$

(v) If $d(x,y) = 0 \Rightarrow x = y, \ \ \forall x, y \in X$.

$$\Rightarrow \left(\sum_{i=1}^{\infty} |x_i - y_i|^{1/p}\right) = 0 \Rightarrow \sum_{i=1}^{\infty} |x_i - y_i|^p = 0 \Rightarrow |x_i - y_i|^p = 0$$

$$\Rightarrow |x_i - y_i| = 0 \Rightarrow x_i - y_i = 0 \Rightarrow x_i = y_i$$

$$\Rightarrow x = y \quad \forall x, y \in X .$$

Hence, l^p is a metric space.

4.8 Hilbert-Sequence Space l^2

The Hilbert-sequence space l^2 is a fundamental example of a Hilbert space, which is a complete inner product space. If we put $p = 2$, in sequence space l^p then the metric becomes

$$d(x, y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^2 \right), \quad \forall x, y \in X .$$

This is known as Hilbert-sequence space l^2. Now we shall show that the function d satisfies all the following axioms of a metric on X .

(1) $d(x, y) \geq 0, \ \forall x, y \in X$

$$\Rightarrow \left(\sum_{i=1}^{\infty} |x_i - y_i|^2 \right)^{1/2} \geq 0 \text{ is always a non-negative real number.}$$

(2) $d(x, y) = d(y, x), \quad \forall x, y \in X$

$$\Rightarrow \left(\sum_{i=1}^{\infty} |x_i - y_i|^2 \right)^{1/2} = \left(\sum_{i=1}^{\infty} |y_i - x_i| \right)^{1/2} \quad \forall x, y \in X .$$

(3) $d(x, y) \leq d(x, z) + d(z, y) \forall x, y, z \in X$

$$\Rightarrow \left(\sum_{i=1}^{\infty} |x_i - y_i|^2 \right)^{1/2} = \left(\sum_{i=1}^{\infty} \left|(x_i - z_i) + (z_i - y_i)\right|^2 \right)^{1/2}$$

(Using Minkowski inequality)

$$\le \left(\sum_{i=1}^{\infty} |x_i - z_i|^2 \right)^{1/2} + \left(\sum_{i=1}^{\infty} |z_i - y_i|^2 \right)^{1/2} \quad \forall x, y, z \in X$$

(4) If $x = y \Rightarrow d(x, y) = 0 \ \ \forall x, y \in X$

$$\Rightarrow x = y \Rightarrow x_i = y_i \Rightarrow x_i - y_i = 0$$

$$\Rightarrow |x_i - y_i| = 0 \Rightarrow |x_i - y_i|^2 = 0 \Rightarrow \left(\sum_{i=1}^{\infty} |x_i - y_i|^2 \right)^{1/2} = 0 \quad \forall x, y \in X$$

(v) If $d(x, y) = 0 \Rightarrow x = y \ \ \forall x, y \in X$

$$\Rightarrow \left(\sum_{i=1}^{\infty} |x_i - y_i|^2 \right)^{1/2} = 0 \Rightarrow \sum_{i=1}^{\infty} |x_i - y_i|^2 = 0$$

$$\Rightarrow |x_i - y_i|^2 = 0 \Rightarrow |x_i - y_i| = 0$$

$$\Rightarrow x_i - y_i = 0 \Rightarrow x_i = y_i \Rightarrow x = y \quad \forall x, y \in X$$

Hence l^2 is a Hilbert space.

4.9 Open Ball, Closed Ball and Sphere

Let (X, d) be a metric space. Let $x_0 \in X$ and $r > 0$ the $B(x_0, r) = \{x \in X : d(x, x_0) < r\}$ is an open ball centered at x_0 with radius r. It is also denoted by $S_r(x_0)$.

Let (X, d) be a metric space. Let $x_0 \in X$ and $r > 0$ then $B(x_0, x) = \{x \in X : d(x, x_0) \le r\}$ is a closed ball centered at x_0 with radius r. It is also denoted by $s_r(x_0)$ or $\overline{B}(x_0, r)$.

Let (X,d) be a metric space. Let $x_0 \in X$ and $r > 0$ then $S(x_0,r) = \{x : X : d(x,x_0) \le r\}$

is a shpere centered at x_0 with radius r.

Distance Between Sets

Let A and B is two non-empty subset of a metric space (X,d). The distance between A and B denoted by $d(A,B)$ and defined as

$$d(A,B) = \inf\ \{d(a,b) : a \in A, b \in B\}$$

Obviously,

(i) $d(A,B) \ge 0$

(ii) $d(A,B) = 0$ if $A \cap B \ne \phi$

However, it is not necssary that if $d(A,B) = 0$ then $A \cap B \ne \phi$. For example, let a metric space (R,d), where d is a usual metric on R and let $A = (0,1)$ and $B = (1,2)$. We have $d(A,B) = 0$ but $A \cap B = \phi$

Distance of a point from a given set

Let (X,d) be a metric space and $A \subset X$. Let $x \in X$ be arbitrary. Then the distance between x and the set A is denoted by $d(x,A)$ and defined as

$$d(x,A) = \inf\ \{d(x,a) : a \in A\}$$

i.e., $d(x,A)$ is the greatest lower bound of the distance between x and point of A. Obviously,

(i) $d(x,A) \geq 0$

(ii) $d(x,A) = \text{if } x \in A$

However, it is not necessary that if $d(x,A) = 0$ then $x \in A$. For example, let a metric space (R,d), where d is usual metric on R and let $A = \{x \in R : 0 < x \leq 1\}$. We have $d(0,A) = 0$ but $0 \notin A$.

Diameter of A set

Let (X,d) be a metric space. Let A be a non-empty subset of X. Then the diameter of A is denoted by $d(A)$ and defined as

$$d(A) = \sup\{d(a_1,a_2) : a_1, a_2 \in A\}$$

Obviously,(i) $d(A) \geq 0$

(ii) If $d(A)$ is finite then A is said to be bounded otherwise unbounded.

Examples

Example.1. Let $X = \{0,1,2,3,4,5\}$ and $A = \{2,3,4\}$. Find the distance between 1 and A.

Solution: We know that $d(x,A) = \inf\{d(x,a) : a \in A\}, \forall x \in X$

Here $d(1,2) = |2-1| = 1, d(1,3) = |3-1| = 2, d(1,4) = |4-1| = 3$

$$d(1, A) = \inf \{d(1,2), d(1,3), d(1,4)\}$$

$$= \inf \{1, 2, 3\}$$

$$= 1.$$

Example.2: Let d be the usual metric defined on d i.e.,

$$d(x, y) = |x - y|, \quad \forall x, y \in R^0$$

If $A = [1,2]$ and $B = [3, 5]$, find the diameters of A and B.

Solution: Given that $A = [1,2]$ and $B = [3,5]$

$$d(A) = \sup\{d(a_1, a_2) : a_1, a_2 \in A\}$$

$$= \sup\{d(a_1 a_2) : a_1, a_2 \in [1,2]\}$$

$$= 1$$

$$d(B) = \sup\{d(b_1, b_2) : b_1, b_2 \in B\}$$

$$= \sup\{d(b_1, b_2) : b_1, b_2 \in [3,5]\}$$

$$= 2.$$

4.10 Neighbourhood of a point

Let (X, d) be a metric space. Let $x \in X$. A subset N of x is said to be a neighbourhood of x if there exist an open set G such that

$$x \in G \subseteq N$$

or

Let $x \in X$ then N is said be a neighbourhood of x, if $r > 0$ and $s_r(x_0)$ is an open set such that

$$x \in S_r(x_0) \subseteq N$$

4.11 Limit point of aset

Let A be a subset of a metric space (X,d). Then a point x of X (which may be on may not be a point A) is called an limit point of A if each open ball centered at x contains at least one point of A different from x.

Derived set

The collection of all limit points of a set A is called derived set of A. It is denoted by $D(A)$ or A'.

Isolated set

Let A be a subset of a metric space (X,d). A point $x \in A$ is said to be an isolated point of A if and only if it is not an limit point of A.

Discrete set

Let A be a subset of a metric space (X,d). A set A is said to be a discrete set if each point of

A is an isolated point of A.

Dense-In-Itself

Let A be a subset of a metric space (X, d). A is said to be dense-in-itself if and only if every point of A is a limit point of A.

Perfect Set

Let A be a subset of a metric space (X, d). Then A is said to be perfect set if and only if

$$A = A'\{\text{or } D(A)\}$$

Closure

Let A be a subset of a metric space (X, d) the closure of A, is the union of A and all its limit points, i.e.,

$$\overline{A} = A \cup A' \quad \text{or} \quad A \cup D(A)$$

Note:

1. $\overline{A} = \cap\ \{\text{of all closed set containing A}\}$
2. A is closed if and only if $\overline{A} = A$
3. $\overline{\overline{A}} = A$
4. $\overline{A}$ is the set of all adherent points of a given subset A of X.
5. $\overline{A}$ is the smallest closed set containing A.

Interior Point

Let A be a subset of a metric space (X,d). The interior of A is the union of all open sets contained in A, *i.e.*,

$$A^{\circ} = \cup \{\text{of all open sets contained in } A\}$$

Note:

1. A is open set if and only if $A^{\circ} = A$.
2. $(A^{\circ})^{\circ} = A$.

Exterior Point

Let A be a subset of a metric space (X,d). The exterior point of A, is the interior of the complement of A, *i.e.*,

$$\text{ext}(A) = (\mathrm{X} \sim \mathrm{A})^{\circ}$$

or

$$\text{ext}(A) = \mathrm{X} \sim \overline{\mathrm{A}}.$$

Boundary point

Let A be a subset of a metric space (X,d). The boundary point of A, is the set of all those elements of A which neither belong to A° or nor to exterior.

Dense set and Separable Space

Let A be a subset of a metric space. Then A is said to be dense in X if $\overline{A} = X$. And X is said to be separable if it has a countable subset which is dense in X .

Examples

Example.3: Show that the closure $\overline{B(x_0, r)}$ of an open ball $B(x_0, r)$ in a metric space can differ from the closed ball $B[x_0, r]$.

Solution: We know that the distance of a point $x \in X$ from a set A is given by

$$d(x, A) = \inf \{d(x, a) : a \in A\}$$

$$\Rightarrow d(x, A) = 0 \text{ if } x \in A$$

Let (X, d) be a discrete metric space

$$d(x, y) = \begin{cases} 1 & \text{if } \quad x \neq y \\ 0 & \text{if } \quad x = y \end{cases}$$

Suppose $x_0 \in X$ if $r > 0, r < 1$ we have

$$B(x_0, r) = \{x \in X : d(x, x_0) < r\}$$

i.e., $d(x, x_0) = 0$ or 1 for each of which is $< r$

$$\Rightarrow x \in X \Rightarrow x \in B(x_0, r), r \leq 1$$

$$\Rightarrow B(x_0, r) = \{x_0\}$$

$$\left\{\begin{array}{ll} \because d\left(x_0, x_0\right)=0 & <r \\ d\left(x_0, x_r\right)=1 & \forall r \text{ if } x \neq x_0 \end{array}\right\}$$

To show that closure $\overline{B(x_0, r)}$ of an open ball $B(x_0, r)$ in a metric space can differ from the closed ball $B[x_0, r]$. Let (X, d) be a discrete metric space and $x_0 \in X$ then

$$B(1, x_0)=\{x_0\} \qquad \ldots(1)$$

But the closed ball $B[1, x_0]=X$

$$\Rightarrow \overline{B(1, x_0)} \neq B[1, x_0]$$

Also consider any other metric space (X_1, d_1)

Let $\quad x \in \overline{B(x_0, r)} \qquad \ldots(2)$

Then $\quad d\left(x, B(x_0, r)\right)=0$

This implies that there exist any $\in>0$ and $y \in B(x_0, r)$ such that

$$d(x, y)<\in$$

$$\Rightarrow d(x, x_0) \leq d(x, y)+d(y, x_0)$$

$$\leq \in + r$$

$$\Rightarrow d(x, x_0) \leq r$$

$$\Rightarrow x \in B[x_0, r] \qquad (3)$$

$$\Rightarrow \overline{B(x_0,r)} \subset B[x_0,r]$$

This implies closure of an open ball $\overline{B(x_0,r)}$ is subset of closed ball but not both are equal.

4.12 Equivalent Metrics

Two metrics d_1 and d_2 on the same set X are said to be equivalent metrics if and only if every d_1-open set is d_2-open and every d_2-open set is d_1-open.

For example: Let (X,d) be a metric space and let

$$d_1(x,y)=\frac{d(x,y)}{1+d(x,y)} \quad \forall x,y \in X$$

Then d_1 is also a metric on X and the two metrics d and d_1 are equivalent.

Examples

Example.4: The space l^∞ is not separable.

Solution: Let $y=(y_1,y_2,y_3,....)$ be a sequence of zeros and ones. They $y \in l^\infty$ with y, we associate a real number y_1 whose binary representation is $\frac{x_1}{2}+\frac{x_2}{2}+\frac{x_3}{2}+.....$ Now we consider the set of points in interval $[0,1]$ is uncountable each $y_1 \in [0,1]$ has a binary representation and different y_1's have different binary representation. Hence, there are uncountably many sequences of zeros and ones. The metric on l^p shows that any two of them which are not equal must be of distance 1 apart. If we let each of these sequences be the center of a small ball say, of radius 1/3, these balls do not interest and we have uncountable many of them. If M is any dense set in l^{∞} each of these non-intersecting balls must contains an element of M .

Hence, M cannot be countable. Since M was an arbitrary dense set, this show that l^{∞} cannot have dense subsets which are countable consequently, l^{∞} is not separable.

Example.5: The space l^p with $1 \leq p < +\infty$ is separable.

Solution: Let S be a set of all sequences Y of the form

$$Y = (y_1, y_2, y_3, ..., y_n, 0, 0, ...)$$

Where n is a positive integer and the y_i's are rational. Let M is countable. To show that M is dense in l^p. Let $x \in (x_i) \in l^P$ be arbitrary.

Then for every $\in > 0$ there is an n (depends on $\in$) such that

$$\sum_{i=n+1}^{\infty} |x_i|^p < \frac{\in^p}{2}$$

Since the rationals are dense in R, for each x_i there is a rational y_i close to it. Hence, we can find $y_i \in M$ satisfies.

$$\sum_{i=1}^{n} |x_i - y_i|^p < \frac{\in^p}{2}$$

It follows that

$$\left[d(x,y)\right]^p = \sum_{i=1}^{n}|x_i - y_i|^p + \sum_{i=n+1}^{n}|x_i|^p < \in^p$$

Thus, we obtained $d(x,y) < \in$ and M is dense in l^p .

4.13 Summary

Let (X,d) be a metric space. Let $x_0 \in X$ and $r > 0$ the $B(x_0,r) = \{x \in X : d(x,x_0) < r\}$ is an open ball centered at x_0 with radius r . It is also denoted by $S_r(x_0)$.

Let (X,d) be a metric space. Let $x_0 \in X$ and $r > 0$ then $B(x_0,x) = \{x \in X : d(x,x_0) \le r\}$ is a closed ball centered at x_0 with radius r . It is also denoted by $s_r(x_0)$ or $\overline{B}(x_0,r)$.

Let (X,d) be a metric space. Let $x_0 \in X$ and $r > 0$ then $S(x_0,r) = \{x : X : d(x,x_0) \le r\}$ is a shpere centered at x_0 with radius r .

Let A and B is two non-empty subset of a metric space (X,d) . The distance between A and B denoted by $d(A,B)$ and defined as $d(A,B) = \inf \{d(a,b) : a \in A, b \in B\}$

Let (X,d) be a metric space and $A \subset X$. Let $x \in X$ be arbitrary. Then the distance between x and the set A is denoted by $d(x,A)$ and defined as $d(x,A) = \inf\{d(x,a) : a \in A\}$

Let (X,d) be a metric space. Let A be a non-empty subset of X . Then the diameter of A is denoted by $d(A)$ and defined as $d(A) = \sup\{d(a_1,a_2) : a_1, a_2 \in A\}$

Let (X,d) be a metric space. Let $x \in X$. A subset N of X is said to be a neighborhood of x if there exist an open set G such that $x \in G \subseteq N$.

Let A be a subset of a metric space (X, d). Then a point x of X (which may be on may not be a point A) is called an limit point of A if each open ball centered at x contains at least one point of A different from x.

The collection of all limit points of a set A is called derived set of A. It is denoted by $D(A)$ or A'.

Let A be a subset of a metric space (X, d). A point $x \in A$ is said to be an isolated point of A if and only if it is not an limit point of A.

Let A be a subset of a metric space (X, d). A set A is said to be a discrete set if each point of A is an isolated point of A.

Let A be a subset of a metric space (X, d). A is said to be dense-in-itself if and only if every point of A is a limit point of A.

Let A be a subset of a metric space (X, d). Then A is said to be perfect set if and only if

$$A = A' \{\text{or } D(A)\}$$

Let A be a subset of a metric space (X, d) the closure of A, is the union of A and all its limit points, i.e., $\overline{A} = A \cup A' \text{or } A \cup D(A)$

Let A be a subset of a metric space (X, d). The interior of A is the union of all open sets contained is A, i.e., $A^{\circ} = \cup \{\text{of all open set contained in } A\}$

Let A be a subset of a metric space (X, d). The exterior point of A, is the interior of the complement of A, i.e., $\operatorname{ext}(A) = (\mathrm{X} \sim \mathrm{A})^{\circ}$ or $\operatorname{ext}(A) = \mathrm{X} \sim \bar{\mathrm{A}}$.

Let A be a subset of a metric space (X, d). The boundary point of A, is the set of all those elements of A which neither belong to A° or nor to exterior.

Let A be a subset of a metric space. Then A is said to be dense in X if $\bar{A} = X$. And X is said to be separable if it has a countable subset which is dense in X.

Two metrics d_1 and d_2 on the same set X are said to be equivalent metrics if and only if every d_1-open set is d_2-open and every d_2-open set is d_1-open.

4.14 Terminal Questions

Q.1. To show that l^{∞} is a metric space.

Q.2. Define Hilbert-sequence space.

Q.3. Show that another metric d, on the set X in $d(x, y) = \max_{t \in I} |x(t) - y(t)|, I = [a, b]$ is defined by $d_1(x, y) = \int_a^b |x(t) - y(t)| dt$

Q.4. Let $C[0,1]$ denote the family of all Riemann integrable function from $[0,1]$ into R. show that the mapping $d : C[0,1] \times C[0,1] \to R$ defined by

$$d(f, g) = \int_0^1 |f - g|(x) dx = \int_0^1 |(f(x) - g(x)| dx$$

Where $f, g \in C[0,1]$ is a pseudometric on $C[0,1]$ but not a metric on $C[0,1]$.

Q.5. Show that the set C of all complex numbers is a metric space under

$$d(z_1, z_2) = \frac{|z_1 - z_2|}{\left(1+|z_1|^2\right)^{1/2}\left(1+|z_2|^2\right)^{1/2}}$$

Q.6. Prove that the sequence space l^p is a metric space.

Q.7. Define open and closed balls.

Q.8. In a metric space, every open ball is an open set.

Q.9. Show that a finite set in a metric space has no limit point.

Q.10. Let (X, d) be a metric space. Let $d_1(x, y) = \min\{1, d(x, y)\}$. Show that d_1 is a metric for X. Also show that the two metrics d and d_1 are equivalent.

UNIT-5: Sequence in a Metric Space

Structure

5.1 Introduction

Cauchy sequence is one in which the terms become arbitrarily close to each other as the sequence progresses. This concept is important in the study of metric spaces, particularly in understanding the completeness of a metric space. A metric space is said to be complete if every Cauchy sequence in the space converges to a point in the space. A function $f : X \to Y$ between metric spaces is a homeomorphism if it is bijective, continuous, and its inverse f^{-1} is also continuous. In simpler terms, a homeomorphism is a function that preserves both continuity and openness, meaning that it maps open sets to open sets and vice versa.

Continuity in metric spaces have a great importance in analysis and topology, as it helps in studying the properties of functions and spaces, including the convergence, limits, and topological properties.

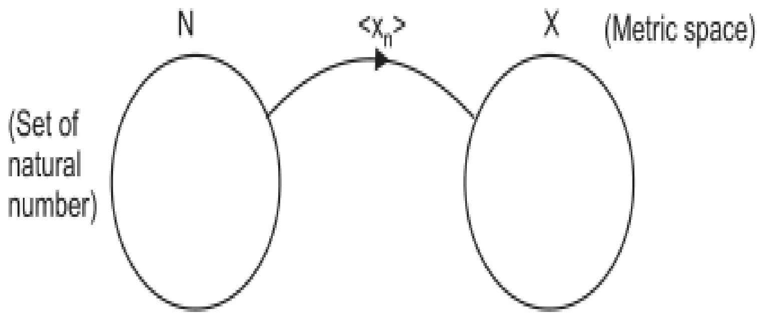
5.2 Objectives

After reading this unit the learner should be able to understand aboutthe

- Sequence in metric space
- Convergent sequence in metric space
- Bounded Set and their important theorem
- Cauchy Sequence and important theorems
- Continuity in Metric Spaces and important theorems
- Open mapping, Closed mapping and Bicontinuous mapping
- Homomorphism andHomomorphism Spaces

5.3 Sequence in Metric Space

Let (X, d) be a metric space. A sequence $< x_n >$ in X is a function from N to X.



The sequence $< x_n >$ is also denoted by $\{x_n\}$ or $< x_1, x_2, x_3, \ldots, x_n, \ldots >, \forall x_1, x_2$, etc., $\in X$ and they need not be distinct.

5.4 Convergent Sequence in a Metric Space

Let (X, d) be a metric space. A sequence $< x_n >$ in X is said to be convergent sequence if it converges to a point $x \in X$ such that

$$\lim_{n\to\infty} d(x_n, x) = 0$$

and x is called the limit of $\{x_n\}$.

Now we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

5.5 Bounded Set

A non-empty subset $M \subset X$ is a bounded set if its diameter

$$\delta(M) = \sup_{x,y \in M} d(x,y) \text{ is finite.}$$

Theorem 1: Let (X,d) be a metric space. Then

(i) A convergent sequence in X is bounded and its limit is unique.

(ii) If $x_n \to x$ and $y_n \to y$ in X then

$$d(x_n, y_n) \to d(x,y)$$

Proof: (i) Let $< x_n >$ be a convergent sequence in X, *i.e.*, $x_n \to x$. Then we take 1, find an n such that

$$d(x_n, x) < 1, \quad \forall n \in N$$

Using triangular inequality $\forall n$, we have

$$d(x_n, x) < 1 + a$$

Where $a = \max\{d(x_1, x), d(x_2, x), ..., d(x_n, x)\}$

i.e., $\{x_n\}$ is bounded.

Now we assume

$$x_n \to x \text{ and } x_n \to z$$

Using triangular inequality, we have

$$0 \le d(x,z) \le d(x,x_n) + d(x_n,z) \to 0+0$$

$\Rightarrow x = z,$ *i.e.*, uniqueness of limit point. Hence, a convergent sequence in X is bounded and its limit is unique.

(ii) It is given that $x_n \to x$.

This implies, for a given $\in > 0$ there exits a positive integer n_0 such that

$$d(x,x_n) < \in /2 \text{ for } n \ge n_0$$

Also given that $y_n \to y$.

This implies for a given $\in > 0$, there exist a positive integer m_0 such that

$$d(y,y_n) < \in /2 \text{ for } n \ge m_0$$

If $p = \max(n_0, m_0)$ then

$$d(x,x_n) < \in /2 \text{ and}$$

$$d(y,y_n) < \in /2 \text{ for } n \ge p$$

Now we have

$$\left|d\left(x_n, y_n\right) - d\left(x, y\right)\right| = \left|d\left(x_n, y_n\right) - d\left(x_n, y\right) + d\left(x_n, y\right) - d\left(x, y\right)\right|$$

$$\leq \left|d\left(x_n, y_n\right) - d\left(x_n - y\right)\right| + \left|d\left(x_n, y\right) - d\left(x, y\right)\right|$$

$$\leq d\left(y, y_n\right) + d\left(x, x_n\right)$$

$$< \in /2 + \in /2 = \in$$

Hence $d\left(x_n, y_n\right)$ converges to $d\left(x, y\right)$.

5.6 Cauchy Sequence

Let $< x_n >$ be a sequence in a metric space $\left(X, d\right)$ then $< x_n >$ is said to be cauchy sequence if given any $\in > 0$ there exist $n_0 \in N$ such that

$$m, n \geq n_0 \quad \Rightarrow \quad d\left(x_m, x_n\right) < \in .$$

Theorem 2: Every convergent sequence in a metric space is a Cauchy sequence.

Proof: Let $< x_n >$ be a convergent sequence in a metric space $\left(X, d\right)$.

To show that $< x_n >$ is also a Cauchy sequence. Since $< x_n >$ a convergent sequence then $< x_n >$ converges to a point say $x \in X$ i.e., x is the limit point of x_n.

Then for a given $\in > 0$ there exist $n_0 \in N$ such that

$$n \geq n_0 \Rightarrow d\left(x_n, x\right) < \in /2$$

For $n, m \geq n_0$, using triangular inequality, we have

$$d\left(x_n, x_m\right) \leq d\left(x_n, x\right) + d\left(x, x_m\right)$$

$$< \in /2 + \in /2 = \in$$

$\therefore d\left(x_n, x_m\right) < \in \quad \forall n, m \geq n_0$

Thus for any given $\in > 0$, there exist $m \in N$ such that

$$d\left(x_n, x_m\right) < \in \forall n, m \geq n_0$$

Hence, $< x_n >$ is a cauchy sequence in X.

Examples

Example.1: To show that Cauchy sequence is not necessarily convergent.

Solution: Let $X = R - \{0\}$ and $d(x, y) = |x - y|$

Consider a sequence $< x_n >$, where

$x_n = \dfrac{1}{n}$ is sequence in X.

To show that $< x_n >$ is a Cauchy sequence but it does not converges in X .

Let $\in > 0$ and n_0 be a positive number such that

$$n_0 > \frac{2}{\in}$$

Now $$d(x_m, x_n) = |x_m - x_n|$$

$$= |x_m + (-x_n)|$$

$$\leq |x_m| + |x_n|$$

$$d(x_m, x_n) \leq \frac{1}{m} + \frac{1}{n}$$

If $m \geq n_0 \Rightarrow m > \frac{2}{\in}$ so that $\frac{1}{m} = \frac{\in}{2}$

Similarly, we have $$\frac{1}{n} = \frac{\in}{2}$$

$$\therefore \quad d(x_m, x_n) \leq \frac{1}{m} + \frac{1}{n} = \frac{\in}{2} + \frac{\in}{2} = \in$$

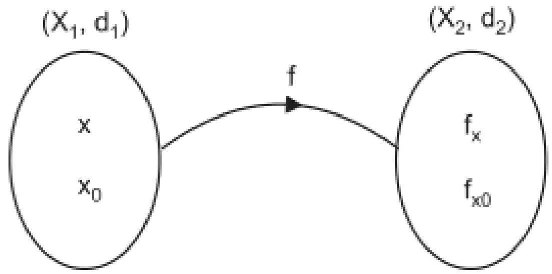
Thus, $$d(x_m, x_n) \leq \in$$

Hence, $< x_n >$ is a Cauchy sequence. Obviously the limit of this sequence is 0 which does not belong to X, i.e., $< x_n >$ does not converge in X .

5.7 Continuity in Metric Spaces

Let (X_1, d_1) and (X_2, d_2) be two metric spaces and $f : X_1 \to X_2$ be a mapping of X_1 into X_2. Then f is said to be continuous at a point $x_0 \in X_1$ if for every $\in > 0$, there exist $\delta > 0$ such that

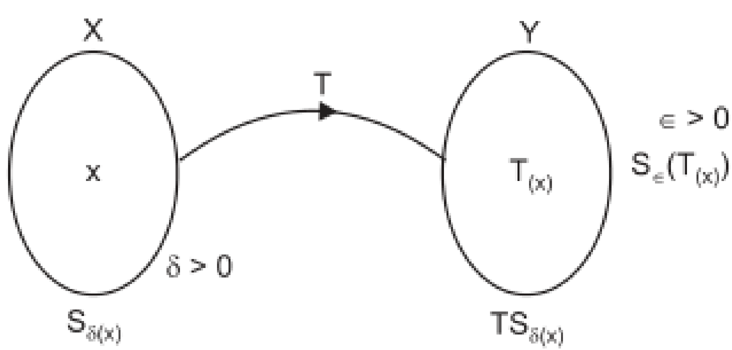
$$d_1(x, x_0) < \delta \qquad \Rightarrow \qquad d_2(f(x), f(x_0)) < \in$$



or

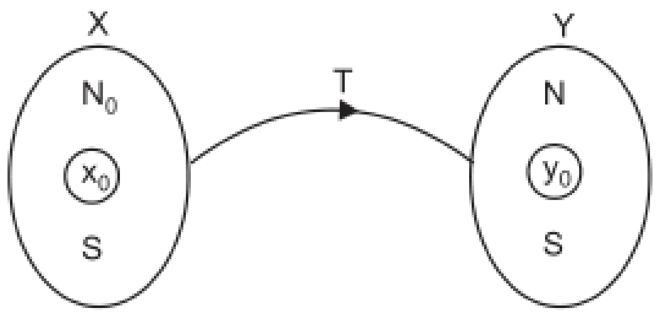
Let X and Y be two metric spaces. A mapping $T : X \to Y$ is said to be continuous, if at a point $x \in X$.

$$T(S_\delta(x)) = S_\in(T(x))$$



Theorem.3: A mapping T of a metric space X into a metric space Y is continuous if and only if the inverse image of any open subset of Y is an open subset of X.

Proof: Let T be continuous and let $S \subset Y$ be open and $S_0 = T^{-1}(s)$ be the inverse of S.



To show that S_0 is open set in X.

(1) Suppose if $S_0 = \phi$ then it is open set.

(2) If $S_0 \neq \phi$ then $x_0 \in S_0 \Rightarrow T_{x_0} \in S$ or $y_0 = T_{x_0} \in S$

It is given that S is an open set then it contains an $\in$-neighbouhood N of $y_0 = T_{x_0}$. Since T is continuous then x_0 has a δ-neighbourhood which is mapped into N. Since $N \subset S$, we have $N_0 \subset S_0$. So that S_0 is open in X because $x_0 \in S_0$ coversely, let the inverse image of every open set in Y is an open set in X.

Then for every $x_0 \in X$ and any $\in$-neighbourhood N of T_{x_0} the inverse image of N_0 of N is open. Since N is open and N_0 contains x_0. Thus, N_0 also contains a $\delta-$neighbourhood of x_0 which is mapped into N, because N_0 is mapped into N. Using definition, T is continuous at x_0, since $x_0 \in X$ was arbitrary.

Hence, T is a continuous mapping from X into Y.

Theorem.4: A mapping $f : X \to Y$ of a metric space (X, d_1) into a metric space (Y, d_2) is continuous at a point $x_0 \in X$ if and only if

$$x_n \to x_0 \quad \Leftrightarrow \quad f(x_n) \to f(x_0)$$

i.e., f is continuous iff f is sequentially continuous.

Proof: Let $f : X \to Y$ be continuous at $x_0 \in X$. Let $< x_n >$ be a sequence in X such that

$$\lim x_n = x_0$$

To show that f is sequentially continuous or $f(x_n) \to f(x_0)$. Using continuity of f at x_0, we have given $\in > 0$, there exist $\delta > 0$ such that

$$d_1(x, x_0) < \delta \Rightarrow d_2(f(x), f(x_0) < \in \qquad \text{....(1)}$$

Since $x_n \to x_0$

So given $\delta > 0$, there exists $n_0 \in N$ such that

$$d_1(x_n, x_0) < \delta \quad \forall n \geq n_0 \qquad \text{....(2)}$$

Using equation (1) and (2), we get the given $\in > 0$ this implies there exist $\delta > 0$ such that

$$d_1(x_n, x_0) < \delta$$

$$\Rightarrow d_2(f(x_n), f(x_0)) < \in \quad \forall n \geq n_0$$

Using definition, we have $f(x_n) \to f(x_0)$.

Conversely, let $x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$.

To show that f is continuous at x_0 suppose if possible, let f is not continuous at x_0 then given $\in > 0$ there exist $\delta > 0$ such that

$$d_1(x_n, x_0) < \delta$$

$$\Rightarrow d_2\left(f(x_n), f(x_0)\right) \not< \in$$

or $$d_1(x_n, x_0) < \delta$$

$\Rightarrow$ $$d_2\left(f(x_n), f(x_0)\right) \geq 0$$

Now consider the sequence of open ball such that

$$< S\left(x_0, \frac{1}{n}\right) : n \in N >$$

Take $x_n \in S\left(x_0, \frac{1}{n}\right)$ but $d_2\left(f(x_n), f(x_0)\right) \geq 0$

From the sequence $< x_n >$, we have $x_n \to x_0$

As $$d_1(x_n, x_0) < \frac{1}{n}$$

Also $\lim_{n \to 0} f(x_n) \neq f(x_0)$. This is contradiction. Hence f is continuous at x_0.

5.8 Open Mapping, Closed Mapping and Bicontinuous Mapping

Let (X_1, d_1) and (X_2, d_2) be two metric spaces. A mapping $f : X_1 \to X_2$ is said to be open mapping if $f[G]$ is d_2-open where G is d_1-open.

Let (X_1, d_1) and (X_2, d_2) be two metric spaces. A mapping $f : X_1 \to X_2$ is said to be closed mapping if $f(F)$ is d_2-closed whenever F is d_1-closed.

Let (X_1, d_1) and (X_2, d_2) be two metric spaces. A mapping $f : X_1 \to X_2$ is said to be bicontinuous mapping if f is open and continuous.

5.9 Homomorphism

Let (X_1, d_1) and (X_2, d_2) be two metric spaces. A mapping $f : X_1 \to X_2$ is said to be homeomorphism if

(i) f is one-one, onto or f is bijective.

(ii) f is continuous

(iii) f^{-1} is continuous.

5.10 Homomorphism Spaces

Let (X_1, d_1) and (X_2, d_2) be two metric spaces. A space X_1 is said to be homeomorphic to another space X_2 if there exists a homeomorphism of X_1 onto X_2 and then X_2 is said to be homeomorphic image of X_1 or simply a homeomorph of X_1. If X_1 is homeomorphic to X_2, we write $X_1 = X_2$.

Note: 1. Let (X_1, d_1) and (X_2, d_2) be two metric spaces and let f be a bijective mapping of X, onto X_2. Then the following statements are equavelent

(i) f is homeomorphism

(ii) f is continuous and open

(iii) f is continuous and closed

2. Homeomorphism is an equivalence relation in the collection of all metric spaces.

5.9 Summary

A non-empty subset $M \subset X$ is a bounded set if its diameter $\delta(M) = \sup_{x,y\in M} d(x,y)$ is finite. Let $<x_n>$ be a sequence in a metric space (X,d) then $<x_n>$ is said to be Cauchy sequence if given any $\in > 0$ there exist $n_0 \in N$ such that $m,n \geq n_0 \Rightarrow d(x_m, x_n) < \in$.

Every convergent sequence in a metric space is a Cauchy sequence.

A mapping $f: X \to Y$ of a metric space (X, d_1) into a metric space (Y, d_2) is continuous at a point $x_0 \in X$ if and only if $x_n \to x_0 \Leftrightarrow f(x_n) \to f(x_0)$. Let (X_1, d_1) and (X_2, d_2) be two metric spaces. A mapping $f: X_1 \to X_2$ is said to be open mapping if $f[G]$ is d_2-open where G is d_1-open.

Let (X_1, d_1) and (X_2, d_2) be two metric spaces. A mapping $f: X_1 \to X_2$ is said to be closed mapping if $f(F)$ is d_2-closed whenever F is d_1-closed. Let (X_1, d_1) and (X_2, d_2) be two metric spaces. A mapping $f: X_1 \to X_2$ is said to be bicontinuous mapping if f is open and continuous.

Let (X_1, d_1) and (X_2, d_2) be two metric spaces. A mapping $f: X_1 \to X_2$ is said to be homeomorphism if (i) f is one-one onto or f is bijective (ii) f is continuous (iii) f^{-1} is continuous.

Homeomorphism is an equivalence relation in the collection of all metric spaces.

5.10 Terminal Questions

Q.1. Define the convergent sequence in metric space.

Q.2. Explain the Cauchy sequence in metric space.

Q.3. Give an example of a function which is continuous and closed but not open.

Q.4. Let (X_1, d_1) and (X_2, d_2) be two metric spaces and let f be a mapping of X_1 into X_2. Then f is continuous if and only if the inverse image of under f of every d_2-closed set is d_1-closed set.

Q.5. Let (X_1, d_1) and (X_2, d_2) be two metric space and let the mapping $f: X_1 \to X_2$ be one-one onto. Then f is a homeomorphism if and only if

$$f\left(\overline{A}\right) = \overline{f(A)} \text{ for every } A \subset X_1.$$

Answer

3. A function $f: R \to R$ such that $f(x) = 1, \ \forall x \in R$.

UNIT-6: Complete Metric Space

Structure

6.1 Introduction

The completeness of a metric space is a fundamental property with several important consequences and applications in mathematics, particularly in analysis and topology. A metric space is said to be complete if every Cauchy sequence in the space converges to a point in the space. Completeness is an important property in analysis and topology, as it allows for certain convergence properties to hold, and it helps distinguish between different types of metric spaces. For example, the real numbers R are complete, while the rational numbers Q are not complete. Completeness is closely related to the topological properties of a space. Complete metric spaces are often easier to work with in terms of continuity, compactness, and connectedness. Many fundamental results in analysis, such as the Bolzano-Weierstrass Theorem and the Heine-Borel Theorem rely on the completeness of R^n. Completeness of a metric space is a fundamental concept that underpins many important results and ideas in mathematics, making it a central notion in the study of spaces and functions.

6.2 Objectives

After reading this unit the learner should be able to understand about the:

- Complete metric space and Incomplete metric space
- Contor's Intersection theorem
- Baire Category Theorem
- Contracting Mapping
- Banach Fixed Point Theorem
- Isometric Mapping
- Completeness of C

6.3 Complete Metric Space

Let (X,d) be a meric space. Then (X,d) is said to be complete metric space if and only if every Cauchy sequence $<x_n>$ in X converges to a point in X.

For example: Let R be the set of real numbers and d be the usual metric on R *i.e.*,

$$d(x,y)=|x-y|, \quad \forall x,y\in R$$

The metric space (R,d) is a complete metric space.

6.4 Incomplete Metric Space

Let (X,d) be a metric space. Then is said to be incomplete metric space if and only if there exists some cauchy sequence $<x_n>$ in X which does not converge to a point in X.

For example: Let $X=(0,1]$ and d be the usual metric on X *i.e.*,

$$d(x,y)=|x-y|, \quad \forall x,y\in X$$

The metric space (X,d) is incomplete.

Theorem.1: Let Y be a sub-space of a complete metric space (X,d). Then Y is complete if and only if Y is closed.

Proof: Let (X,d) be a complete metric space. Let $Y\subset X$ be also complete.

To show that Y is closed in X

i.e., show that $D(Y) \subset Y$.

Suppose $y \in D(y)$, to show that $y \in y$

Now we assume y is a limit point of y so that

$$\left[S_{\in}(y)-\{y\}\right] \cap Y = \phi, \quad \forall \in R$$

y is a limit point of $Y \Rightarrow y$ is the limit of the sequence $< y_n >$.

Using a theorem, let (X,d) be matrix space and $A \subset X, x_0$ is said to be limit of A if and only if there exist sequence of distinct point A converge to x_0.

$\Rightarrow \quad < y_n >$ is convergent sequence which converges to y.

$\Rightarrow \quad < y_n >$ is a Cauchy sequence which converges to y.

$y \subset X, < y_n > \in Y : n \in N \quad \Rightarrow \quad < y_n > \in X : n \in N >$

In a metric space X, a Cauchy sequence $< y_n >$ converges to $y \Rightarrow y \in X$. $\because (X,d)$ is a complete metric space. A Cauchy sequence $< y_n >$ in Y is complete $\Rightarrow y \in X$. Also Y is complete $y \in Y$. Hence Y is closed.

Conversely, let Y is closed subset of a complete metric space. To show that Y is complete. Let $< y_n >$ be a cauchy sequence in Y. Then $< y_n >$ is also a cauchy sequence in X for $Y \subset X$.

Since (X,d) is complete and $< y_n >$ converges to some point, say $y \in X$.

Case (1): When $< y_n >$ has infinity many distinct points $< y_n >$ converges to y. This implies the limit of the sequence $<< y_n >$ is $y \in X$.

$\Rightarrow$ the limit point of the set $\{y_n : n \in N\}$ is y

$$y \in D\{y_n : n \in N\} \subset D(Y)$$

$$\because A \subset B \Rightarrow D(A) \subset D(B)$$

$$\Rightarrow y \in D(Y) \subset Y \qquad [\because Y \text{ is closed}]$$

$$\Rightarrow y \in Y.$$

Case 2: When $< y_n >$ has finitely many distinct points $\because$ y is repeated an infinite number of times in the sequence $< y_n \in Y : n \in N >$. This implies $y \in Y$.

$\therefore$ In either case $y \in Y$. Thus, an arbitrary Cauchy sequence $< y_n >$ in Y converges to a point $y \in Y$. This implies every Cauchy sequence in Y is convergent. Using definition, this prove that Y is complete metric space.

6.5 Contor's Intersection Theorem

Let $< x_n >$ be a decreasing sequence of non-empty closed subsets of a complete metric space (X, d) such that $d(x_n) \to 0$ as $n \to \infty$. Then $\bigcap_{n=1}^{\infty} x_n$ contains exactly one point.

Proof: Let (X, d) be a complete metric space and $< x_n >$ be a decreasing sequence.

We know that a sequence $< x_n >$ of subsets of X is said to be monotonic decreasing sequence if and only if

$$x_1 \supset x_2 \supset x_3 \supset x_4 \supset .....$$

Since $d(x_n) \to 0$, then $\bigcap_{x=1}^{\infty} x_n$ cannot contain more than one point. So we have to show that $\bigcap_{x=1}^{\infty} x_n$ is non-empty.

Since each x_n is non-empty there exists a sequence $< x_n >$ such that $y_n \in x_n$ for $n = 1, 2, 3, ...$ For $d(x_n) \to 0 \Rightarrow$ for $\in > 0$, there exist a positive integer say m_0 such that

$$d(x_{m0}) < \in$$

Again, because $< x_n >$ is a decreasing sequence.

$$\therefore \qquad m, n \geq m_0 \Rightarrow x_n, x_m \subset x_{m0}$$

$$\Rightarrow y_n, y_m \in x_{m0}$$

$$\Rightarrow d(y_n, y_m) < \in$$

i.e., $< y_n >$ is a Cauchy sequence.

It is given X is complete, $< y_n >$ must converge to some point, say x_0 in X. We show that

$$x_0 \in \bigcap_{n=1}^{\infty} x_n$$

Suppose, if possible $x_0 \notin \bigcap_{n=1}^{\infty} x_n$

$$\Rightarrow \qquad x_0 \notin x_{n_0} \text{ for some } n_0 \in N$$

Since each x_n is a closed set, x_{n_0} is also a closed set, therefore x_0 cannot be a cluster point of x_{n_0}, and so

$$d(x_0, x_{n_0}) \neq 0$$

Suppose $$d(x_0, x_{n_0}) = \delta > 0$$

Then $$x_{n_0} \subset S_{\frac{1}{2}\delta}(x_0) = \phi$$

Thus, $$y_n \in x_{n_0} \text{ for } n \geq n_0$$

$$\Rightarrow \quad y_n \notin S_{\frac{1}{2}\delta}(x_0)$$

But it is impossible for $y_n \to x_0$.

Hence, $x_0 \in \cap\{x_n : n \in N\}$.

6.6 Baire Category Theorem

Let (X, d) be a metric space. A subset A of a metric space is said to be the first category if and only if it can be written as the union of a countable family of nowhere dense sets; otherwise is known as second category.

Before Baire category theorem, we give some preliminary theorems:

(1) Let A be a subset of a metric space (X, d). Then the following statements are equivalent

(a) A isnon-dense in X.

(b) $\overline{A}$ contains no neighbouhood.

(c) $(\overline{A})'$ is dense in X.

(2) If A is nowhere dense, then $\overline{A}$ is not the entire space X.

(3) The union of a finite number of nowhere dense sets in nowhere dense.

(4) If A is non-dense in X, then each open sphere contains a closed sphere which contains no points of A.

Theorem.2: Every complete metric space is of the second category as a subset of itself.

Proof: Suppose (X,d) is a complete metric space. To show that X is of second category. Let, if possible X is not of second category, then X may be of first category so that X is the union of a countable family of nowhere-dense sets. Suppose this family denoted as $<x_n>$. Since x_1 is non-dense using theorem (iv), there exists a closed sphere p_1 with radius $r_1 < 1$ such that $p_1 \cap x_1 = \phi$

Let S_1 denoted the open sphere having the same centre and radius as p_1. In S_1, we can determine a closed sphere p_2 of radius $r_2 < 1/2$ such that

$$p_2 \cap x_2 = \phi \text{ and so on.}$$

In this manner, we construct a nested sequence $< p_n >$ of closed sphere having the following two properties:

(i) For each positive integer n, p_n does not intersect $x_1, x_2, \ldots, x_n$.

(ii) The radius of $p_n \to 0$ as $n \to \infty$.

It is given that X is complete, it follows by using cantor's intersection theorem that $\bigcap_{n=1}^{\infty} p_n$ consists of a single point x_0 which does not belong to any of the nowhere dense sets A_n by (i). But this is not possible since X is the union of this family. It follows that the metric space X is not of first category. Hence X must be of second category.

6.7 Contracting Mapping

Let (X,d) be a complete metric space. A mapping $f : X \to X$ is said to be contracting mapping if there exist a real number λ with $0 \le \lambda < 1$ such that

$$d\left(f(x), f(y)\right) \le \lambda d(x,y) < d(x,y) \quad \forall x, y \in X \ .$$

6.8 Banach Fixed Point Theorem

Let (X, d) be a complete metric space and f is a contracting mapping on X. Then there exist one and only one point x in X such that

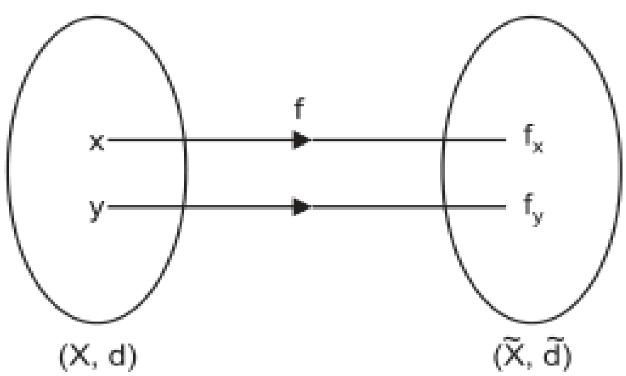
$$f(x) = x \, .$$

6.9 Isometric Mapping

Let (X, d) and $(\tilde{X}, \tilde{d})$ be two metric space. A mapping $f : X \to \tilde{X}$ is said to be isometric on an isometry of f preserves distances *i.e.*, if $\forall x, y \in X$.

$$\tilde{d}\left(f_x, f_y\right) = d(x,y)$$

Where f_x and f_y are the images of x and y respectively.



Theorem.3: The metric space (R,d) is complete, where d is usual on R.

Proof: Let $x_1, x_2 \in R$ be arbitrary then d is defined.

$$d(x_1, x_2) = |x_1 - x_2|, \quad \forall x_1, x_2 \in R \qquad \ldots(1)$$

Suppose $< x_n >$ is a cauchy sequence in R. We define a sequence $< n_k >$ of positive numbers by induction as follows:

$$m, n \geq n_k \Rightarrow |x_n - x_m| < \frac{1}{2^{k+1}}$$

$$\Rightarrow |x_{n_k} - x_m| < \frac{1}{2^{k+1}} < \frac{1}{2^k}$$

$$\Rightarrow |x_{n_k} - x_m| < \frac{1}{2^k} \qquad \ldots.(2)$$

This is possible because $< x_n >$ is a Cauchy sequence. Let I_k be the closed interval

$$\left[S_{n_k}^{-2^{-k}}, S_{n_k} + 2^k \right]$$

Then I_k is closed interval. Let, $|I_k|$ denote length of I_k. Then

$$|I_{k+1}| = |x_n - x_{n_{k+1}}| = \frac{2}{2^{k+1}} = \frac{1}{2^k} < \frac{1}{2^{k-1}} = |I_k|$$

or $\qquad |I_{k+1}| < 1$ so that $I_{k+1} \subset I_k$

This implies $\cap\{I_k : k \in N\}$ consist of exactly only one point say $a \in R$.

Using Cantor's intersection theorem, we have

$$\cap[I_k : k \in N] = \{a\}$$

This implies $a \in I_n \ \forall n$

$$\Rightarrow \qquad \left|a - x_{n_k}\right| < \frac{1}{2^k} \quad \forall k \in N \qquad \text{….(3)}$$

Using equations (2) and (3), we have $\forall m > n_k$

$$\left|a - x_m\right| = \left|a - x_{n_k} + x_{n_k} - x_m\right|$$

$$\leq \left|a - x_{n_k}\right| + \left|x_{n_k} - x_m\right|$$

$$\leq \left|a - x_{n_k}\right| + \left|x_{n_k} - x_m\right|$$

$$\leq \frac{1}{2^k} + \frac{1}{2^k} = \frac{1}{2^{k-1}} = \in \text{(say)}$$

Thus, $\qquad \left|a - x_m\right| < \in \quad \forall m > n_k$

This implies $\qquad \lim_{m \to \infty} x_m = a \in R$

i.e., every Cauchy sequence $< x_n >$ in R converges to a point in R. Hence, (R, d) is complete.

Theorem.4: The set C of complex numbers with usual metric is complete metric space.

Proof: Suppose $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be arbitrary elements of C so that $x_1, x_2, y_1, y_2 \in R$. The usual metric is defined by

$$d(z_1, z_2) = \left|z_1 - z_2\right|, \quad \forall z_1, z_2 \in C.$$

Suppose $< z_n >$ is a cauchy sequence in C. So that given $\in > 0$, there exist $n_0 \in N$ such that

$$n, m \geq n_0$$

$$\Rightarrow \qquad |z_n - z_m| < \in$$

$$\Rightarrow \qquad |x_n + iy_n - (x_m + iy_m)| < \in$$

$$\Rightarrow \qquad |(x_n - x_m) + i(y_n - y_m)|^2 < \in^2$$

$$\Rightarrow \qquad (x_n - x_m)^2 + (y_n - y_m)^2 < \in^2$$

From this we can conclude that

$$x_n - x_m < \in', y_n - y_m < \in' \text{where } \in' = \in/\sqrt{2}$$

It follows that $< x_n >$ and $< y_n >$ are Cauchy sequence in R . But R is complete so that

$$x_n \to x \in R, \quad y_n \to y \in R$$

Consequently, $\qquad z_n \to x + iy = z \in C$.

Therefore every Cauchy sequence $< z_n >$ in C converges to a point $z \in C$.

Hence, (C, d) is complete metric space.

Examples

Example.1: The metric space (R^n, d) is complete, where d is usual metric on R^n .

Solution: Let $x = (x_1, x_2, ...x_n), y = (y_1, y_2, ...y_n)$ be arbitrary elements of R^n . Then d is defined as

$$d(x,y)=\left(\sum_{r=1}^{n}(x_r-y_r)^2\right)^{1/2} \qquad \text{....(1)}$$

Suppose an element $(x_1, x_2, ...x_n,) \in R^n$ can be regarded as a real function defined on $\{1,2,3,\ldots,n\}$. Thus, for we write

$$f_x=(x_1,x_2,...,x_n),\ \ \forall x\in R$$

Now suppose $< f_n >$ be a cauchy sequence in R^n so that given $\in > 0$ there exist $n_0 \in N$ such that

$$p,q\geq n_0 \Rightarrow d\left(f_p, f_q\right)<\in$$

$$\Rightarrow d^2\left(f_p, f_q\right)<\in^2$$

This implies $\left[\sum_{r=1}^{n}\left(f_p(r)-f_q(r)\right\}^2\right]<\in^2$

i.e., we have $\left|f_p(r)-f_q(r)\right|<\in,$ where $r=1,2,3,....,n$.

This implies $< f_p(r) >$ is a cauchy sequence and also R is complete.

Then the sequence $< f_p(r) >$ converges point-wise to a limit function, say $f(r)\in R$.

This implies $\lim\limits_{p\to\infty} f_n(r)=f(r)$ for $r=1,2,....,n$

Since $\{1,2,3,\ldots,n\}$ is finite and hence this convergence is uniform.

$$\Rightarrow\left|f_p(r)-f(r)\right|<\in/\sqrt{n} \quad \forall_p\geq n_0$$

Taking sum with squaring and adding, we get.

$$\left[\sum_{r=1}^{n}\left\{f_p(r)-f(r)\right\}^2\right]^{1/2}<\left(\frac{\in^2}{n}.n\right)^{1/2}=\in$$

$$\Rightarrow \qquad d\left(f_p, f\right)<\in \forall p \geq n_0$$

This show that the Cauchy sequence $<f_n>$ in R^n converges to $f \in R^n$. Hence, $\left(R^n, d\right)$ is complete metric space.

Example.2: The space l^∞ is complete.

Solution: Suppose $\{x_m\}$ be a Cauchy sequence in l^∞, where

$$x_m=\left(\xi_i^{(m)}, \xi_2^m, \ldots\right) \text{ such that}$$

$$\sup_{1\leq i<\infty}\left|\xi_i^{(m)}\right|<\infty, \ \ (m=1,2,3,\ldots)$$

Then for each $\in>0$, there exist a positive integer N such that

$$d\left(x_m, x_n\right)=\sup_{1\leq i<\infty}\left|\xi_i^{(m)}-\xi_i^{(n)}\right|<\in \quad \forall m, n \geq N$$

$$\Rightarrow \qquad \left|\xi_i^{(m)}-\xi_i^{(n)}\right|<\in \forall m, n \geq N\left(i=1,2,3,\ldots\right) \qquad \ldots(1)$$

This implies for each fixed $\left(1 \leq i<\infty\right)$, the sequence $\left(\xi_i^{(1)}, \xi_i^{(2)}, \ldots\right)$ is a Cauchy sequence in k $\left(R \text{ or } C\right)$. Since k is complete, it converges in k. Let $\xi_i^{(m)} \rightarrow \xi_i$ as $m \rightarrow \infty$. Now we define $x=\left(\xi_i, \xi_2, \ldots\right)$ and show that $x \in l^\infty$ and $x_m \rightarrow x$

Suppose $n \rightarrow \infty$ in equation (1), we get

$$\left|\xi_i^{(m)}-\xi_i\right| \leq \in \quad \forall m \geq N \quad \{i=1,2,\ldots.\}$$

Since $x_m = \{\xi_i^{(1)}, \xi_1^{(2)}, ...\} \in l^\infty$, there is a real number k_m such that $\left|\xi_i^{(m)}\right| \le k_m \quad \forall i$

Therefore, we have

$$|\xi_i| = \left|\xi_i - \xi_i^{(m)} + \xi_i^{(m)}\right|$$

$$\le \left|\xi_i^{(m)} + \xi_i\right| + \left|\xi_i^{(m)}\right| \qquad \text{(Using traingular in equality)}$$

$$\le \in + k_m \quad \forall m \ge N \,(i = 1, 2, ...)$$

This inequality is true for each i and right hand side is independent of i. It means $\{\xi_i\}$ is a bounded sequence of numbers

$$\Rightarrow \qquad x \in \{\xi_i\} \in l^\infty$$

Using equation (2), we have

$$d(x_m, x) = \sup_{1 \le i < \infty} \left|\xi_i^{(m)} - \xi_i\right| \le \in \quad \forall m \ge N$$

$$\Rightarrow \qquad x_m \to x \text{ in } l^\infty$$

Hence, l^∞ is a complete metric space.

6.10 Completeness of C

The space consists of all convergent sequences $x = (\xi_i)$ of complex number, with the metric induced from the space l^∞.

Example.3: The space C is complete.

Solution: We know that C is a subspace of l^∞. To show that C is closed in l^∞ .

Also we know that a subspace M of a complete metric space X is itself complete if and only if the set M is closed in X .

We take $x = (\xi_i) \in \overline{C}$ (the closure of C)

We know that $x \in \overline{M}$ if and only if there is a sequence $< x_n >$ in M such that $x_n \to x$

$$\Rightarrow x_n = \left(\xi_i^{(n)}\right) \in C$$

Such that $x_n \to x$.

Give $\in > 0,$ there is an N such that for $n \geq N$ and for all i, we have

$$\left|\xi_i^{(n)} - \xi_i\right| \leq d(x_n, x) < \frac{C}{3}$$

In particular for $n \in N$ and $\forall i$.

Since $x_n \in C$, its terms $\xi_i^{(N)}$ from a convergent sequence, such a sequence is Cauchy. Hence, there is an N_1 such that

$$\left|\xi_i^{(N)} - \xi_k^{(N)}\right| < \frac{\in}{3}(i, k \geq N_1)$$

Using triangular inequality, $\forall i, k \geq N_1$

$$\left|\xi_i - \xi_k\right| \leq \left|\xi_i - \xi_i^{(N)}\right| + \left|\xi_i^{(N)} - \xi_k^{(N)}\right| + \left|\xi_k^{(N)} - \xi_k\right| < \in$$

This implies $x=(\xi_i)$ is convergent. Hence, $x \in C$. Since $x \in \bar{C}$ was arbitrary $\Rightarrow C$ is closed in l^∞. Hence, the space C complete.

Example.4: The space l^p is complete, here p is fixed and $1 \le p < +\infty$.

Solution: Let $\{x_m\}$ be a Cauchy sequence in l^p, where

$$x_m = \left\{\xi_1^{(m)}, \xi_2^{(m)}, ...\right\} \text{ such that}$$

$$\sum_{i=1}^{\infty}\left[\xi_i^{(m)}\right\}^p < \infty\left(m=1,2,3,....\right)$$

Then for each $\in > 0$, there exist a positive integer N such that

$$d\left(x_m, x_n\right) = \left(\sum_{i=1}^{\infty}\left|\xi_i^{(m)} - \xi_i^{(n)}\right|^p\right)^{1/p} < \in \quad \forall m, n > N \qquad (1)$$

$$\Rightarrow \left|\xi_i^{(m)} - \xi_i^{(n)}\right| < \in \ \forall m, n \ge N\left(i=1,2,3,....\right)$$

This implies for each $i\left(1 \le i < \infty\right)$, the sequence $\xi_i^{(1)}, \xi_i^{(2)}, \xi_i^{(3)}, ...,$ is a Cauchy sequence in R or $C\left(k\right)$ since k is complete it converges in k.

Suppose $\xi_i^{(m)} \to \xi_i$ as $m \to \infty$

Now we define $x = \left(\xi_1, \xi_2, \xi_3, ...\right)$ and show that $x \in l^p$ and $x_m \to x$. Using equation (1) we get

$$\sum_{i=1}^{k}\left|\xi_i^{(m)} - \xi_i^{(n)}\right|^p < \in^p, \ \forall m, n \ge N\left(i=1,2,3,...\right)$$

Letting $n \to \infty$, we get

$$\sum_{i=1}^{k}\left|\xi_i^{(m)}-\xi_i^{(n)}\right|^p <\in^p,\ \ \forall m\geq N\,(i=1,2,3,\ldots)$$

Now suppose $k\to\infty,$ then for $m\geq N$

$$\sum_{i=1}^{\infty}\left|\xi_i^{(m)}-\xi_i\right|^p \leq\in^p,\quad \forall m\geq N$$

$$\Rightarrow x_m - x = \left(\xi_i^{(m)}-\xi_i\right)\in l^p$$

Since $x_m\in l^p,$ it follows that using Minkowski inequality we get

$$x = x_m + (x - x_m)\in l^p$$

$$\Rightarrow x\in l^p$$

From equation (2), we get $\quad d(x_m, x) <\in \ \ \forall m\geq N$

Which verify that $x_m\to x$ in l^p

Hence, l^p (with $1\leq p<\infty$) is a complete metric space.

Example.5: The function space $C[a,b]$ is complete, here $[a,b]$ is any given closed interval on R.

Solution: Suppose $\{x_m\}$ is a Cauchy sequence in $C[a,b]$. Then for each $\in>0$, there exist a positive integer N such that

$$d(x_m,x_n)=\max_{t\in I}\left|x_m(t)-x_n(t)\right|\leq\in\qquad \forall m,n\geq N,\ \text{ where } I=[a,b]$$

For any fixed $t=t_0\in I=[a,b],$ we get

$$\left|x_m(t_0) - x_n(t_0)\right| < \in \quad \forall m, n \geq N \qquad \text{....(1)}$$

This implies $\{x_1(t_0), x_2(t_0), ...\}$ is a Cauchy sequence in R. But R is complete, this sequence converges. Now we let.

$$x_m(t_0) \to x(t_0) \text{ as } m \to \infty$$

We can associate to each $t \in I$ a unique $x(t)$. This show that a function x on I is pointwise and $x \in C[a,b], x_m \to x.$

From equation (1) we have

$$\left|x_m(t) - x_n(t)\right| < \in \; \forall m, n \geq N \text{ and } \forall t \in I$$

Take $n \to \infty$, we get $\left|x_m(t) - x(t)\right| < \in \; \forall m \geq N$ and $\forall t \in I$ ....(2)

This implies the sequence $\{x_m\}$ of continuous functions converges uniformly to the function x on $[a,b]$ and hence the limit function x is a continuous function on as such $x \in C[a,b]$.

Using equation (2), we have

$$\max_{t \in I} \left|x_m(t) - x(t)\right| < \in \; \forall n \geq N$$

$$\Rightarrow d(x_m, x) < \in \quad \forall m \geq N$$

$$\Rightarrow x_m \to x \text{ in } C[a,b]$$

Hence, $C[a,b]$ is a complete metric space.

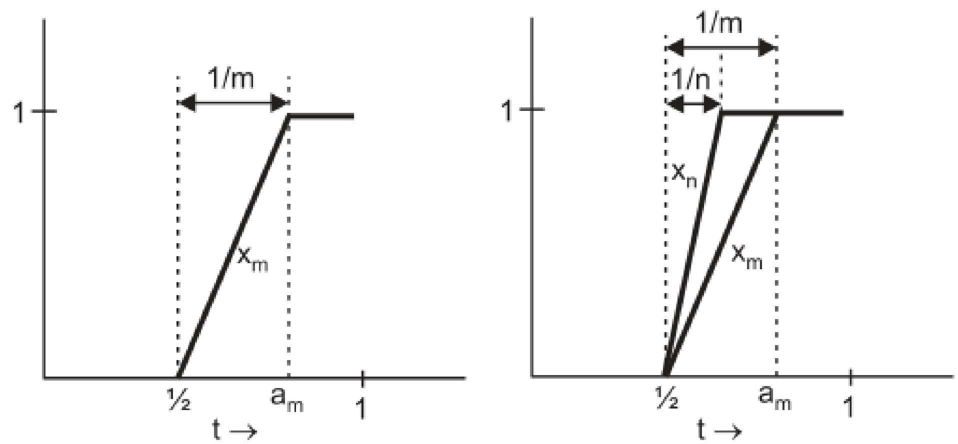
Example.6: Let X be the set of all continuous real valued function on $I = [0,1]$ and let

$$d(x,y)=\int_0^1 |x(t)-y(t)|\,dt$$

Show that this metric space (X,d) is not complete.

Solution: Suppose the function x_m from a Cauchy sequence because $d(x_m,x_n)$ is the area of the triangle and for every given $\in>0$

$$d(x_m,x_n)<\in \quad \text{where } m,n>\frac{1}{\in}$$



To show that this cauchy sequence does not converge we have

$$x_m(t)=0 \text{ if } t\in\left[0,\frac{1}{2}\right]$$

$$x_m(t)=1 \text{ if } t\in[a_m,1]$$

Where $$a_m=\frac{1}{2}+\frac{1}{m},\ \forall x\in X.$$

Now we have

$$d(x_m,x)=\int_0^1 |x_m(t)-x(t)|\,dt$$

$$= \int_0^{1/2} |x(t)|\,dt + \int_{1/2}^{a_m} |x_m(t) - x(t)|\,dt + \int_{a_m}^{1} |1-(x)t|\,dt$$

Since the intergrands are non-negative so is each integral on the right, i.e., $d(x_m, x) \to 0$ woulf imply that each integral approaches to zero. Since x is continuous, we have

$$x(t) = \begin{cases} 0 & \text{if } t \in \left[0, \frac{1}{2}\right) \\ 1 & \text{if } t \in \left(\frac{1}{2}, 1\right] \end{cases}$$

i.e., contradiction for a continuous function. Hence, $< x_m >$ does not converges ot do not have a limit in X .Hence, X is not complete and $\{X, d)$ is not complete metric space.

Example.7: Consider the usual metric d for R^2 and the mapping $f : R^2 \to R^2$ such that

$$f(x) = \frac{x}{2} \forall x \in \boldsymbol{R}^2$$

Where $x = (x_1, x_2)$. Then f is a contraction on R^2.

Solution: Given that

$$f : R^2 \to R^2 \text{ such that}$$

$$f(x) = \frac{x}{2} \forall x \in \boldsymbol{R}^2$$

We have

$$d(f(x), f(y)) = d\left(\frac{x}{2}, \frac{y}{2}\right)$$

$$= d\left(\frac{1}{2}(x_1, x_2), \frac{1}{2}(y_1, y_2)\right)$$

$$= d\left(\left(\frac{x_1}{2}, \frac{x_2}{2}\right), \left(\frac{y_1}{2}, \frac{y_2}{2}\right)\right)$$

$$= \sqrt{\left[\frac{1}{4}(x_1 - x_2)^2 + \frac{1}{4}(y_1 - y_2)^2\right.}$$

$$= \frac{1}{2}\sqrt{\left[(x_1 - x_2)^2 + (y_1 - y_2)^2\right.}$$

$$= \frac{1}{2} d(x, y)$$

Hence, f is contracting mapping because

$$d\left(f(x), f(y)\right) = \lambda\left(d(x, y)\right); \quad \text{where } 0 \le \lambda < 1\left(\lambda = \frac{1}{2}\right).$$

6.11 Summary

Let (X, d) be a meric space. Then (X, d) is said to be complete metric space if and only if every Cauchy sequence $< x_n >$ in X converges to a point in X.

Let (X, d) be a metric space. Then is said to be incomplete metric space if and only if there exists some cauchy sequence $< x_n >$ in X which does not converge to a point in X.

Let $< x_n >$ be a decreasing sequence of non-empty closed subsets of a complete metric space (X, d) such that $d(x_n) \to 0$ as $n \to \infty$. Then $\bigcap_{n=1}^{\infty} x_n$ contains exactly one point.

Let (X,d) be a metric space. A subset A of a metric space is said to be the first category if and only if it can be written as the union of a countable family of nowhere dense sets; otherwise is known as second category.

Let (X,d) be a complete metric space. A mapping $f: x \to X$ is said to be contracting mapping if there exist a real number λ with $0 \le \lambda < 1$ such that

$$d(f(x), f(y)) \le \lambda d(x,y) < d(x,y) \quad \forall x, y \in X .$$

Let (X,d) be a complete metric space and f is a contracting mapping on X. Then there exist one and only one point x in X such that

$$f(x) = x .$$

Let (X,d) and $(\tilde{X}, \tilde{d})$ be two metric space. A mapping $f: X \to \tilde{X}$ is said to be isometric on an isometry of f preserves distances i.e., if $\forall x, y \in X$.

$$\tilde{d}(f_x, f_y) = d(x,y)$$

where f_x and f_y are the images of x and y respectively.

The space consists of all convergent sequence $x = (\xi_i)$ of complex number, with the metric induced from the space l^∞ .

6.12 Terminal Questions

Q.1 Define complete metric space with examples.

Q.2. Explain the Contor's intersection theorem.

Q.3. To show that every contracting mapping is continuous.

Q.4. Let X be the set of all positive integers and $d(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|$. Show that (X, d) is not complete.

Q.5. The metric space of rational numbers with the usual metric is incomplete.

Q.6. Show that the set X of all integers with metric d defined by $d(x, y) = |x - y|$ is a complete metric space.

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Master of Science/ Master of Arts

PGMM-109N
MAMM-109N
Topology

Block

3 Introduction to Topological Spaces

Block-3

Introduction to Topological Spaces

Topology plays a crucial role in various areas of mathematics, science, and engineering due to its ability to capture essential geometric and topological properties of spaces. It is concerned with the study of shapes and spaces, focusing on the concepts of continuity and connectivity. Topology seeks to understand the underlying structure of spaces and the relationships between different spaces, often using concepts like open sets, closed sets, neighborhoods, and continuous functions. In 1872, Cantor introduced the concept of the first derived set, or set of limit points, of a set. He also defined certain closed subsets of the real line as subsets that contain their first derived set. Cantor also introduced the concept of an open set, another fundamental concept in point-set topology. In 1902, Hilbert used the idea of a neighborhood. In 1914, Felix Hausdorff coined the term "Topological Space" and provided the definition for what is now known as a Hausdorff space. In its current usage, a topological space is a slight generalization of Hausdorff spaces, as formalized by Kazimierz Kuratowski in 1922.

In the seventh unit, we shall discussed about Topological Spaces, Trivial topology, Non-Trivial topologies, Comparison of Topologies, Algebra of Topologies, Open Set, Neighbourhood, Usual Topology, Limit Points, Derived Set, Closed Sets, Door Space, and in the eighth unitwe deal with Closure of a Set, Separated Set, Interior points and the Interior of a Set, Exterior of a Set, Boundary Points, Dense Set.

Ninth unit deals with Relative Topology, Subspace, Base for a topology, Sub-bases, Local base, First Countable Space, Second Countable Space, Topologies Generated by Classes of Sets, Separable Space, Cover of a Space, LindelöfSpace. In the tenth unit we shall discuss about the Continuous Function, Open Mapping, Closed Mapping, Bicontinuous Mapping, Bijective Mapping, Sequential Continuity, The pasting Lemma, Homeomorphism.

UNIT-7: Topological Spaces-I

Structure

7.1 Introduction

Topology is a mathematical field that explores the properties of space that remain unchanged under continuous transformations like stretching, twisting, and bending, without tearing or gluing. This unit covers topics such as topological spaces, trivial topology (discrete and indiscrete topology), comparison of topologies, algebra of topologies, open sets, neighborhoods and neighborhood systems of a point, usual topology, limit points, closed sets, and door spaces. In 1872, Cantor introduced the concept of the first derived set, or set of limit points, of a set. He also defined some closed subsets of the real line as subsets containing their first derived set. Cantor also introduced the concept of an open set, another fundamental concept in point set topology.

7.2 Objectives

After reading this unit the learner should be able to understand about the:

- Introduction about Topology and Topological Spaces,
- Trivial topology and Non-Trivial topologies
- Comparison of Topologies
- Algebra of Topologies
- Open Set and Neighbourhood, Neighborhood system of that point
- Usual Topology or standard Topology
- Limit Points and Derived Set
- Closed Sets and Door Space

7.3 Topological Spaces

Topology is a mathematical discipline concerned with the properties of shapes and spaces that remain unchanged under continuous transformations like stretching, bending, and twisting, without tearing or gluing. It is a foundational field with broad applications in mathematics, physics, biology, and computer science. Topological spaces are structures that generalize notions of proximity and include sets of points along with collections of open sets satisfying specific properties. This branch of mathematics provides a framework for analyzing properties that endure through continuous transformations, making it a potent tool in mathematical analysis and various other fields.

Let us consider X be a non-empty set and $\Im$ be the collection of subsets of X. Then $\Im$ is said to be a topology on X, if the following properties are satisfied:

- $X \in \Im$ and $\phi \in \Im$.
- Let a, b $\in \Im$ then $a \cap b \in \Im$, hence it is closed under the operation of finite intersection.
- Let $\{A_i : i \in I\} \in \Im$ then $\cup\{A_i : i \in I\} \in \Im$, hence it is closed under the operation of arbitrary union.

The members of $\Im$ are known as open sets of the topology $\Im$ and the pair $(X, \Im)$ is known as a topological space.

Note: A subset of X may be open, closed, both or neither.

Examples

Example.1. Consider X= {a, b, c} and $\Im$ ={X, ϕ, {a}, {b}, {a, b}}. Then show that $\Im$ is a topology on X.

Solution. It is given that X= {a, b, c} and $\Im=\{X, \phi, \{a\}, \{b\}, \{a, b\}\}$

(i) $X \in \Im$ and $\phi \in \Im$.

(ii) We have $\{a\} \cap \{b\} = \phi \in \Im$, $X \cap \{b\} = \{b\} \in \Im$,

$X \cap \{a\} = \{a\} \in \Im$, $\phi \cap \{b\} = \phi \in \Im$,

$\phi \cap \{a\} = \phi \in \Im$, $X \cap \phi = \phi \in \Im$.

i.e., it is closed under the operation of finite intersection.

(iii) We have $\{a\} \cup \{b\} = \{a, b\} \in \Im$, $X \cup \{b\} = X \in \Im$,

$X \cup \{a\} = X \in \Im$, $X \cup \phi = X \in \Im$,

$\{a\} \cup \phi = \{a\} \in \Im$, $\{b\} \cup \phi = \{b\} \in \Im$.

i.e., it is closed under the operation of arbitrary union. Hence $\Im$ is a topology on X.

Example.2. Consider $X= \{a, b\}$ and $\Im=\{X, \phi, \{a\}, \{b\}\}$. Then show that $\Im$ is a topology on X.

Solution. It is given that $X= \{a, b\}$

and $\Im = \{X, \phi, \{a\}, \{b\}\}$

(i) $X \in \Im$ and $\phi \in \Im$.

(ii) We have $\{a\} \cap \{b\} = \phi \in \Im$, $X \cap \{b\} = \{b\} \in \Im$,

$X \cap \{a\} = \{a\} \in \Im$, $\phi \cap \{b\} = \phi \in \Im$,

$\phi \cap \{a\} = \phi \in \Im$, $X \cap \phi = \phi \in \Im$.

i.e., it is closed under the operation of finite intersection.

(iii)We have $\{a\} \cup \{b\} = X \in \Im$, $X \cup \{b\} = X \in \Im$,

$X \cup \{a\} = X \in \Im$, $X \cup \phi = X \in \Im$,

$\{a\} \cup \phi = \{a\} \in \Im$, $\{b\} \cup \phi = \{b\} \in \Im$.

i.e., it is closed under the operation of arbitrary union. Hence $\Im$ is a topology on X.

Example.3. Determine all the topologies on X= {a, b}.

Solution. It is given that X= {a, b}.

Then $\Im_1 = \{X, \phi\}$,

$\Im_2 = \{X, \phi, \{a\}\}$,

$\Im_3 = \{X, \phi, \{b\}\}$

and $\Im_4 = \{X, \phi, \{a\}, \{b\}\}$ are all topologies on X={a, b}.

Example.4. Consider X= {a, b, c, d, e} and $\Im = \{X, \phi, \{a\}, \{b, c\}, \{c, d, e\}\}$. Then show that $\Im$ is not a topology on X.

Solution. It is given that X= {a, b, c, d, e}

and $\Im = \{X, \phi, \{a\}, \{b, c\}, \{c, d, e\}\}$

(i) $X \in \Im$ and $\phi \in \Im$.

(ii) We have $\{b, c\} \cap \{c, d, e\} = \{c\} \notin \Im$

i.e., it is not closed under the operation of finite intersection.

Hence $\Im$ is not a topology on X.

Example.5. Consider X= {a, b, c} and $\Im = \{\phi, \{a\}, \{b\}, \{a, b\}\}$. Write down why $\Im$ is not a topology on X.

Solution. It is given that X= {a, b, c}

and $\Im = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

Here $\phi \in \Im$ but $X \notin \Im$

i.e.,$\Im$ is not a topology on X because it's not contains X.

Hence $\Im$ is not a topology on X.

7.4 Trivial Topology

The trivial topology satisfies the axioms of a topology, namely that the empty set and the whole space are open, the intersection of any finite number of open sets is open, and the union of any collection of open sets is open.There are two types of trivial topology:

(i) Indiscrete Topology
(ii) Discrete Topology

(i) <u>Indiscrete Topology</u>

Let X be a non empty set. Then the collection $\Im$ ={X, ϕ}, (consisting of only empty set and the whole space) is always a topology for X, is called the *indiscrete topology*. The pair (X,$\Im$) is called an *indiscrete topological space*.

Indiscrete topology is also denoted by I. For any set X, the indiscrete topology I is coarser or smaller or weaker topology.

(ii) <u>Discrete Topology</u>

Let X be a non empty set. Then the collection $\Im$ = {consisting of all the subsets of X} is always a topology for X, called the *discrete topology*. The pair (X,$\Im$) is called *discrete topological space*. Discrete topology is also denoted by D. For any set X, the discrete topology D is finer or stronger or larger topology.

For example, we haveX= {a, b, c} then we have

$\Im_1$ ={X, ϕ} is the indiscrete topology for X and $\Im_2$={X, ϕ, {a}, {b}, {c}, {a, b}, {b, c}, {a, c}} is the discrete topology on X.

$\Im_3$={X, ϕ, {a}}

$\Im_4$={X, ϕ, {b}}

$\Im_5$={X, ϕ, {c}}

$\Im_6$={X, ϕ, {a}, {a, b}}

$\Im_7$={X, ϕ, {a}, {b, c}}

$\Im_8$={X, ϕ, {a}, {a, c}}

$\Im_9$={X, ϕ, {b}, {a, b}}

$\Im_{10}$={X, ϕ, {b}, {b, c}}

$\Im_{11}$={X, ϕ, {b}, {a, c}}

$\Im_{12}$={X, ϕ, {c}, {a, b}}

$\Im_{13}$={X, ϕ, {c}, {b, c}}

$\Im_{14}$={X, ϕ, {c}, {a, c}}

$\Im_{15}$={X, ϕ, {a}, {b}, {a, b}}

$\Im_{16}$={X, ϕ, {a}, {c}, {a, c}}

$\Im_{17}$={X, ϕ, {b}, {c}, {b, c}}

$\Im_{18}$={X, ϕ, {a}, {a, b}, {a, c}}

$\Im_{19}$={X, ϕ, {b}, {a, b}, {b, c}}

and $\Im_{20}$={X, ϕ, {c}, {a, c}, {b, c}} are all non-trivial topologies on X.

7.5 Non-Trivial Topologies

Non-trivial topologies are topologies on a set that are not the trivial topology. In other words, they are topologies that contain open sets other than the empty set and the entire space. Non-trivial topologies are often used to define interesting and useful topological properties on sets.Topologies defined on X other than trivial topology (Indiscrete and Discrete topology) are known as *non-trivial topologies*.

7.6 Comparison of Topologies

Comparing topologies involves understanding how one topology relates to another in terms of their open sets and the properties they induce on a space.

Let X be a non-empty set and $\Im_1$, $\Im_2$ are two topologies on X then either $\Im_1 \subset \Im_2$ and $\Im_2 \subset \Im_1$, the topologies $\Im_1$ and $\Im_2$ are *comparable*. If $\Im_1 \not\subset \Im_2$ and $\Im_2 \not\subset \Im_1$ then the topologies $\Im_1$and $\Im_2$ are not comparable.

Examples

Example.6. Consider X= {*a*, b, c}. Find the three topologies $\Im_1$, $\Im_2$ and $\Im_3$ for such that $\Im_1 \subset \Im_2 \subset \Im_3$.

Solution. Given that X= {*a*, b, c}.

Then $\Im_1$ ={X, ϕ},

$\Im_2$={X, ϕ, {a}}

and $\Im_3$={X, ϕ, {a}, {b}, {a, b}} are topologies on X. Hence $\Im_1 \subset \Im_2 \subset \Im_3$.

Example.7. Let X= {*a*, b, c}. Find indiscrete and discrete topologies on X.

Solution. Given thatX= {a, b, c} then

$\Im_1$ ={X, ϕ} is an indiscrete topology for X

and $\Im_2$={X, ϕ, {a}, {b}, {c}, {a, b}, {b, c}, {a, c}} is a discrete topology on X.

Example.8. Find two mutually comparable topologies for the set X= {*a*, b, c}.

Solution. Given that X= {a, b, c}.

Then $\Im_1$ ={X, ϕ, {a}},

$\Im_2$={X, ϕ, {b}}

and $\Im_3$={X, ϕ, {a}, {b}, {a, b}}.

Hence the above two topologies $\Im_1$ and $\Im_3$, $\Im_2$ and $\Im_3$ are comparable because $\Im_1 \subset \Im_3$ and $\Im_2 \subset \Im_3$.

Example.9. Give five distinct non-trivial topologies for the set X= {*a*, b, c}.

Solution. Given that X= {a, b, c}.

Then $\Im_1$ ={X, ϕ, {a}},

$\Im_2$={X, ϕ, {b}},

$\Im_3$={X, ϕ, {c}}

$\Im_4$={X, ϕ, {a}, {a, b}}

and $\Im_5$={X, ϕ, {a}, {b}, {a, b}} are all five distinct non-trivial topologies on X.

Example.10. Find three mutually non-comparable topologies for the set X = {*a, b, c*}.

Solution. Given that X= {a, b, c}.

Then $\Im_1$ ={X, ϕ, {a}},

$\Im_2$={X, ϕ, {b}}

and $\Im_3$={X, ϕ, {c}} are all three mutually non-comparable topologies for the set X because $\Im_1 \not\subset \Im_2$, $\Im_1 \not\subset \Im_3$ and $\Im_2 \not\subset \Im_3$.

Hence $\Im_1$, $\Im_2$ and $\Im_3$ are non-comparable topologies for the set X.

7.7 Algebra of Topologies

The algebra of topologies refers to the operations and properties that arise when combining or manipulating topological spaces and their associated topologies. These operations include the intersection, union, and product of topologies, as well as the generation of new topologies from existing ones through operations like closure and interior.

Theorem.1. Prove that the finite intersection of two topology on X is again a topology on X.

Proof. Let (X, $\Im_1$) and (X, $\Im_2$) are two topological space. To show that ($\Im_1 \cap \Im_2$) is again a topology on X.

(i) Since $\Im_1$ and $\Im_2$ ate two topology on X such that

$$X \in \Im_1 \text{ and } \phi \in \Im_1$$

Similarly, $X \in \Im_2$ and $\phi \in \Im_2$.

$$\Rightarrow X \in \Im_1 \cap \Im_2 \text{ and } \phi \in \Im_1 \cap \Im_2.$$

(ii) Let a, b$\in\Im_1\cap\Im_2$

$\Rightarrow$ a$\in\Im_1\cap\Im_2$ i.e., a$\in\Im_1$, a$\in\Im_2$

and b$\in\Im_1\cap\Im_2$ i.e., b$\in\Im_1$, b$\in\Im_2$

Now we have a$\in\Im_1$, b$\in\Im_1\Rightarrow$ a$\cap$b $\in\Im_1$

a$\in\Im_2$, b$\in\Im_2\Rightarrow$ a$\cap$b $\in\Im_2$

Since a$\cap$b $\in\Im_1$, a$\cap$b $\in\Im_2$ $\Rightarrow$ a$\cap$b$\in\Im_1\cap\Im_2$

i.e., it is closed under the operation of finite intersection.

(iii) Let $\{A_i : i\in I\}\in\Im_1\cap\Im_2$.

$\{A_i : i\in I\}\in\Im_1$ and $\{A_i : i\in I\}\in\Im_2$

$\Rightarrow\cup\{A_i : i\in I\}\in\Im_1$ and$\cup\{A_i : i\in I\}\in\Im_2$ {since $\Im_1$ and $\Im_2$ are topology on X}

$\Rightarrow\cup\{A_i : i\in I\}\in\Im_1\cap\Im_2$.

i.e., it is closed under the operation of arbitrary union. Hence $\Im_1\cap\Im_2$ is a topology on X.

Theorem.2. Prove that the intersection of any number of topologies on X is again a topology on X.

Proof. Let X be a non-empty set and collection of topologies $\{\Im_i : i\in I\}$ is a topology on X. To show that $\cap\{\Im_i : i\in I\}$ is a topology on X.

(i) Since X, $\phi\in\Im_i$.

$\Rightarrow X\in\cap\{\Im_i : i\in I\}$ and $\phi\in\cap\{\Im_i : i\in I\}$

(ii) Let G_1 and $G_2\in\cap\{\Im_i : i\in I\}$

Since G_1 and G_2 $\in$each $\Im_i$

$\Rightarrow G_1 \cap G_2 \in$ each $\Im_i$

i.e., it is closed under the operation of finite intersection.

(iii) Let $\{G_\alpha : \alpha \in I\} \in \cap\{\Im_i : i \in I\}$

Since $\{G_\alpha : \alpha \in I\} \in$ each $\Im_i$

and each $\Im_i$ is a topology on X.

$\cup\{G_\alpha : \alpha \in I\} \in \cap\{\Im_i : i \in I\}$

i.e., it is closed under the operation of arbitrary union. Hence $\cap\{\Im_i : i \in I\}$ is a topology on X.

Theorem.3. Prove that the union of two topologies on X is again a topology if X consists of at most two elements.

Proof. Suppose if possible X={a, b, c},

$\Im_1 = \{X, \phi, \{a\}\}$

and $\Im_2 = \{X, \phi, \{b\}\}$

Then $\Im_1 \cup \Im_2 = \{X, \phi, \{a\}, \{b\}\}$

(i)Since X, $\phi \in \Im_1$ and X, $\phi \in \Im_2$

$\Rightarrow$ X, $\phi \in \Im_1 \cup \Im_2$

(ii)Let $\{a\}, \{b\} \in \Im_1 \cup \Im_2$

$\Rightarrow \{a\} \cap \{b\} = \phi \in \Im_1 \cup \Im_2$

i.e., it is closed under the operation of finite intersection.

(iii)Let $\{a\}, \{b\} \in \Im_1 \cup \Im_2$

$\Rightarrow \{a\} \cup \{b\} = \{a, b\} \notin \Im_1 \cup \Im_2$

i.e., it is not closed under the operation of arbitrary union.

Hence $\Im_1 \cup \Im_2$ is not a topology on X because it consists more than two elements.

Now we suppose X= {a, b},

$\Im_1=\{X, \phi, \{a\}\}$

and $\Im_2=\{X, \phi, \{b\}\}$

Then $\Im_1 \cup \Im_2 =\{X, \phi, \{a\}, \{b\}\}$

Here (i) and (ii) are satisfied as above.

Now let $\{a\}, \{b\} \in \Im_1 \cup \Im_2$

$\Rightarrow \quad \{a\} \cup \{b\} = X \in \Im_1 \cup \Im_2$

i.e., it is closed under the operation of arbitrary union.

Hence $\Im_1 \cup \Im_2$ is a topology on X if it consists of at most two elements.

7.8 Open Set

The concept of an open set is important because it allows us to define continuity, convergence, and many other fundamental properties in topology.

If X is a non empty set and $\Im$ is a topology on X, then every member of $\Im$ is called *open set*.

<u>Some properties of open set</u>

(i) The empty set ϕ is open.

(ii) The whole space X is open.

(iii) The intersection of two open set is open.

(iv) Arbitrary union of open set is open.

(v) A finite set is not an open set.

(vi) R is an open set.

(vii) Q is neither open nor closed.

(viii) The complement of a closed set is open set.

(ix) Every neighbourhood is an open set.

For example, the set $\{x \in R: x > 3\}$ is open set but the set $\{x \in R: x^2 \geq 1\}$ is not open set.

7.9 Neighborhood

Neighborhoods are important in topology because they allow us to define and study the notion of "closeness" of points in a topological space. In topology, a neighborhood of a point in a topological space is an open set that contains that point.

Let $(X, \Im)$ be a topological space. A subset N of X is said to be a neighborhood of point x if it contain an open set G to with the point $x \in G$ such that $x \in G \subset N$.

or

Let $(X, \Im)$ be a topological space and $x \in X$. A subset N of whole space X is said to be a $\Im$-neighborhood of x if and only if there exist a $\Im$-open set G such that $x \in G \subset N$.

Neighborhood system of that point

The neighborhood system of a point in a topological space is a fundamental concept that describes the local structure of the space around that point. It plays a crucial role in defining continuity, convergence, and other important concepts in topology.The neighborhood system of a point in a topological space is the collection of all neighborhoods of that point.The collection of all the neighborhood of a point is called the *Neighborhood system of that point.*

Note. 1. Since every set is a subset of itself therefore open set is the neighborhood each of its point.

2. A $\Im$–open set is a $\Im$-neighborhood of each of its points but a $\Im$-neighborhood of a point need not be an open set.

3. Let us consider $(X,\Im)$ be a topological space. Then

(i) For each point $x\in X$ and each $\Im$-neighborhood N of x, $x\in N$.

(ii) For each point $x\in X$, there is at least one $\Im$-neighborhood of x.

(iii) If N is any $\Im$-neighborhoods of $x\in X$ and M is a superset of N, then M is also $\Im$-neighborhood of x for all $x\in X$.

(iv) If M and N are any two $\Im$-neighborhoods of $x\in X$ then $M\cap N$ is also $\Im$-neighborhood of x for all $x\in X$.

Theorem.4. Let us consider $(X, \Im)$ be a topological space and $A\subset X$. The set A is a $\Im$-open set if and only if it is a $\Im$-neighborhood of each of its points.

Proof. Let $(X, \Im)$ be a topological space and A be any subset of X.

If A is $\Im$-open set then $\forall x\in A$ and $A\subseteq A$.

Therefore A is neighborhood of each of its points.

Conversely, let A is a neighborhood of each of its points, then we have

(i) $A=\phi$, this implies, it is open.

(ii) $A\neq\phi$, then to each $x\in A$

$\Rightarrow$ there exist an open set A_x of x such that

$$x\in A_x\subset A$$

$\Rightarrow \quad A=\cup\{ A_x : x\in A\}$

$\Rightarrow$ A is a union of open sets.

$\Rightarrow$ A is open.

7.10 Usual Topology or Standard Topology

The term "usual topology" is often used to refer to the standard topology on a particular space, especially in contexts where there may be multiple possible topologies under consideration.

Let R be a set of real numbers and $\boldsymbol{U}$ be the collection of subsets of R and if $\boldsymbol{U}$ satisfies all the properties of topology then $\boldsymbol{U}$ is a topology on R and pair (R, $\boldsymbol{U}$) is called *usual topological space or standard topological space.*

Theorem.5. Let U be a collection of null set and all those subset of R such that $x\in G$ there exist $\in>0$ such that $]x-\in, x+\in[\subset G\subset R$. Then show that U is a topology on R.

Proof. (i) Given that $\phi\in\boldsymbol{U}$ and $R\in\boldsymbol{U}$

(because to each $x\in R$ i.e.,$]x-\in, x+\in[\subset R$)

(ii) Let $G_1, G_2\in\boldsymbol{U}$.

Case.1. If $G_1\cap G_2=\phi$, then $G_1\cap G_2\in\boldsymbol{U}$.

Case.2. If $G_1\cap G_2\neq\phi$, then

$$x\in G_1,\ x\in G_2$$

$\Rightarrow$ there exist $\in_1, \in_2>0$ such that

$]x-\epsilon_1, x+\epsilon_1[\subset G_1,]x-\epsilon_2, x+\epsilon_2[\subset G_2$

Let $\epsilon = \min(\epsilon_1, \epsilon_2)$

$\epsilon>0$ such that $] x-\epsilon, x+\epsilon[\subset G_1 \cap G_2$

$\Rightarrow \quad G_1 \cap G_2 \in \boldsymbol{U}.$

(iii) Let $\{G_i: i \in I\} \in \boldsymbol{U}.$

Since $]x-\epsilon, x+\epsilon[\subset \{G_i: i \in I\}$

Also $\{G_i: i \in I\} \subset \cup \{G_i: i \in I\}$

$\Rightarrow \quad]x-\epsilon, x+\epsilon[\subset \cup \{G_i: i \in I\}$

$\Rightarrow \quad \cup \{G_i: i \in I\} \in \boldsymbol{U}$

Hence $\boldsymbol{U}$ *is a topology on R.*

Theorem.6. Let *f* be a mapping from X to Y, where X is a non-empty set and Y is a topological space if $\Im$ is a topology on Y then prove that $\Im_1=\{f^{-1}(G): G \in \Im\}$ is a topology on X.

Proof. Let us consider $\Im_1 = \{f^{-1}(G): G \in \Im\}$.To show that $\Im_1$ is a topology on X.

(i) Since $\Im$ is a topology on Y then

$Y, \phi \in \Im$, therefore

$f^{-1}(Y) = X \in \Im_1$

and $f^{-1}(\phi) = \phi \in \Im_1$

Therefore $X, \phi \in \Im_1$

(ii) Let $f^{-1}(G_1), f^{-1}(G_2) \in \mathfrak{I}_1$

$\Rightarrow \quad f^{-1}(G_1) \cap f^{-1}(G_2) \in \mathfrak{I}_1$

$\Rightarrow \quad f^{-1}(G_1 \cap G_2) \in \mathfrak{I}_1 \qquad \{\text{because } G_1 \in \mathfrak{I}, G_2 \in \mathfrak{I} \Rightarrow G_1 \cap G_2 \in \mathfrak{I}\}$

i.e., it is closed under the operation of finite intersection.

(iii)Let $f^{-1}\{G_i : i \in I\} \in \mathfrak{I}_1$

Using definition of topology, we have

$\Rightarrow \qquad \cup f^{-1}\{G_i : i \in I\} \in \mathfrak{I}_1$

i.e., it is closed under the operation of arbitrary union. Hence $\mathfrak{I}_1$ is a topology on X.

7.11 Limit Point

Limit points are important in topology because they tell us about how points in a set relate to each other and to the space around them. They help us to understand when a set is closed, which is a key concept in topology. Limit points also help us to define when functions are continuous and when sets are compact.

Let $(X, \mathfrak{I})$ be a topological space and A be a subset of X. Then a point $x \in X$ is a limit point of A if each neighborhood of x contains at least one point of A other than x.

7.12 Derived Set

A derived set, also known as the derived set or set of limit points, of a set A in a topological space X is the set of all limit points of A. The derived set is important in topology because it helps us to define the closure of a set. The derived set also plays a role in characterizing the properties of a set, such as whether it is closed or dense in the space X.A collection of all the limit points of a set A is called the*derived set of A* denoted by A' or D (A).

Note. A limit point may or may not belong to the set.

Theorem.7. Let A and B be subsets of topological space (X, $\Im$). Then $(A\cup B)' = A' \cup B'$.

Proof. Let (X, $\Im$) be a topological space and $A,B \subset X$.

We know that $A \subset A\cup B \Rightarrow A' \subset (A\cup B)'$. $\{A \subset B$ then $A' \subset B'\}$

$B \subset A\cup B \Rightarrow B' \subset (A\cup B)'$.

$\Rightarrow A' \cup B' \subset (A\cup B)'$. ...(1)

Now to show that $(A\cup B)' \subset A' \cup B'$

Let if possible $x \notin A' \cup B' \Rightarrow x \notin A'$ and $x \notin B'$

$\Rightarrow$ x is not limit point of A and B

$\Rightarrow$ there exist a neighborhood U_x which contains no point of A other then x (by definition) and V_x which contains no point of B other then x (by definition)

i.e., $x \notin (A' \cup B')$

Now if $x \in A' \cup B' \Rightarrow x \in (A' \cup B')$

Therefore $(A\cup B)' \subset A' \cup B'$ ...(2)

Uging (1) and (2), we get $(A\cup B)' = A' \cup B'$.

Examples

Example.11. Let X= {a, b} and $\Im$={X, ϕ, {a}, {b}}. Find $\Im$-neighborhood of (i) a and (ii) b.

Solution. Given thatX= {a, b} and $\Im$={X, ϕ, {a}, {b}}.

(i) $\Im$-open sets containing a are X, {a}

Superset of X is X.

Superset of {a} are {a}, X.

Hence $\Im$-neighborhoods of a are {a}, X.

(ii) $\Im$-open sets containing b are X, {b}

Superset of X is X.

Superset of {b} are {b}, X.

Hence $\Im$-neighborhoods of b are {b}, X.

Example.12. Let X= {a, b, c, d} and $\Im$={X, ϕ, {b}, {a, b}, {a, b, d}}. Find $\Im$-neighborhood of (i) a (ii) b and (iii) c.

Solution. Given thatX= {a, b, c, d} and $\Im$={X, ϕ, {b}, {a, b}, {a, b, d}}.

(i) $\Im$-open set containing a are X, {a, b}, {a, b, d}.

Superset of X is X.

Superset of {a, b} are {a, b}, {a, b, c}, {a, b, d}, X.

Superset of {a, b, d} are {a, b, d}, X.

Hence $\Im$-neighborhoods of a are

{a, b}, {a, b, c}, {a, b, d}, X.

(ii) $\Im$-open set containing b are X, {b}, {a, b}, {a, b, d}.

Superset of X is X.

Superset of {b} are {b}, {a, b}, {b, c}, {b, d}, {a, b, c}, {a, b, d}, {b, c, d}, X.

Superset of {a, b, d} are {a, b, d}, X.

Hence $\Im$-neighborhoods of b are

{b}, {a, b}, {b, c}, {b, d}, {a, b, c}, {a, b, d}, {b, c, d}, X.

(iii) $\Im$-open set containing c is X.

Superset of X is X.

Hence $\Im$-neighborhoods of c is X.

Example.13. Let X= {a, b, c} and $\Im$={X, ϕ, {a},{b}, {c}, {a, b}, {b, c}, {a, c}} is a discrete topology on X. Find $\Im$-neighborhood system of (i) a (ii) b and (iii) c.

Solution. Given thatX= {a, b, c}

and $\Im$={X, ϕ, {a},{b}, {c}, {a, b}, {b, c}, {a, c}}

(i) $\Im$-open sets containing a are X, {a}, {a, b}, {a, c}

$\Im$-neighborhoods of a are {a}, {a, b}, {a, c}, X.

Hence neighborhood system of a is

$\aleph_a$= {{a}, {a, b}, {a, c}, X}.

(ii) $\Im$-open sets containing b are X, {b}, {a, b}, {b, c}

$\mathfrak{I}$-neighborhoods of b are {b}, {a, b}, {b, c}, X.

Hence neighborhood system of b is

$\aleph_b$= {{b}, {a, b}, {b, c}, X}.

(iii) $\mathfrak{I}$-open sets containing c are X, {c}, {a, c}, {b, c}

$\mathfrak{I}$-neighborhoods of a are {c}, {a, c}, {b, c}, X.

Hence neighborhood system of c is

$\aleph_c$= {{c}, {a, c}, {b, c}, X}.

Example.14. Let X= {a, b, c, d, e} and $\mathfrak{I}$={X, ϕ, {a}, {a, b}, {a, b, c}, {a, c, d}, {a, b, c, d}, {a, b, e}}. Find $\mathfrak{I}$-neighborhood system of (i) c and (ii) e.

Solution. Given thatX= {a, b, c, d, e}

and $\mathfrak{I}$={X, ϕ, {a}, {a, b}, {a, b, c}, {a, c, d}, {a, b, c, d}, {a, b, e}}.

(i) $\mathfrak{I}$-open set containing c are X, {a, b, c}, {a, c, d}, {a, b, c, d}.

Superset of X is X.

Superset of {a, b, c} are {a, b, c}, {a, b, c, d}, {a, b, c, e}, X.

Superset of {a, c, d} are {a, c, d}, {a, b, c, d}, {a, c, d, e}, X.

Superset of {a, b, c, d} are {a, b, c, d}, X.

$\mathfrak{I}$-neighborhoods of c are {a, b, c}, {a, c, d}, {a, b, c, d}, {a, b, c, e}, {a, c, d, e}, X.

Hence neighborhood system of c is

$\aleph_c$= {{a, b, c}, {a, c, d}, {a, b, c, d}, {a, b, c, e}, {a, c, d, e}, X}.

(ii) $\mathfrak{I}$-open set containing e are X, {a, b, e}.

Superset of X is X.

Superset of {a, b, e} are {a, b, e}, {a, b, c, e}, {a, b, d, e}, X.

$\Im$-neighborhoods of e are a, b, e}, {a, b, c, e}, {a, b, d, e}, X.

Hence neighborhood system of e is

$\aleph_c$= {{a, b, e}, {a, b, c, e}, {a, b, d, e}, X}.

Example.15. Let X= {a, b, c}, $\Im$={X, ϕ, {a}, {b}, {c}, {a, b}, {b, c}, {a, c}} and A = {a, b}. Find all the limit points of A.

Solution. Given thatX= {a, b, c}, $\Im$={X, ϕ, {a},{b}, {c}, {a, b}, {b, c}, {a, c}}

and A ={a, b}.

(i) Neighborhoods of a are X, {a}, {a, b}, {a, c}.

Here a is not a limit point of A because {a} is singleton and c$\notin$A.

(ii) Neighborhoods of b are X, {b}, {a, b}, {b, c}.

Here b is not a limit point of A because {b} is singleton and c$\notin$A.

(iii) Neighborhoods of c are X, {c}, {b, c}, {a, c}

Here c is not a limit point of A because {c} is singleton.

Example.16. Let X= {a, b, c}, $\Im$={X, ϕ, {a}, {a, b}, {a, c}} and A ={a, c}. Find all the limit points of A and hence determine A'.

Solution. Given thatX= {a, b, c}, $\Im$={X, ϕ, {a}, {a, b}, {a, c}} and A ={a, c}.

(i) Neighborhoods of a are X, {a}, {a, b}, {a, c}.

Here a is not a limit point of A because {a} is singleton.

Neighborhoods of b are X, {a, b}.

Here b is a limit point of A.

Neighborhoods of c are X, {a, c}.

Here c is a limit point of A.

(ii) The derived set of A is A' = {b, c}

Example.17. Let X= {a, b, c, d, e}, ℑ={X, ϕ, {a}, {c, d}, {a, c, d}, {b, c, d, e}} and A={a, b, c} Find the derived set of A.

Solution. Given thatX= {a, b, c, d, e}, ℑ={X, ϕ, {a}, {c, d}, {a, c, d}, {b, c, d, e}}

and A={a, b, c}.

ℑ-neighborhoods of a are X, {a}, {a, c, d}.

Here a is not a limit point of A because {a} is singleton.

ℑ-neighborhoods of b are X, {b, c, d, e}.

Here b is a limit point of A.

ℑ-neighborhoods of c are X, {c, d}, {a, c, d}, {b, c, d, e}.

Here c is not a limit point of A because d$\notin$A.

ℑ-neighborhoods of d are X, {c, d}, {a, c, d}, {b, c, d, e}.

Here d is a limit point of A.

ℑ-neighborhoods of e are X, {b, c, d, e}

Here e is a limit point of A.

The derived set of A is i.e., A' ={b, d, e}.

Example.18. Let X= {a, b, c, d, e}, $\Im$={X, ϕ, {a}, {a, b}, {a, c, d}, {a, b, c, d}, {a, b, e}},

(i) A ={ c, d, e} and (ii) B={b}. Find A' and B'.

Solution. Given thatX= {a, b, c, d, e},

$\Im$={X, ϕ, {a}, {a, b}, {a, c, d}, {a, b, c, d}, {a, b, e}}

(i) It is also given that A ={c, d, e}.

$\Im$-neighborhoods of a are X, {a}, {a, b}, {a, c, d}, {a, b, c, d}, {a, b, e}.

Here a is not limit point of A because {a} is singleton and b$\notin$A.

$\Im$-neighborhoods of b are X, {a, b}, {a, b, c, d}, {a, b, e}.

Here b is not limit point of A because a$\notin$A.

$\Im$-neighborhoods of c are X, {a, c, d}, {a, b, c, d}.

Here c is limit point of A.

$\Im$-neighborhoods of d are X, {a, c, d}, {a, b, c, d}.

Here d is limit point of A.

$\Im$-neighborhoods of e are X, {a, b, e}.

Here e is not limit point of A because a, b$\notin$A.

Hence A' ={c, d}.

(ii) It is also given that B ={b}.

Using (i) we see that a, b, c, and d are not limit point of B.

$\Im$-neighborhoods of e are X, {a, b, e}

Here e is limit point of B. Hence B' ={e}

7.13 Closed Sets

Closed sets are a fundamental concept in topology that complement the notion of open sets. A set is considered closed if it contains all its limit points. In other words, a set is closed if it includes the points it converges to.Closed sets is crucial in topology as they help define the structure of a space and its relationship with its subsets. They also play a key role in defining continuity, compactness, and other important concepts in topology.

A set *A* is said to be *closed* if every limit point of *A* belong to the set *A* itself i.e., a set *A* is said to be *closed* if $D(A) \subset A$.

Some properties of closed set

(i) Every singleton is a closed set.

(ii) The empty set ϕ is closed for $D(\phi) \subset \phi$.

(iii) The complements of open sets are closed sets.

(iv) A closed interval is always a closed set.

(v) The set A=$\left[\frac{2}{5}, \frac{1}{3}, \frac{4}{11}, \ldots\ldots\ldots\ldots, \frac{2n}{2n+1}, \ldots\ldots\ldots\right]$ is a closed set.

(vi) The set A=$\left[0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots\ldots\ldots\ldots, \frac{1}{n}, \ldots\ldots\ldots\right]$ is aclosed set.

7.14 Door Space

A door space is a topological space that has a special type of open set called a "door". In a door space, each point has a neighborhood that behaves like a closed interval in the real numbers.

Let $(X,\mathfrak{I})$ be a topological space. Then $(X,\mathfrak{I})$ is said to be *door space* if every subset of X is either open or closed.

Theorem.8. Prove that the intersection of arbitrary number of closed sets is closed and the union of finite number of closed sets is closed.

Proof. Let $\{A_i : i \in I\}$ is a closed set in X.

To show that $\cap\{A_i : i \in I\}$ is closed in X.

We have $\{X \sim A_i : i \in I\}$ is open set in X

$\Rightarrow \cup\{X \sim A_i : i \in I\}$ is open set in X

Now $X \sim \cap\{A_i : i \in I\} = \cup\{X \sim A_i : i \in I\}$ (using demorgan law)

$\Rightarrow X \sim \cap\{A_i : i \in I\}$ is open set in X

$\Rightarrow \cap\{A_i : i \in I\}$ is closed set in X.

i.e., intersection of arbitrary number of closed sets is closed.

Now we let $\{A_i : i \in J_n\}$ is a closed set in X

To show that $\cup\{A_i : i \in J_n\}$ is closed in X.

We have $\{X \sim A_i : i \in J_n\}$ is open set in X

$\cap\{X \sim A_i : i \in J_n\}$ is open set in X

$X \sim \cup\{A_i : i \in J_n\}$ is open set in X (using demorgan law)

$\cup\{A_i : i \in J_n\}$ is closed set in X.

i.e., union of finite number of closed sets is closed.

Theorem.9. Prove that in a topological space $(X,\Im)$, a subset Hof X is closed if and only if it contains the set of its limit point.

Proof. Let $(X,\Im)$ be a topological space. Given H is closed if and only if $H' \subset H$.

We will prove this theorem by contradiction method.

Let $x \in H'$ and suppose if possible $x \notin H$.

Since $x \notin H \Rightarrow x \in X \sim H$

H is closed set $\Rightarrow$ $X \sim H$ is open set.

Using definition, every open set contains neighborhood of each of its point therefore there exist a neighborhood U_x of x such that $U_x \subset X \sim H$

$\Rightarrow$ U_x contains no point of H

$\Rightarrow$ x is not limit point of H

$\Rightarrow x \in H'$

$\Rightarrow H' \subset H.$

Conversely, suppose $H' \subset H.$ To show that H is closed or X~H is open set.

Let x∈X~H $\Rightarrow$ x∉H. It is also given that $H' \subset H \Rightarrow$ x∉ H'

$\Rightarrow$ x is not limit point of H

$\Rightarrow H' \cap H = \phi$ {If A∩B=ϕ then A⊂X∩B}

$\Rightarrow$ U_x⊂X~H

Using definition, X~H is open set

$\Rightarrow$ H is open set. Hence His closed if and only if $H' \subset H.$

Theorem.10. Prove that in a topological space (X,ℑ), the union of a set A and the set of its limit points is closed i.e., A∪ A' is closed.

Proof. Let (X,ℑ) be a two topological space and A be any subset of X. To show thatA∪ A' is closed or X~(A∪ A') is open set.

Let x∉ A∪ A' *i.e.*, x∉A, x∉ A'

$\Rightarrow$ x is not limit point of A

$\Rightarrow$ there exist a neighborhood U_x of x which contains no point of A.

$\Rightarrow$ U_x∩A=ϕ and U_x∩ A' =ϕ

$\Rightarrow$ U_x∩(A∪ A')=ϕ

$\Rightarrow$ U_x⊂X~ (A∪ A')

$\Rightarrow$ X~(A$\cup$ A') is open set because U_x is open set.

$\Rightarrow$ A$\cup$ A' is closed.

Examples

Example.19. Give two examples of a proper non-empty subset of a topological space which are both open and closed.

Solution. (i) Suppose thatX= {a, b, c},

$\Im$={X, ϕ, {a}, {b, c}} and (X,$\Im$) is a topological space.

We know that every member of $\Im$- are open sets

X, ϕ, {a}, {b, c}.

And we know that complements of open sets are closed sets

ϕ, X, {b, c}, {a}.

Here X, ϕ, {a}, {b, c}are both open and closed.

Hence all proper subsets of (X,$\Im$) are open and closed.

(ii) Suppose thatX= {a, b, c}, $\Im$={X, ϕ, {a}, {b}, {c}, {a, b}, {b, c},{a, c}} and (X,$\Im$) is a topological space.

We know that every member of $\Im$- are open sets

$X, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}$.

And we know that complements of open sets are closed sets

$\phi, X, \{b, c\}, \{a, c\}, \{a, b\}, \{c\}, \{a\}, \{b\}$.

Hence each proper subsets of $(X,\Im)$ are open as well as closed.

7.7 Summary

Let X be a non-empty set and $\Im$ be a collection of subset of X. Then $\Im$ is a topology for X, if the following properties are satisfied: (i) $X \in\Im$ and $\phi\in\Im$. (ii) Let a, b $\in\Im$ then $a\cap b\in\Im$, hence it is closed under the operation of finite intersection. (iii) Let $\{A_i: i\in I\}\in\Im$ then $\cup\{A_i: i\in I\}\in\Im$, hence it is closed under the operation of arbitrary union.

The members of $\Im$ are called open sets of the topology $\Im$ and the pair $(X,\Im)$ is called a topological space.There are two types of trivial topology: (i) Indiscrete Topology (ii) Discrete Topology. Let X be a non empty set. Then the collection $\Im =\{X, \phi\}$, (consisting of only empty set and the whole space) is always a topology for X, is called the *indiscrete topology*. The pair $(X,\Im)$ is called an *indiscrete topological space*.

Let X be a non empty set. Then the collection $\Im$ = {consisting of all subsets of X} is always a topology for X, called the *discrete topology*. The pair $(X,\Im)$ is called *discrete topological space*. Topologies defined on X other than trivial topology (Indiscrete and Discrete topology) are known as *non-trivial topologies*.

Let X be a non-empty set and $\Im_1$, $\Im_2$ are two topologies on X. Then either $\Im_1\subset\Im_2$ and $\Im_2\subset\Im_1$, the topologies $\Im_1$ and $\Im_2$ are *comparable*. If $\Im_1\not\subset\Im_2$ and $\Im_2\not\subset\Im_1$ then the topologies $\Im_1$and $\Im_2$ are not comparable.Let X be a non empty set and $\Im$ be a topology on X. Then every member of $\Im$ is called *open set*.Let (X,$\Im$) be a topological space. A subset N of X is said to be a neighborhood of a point x if it contains an open set G to with the point $x\in G$ such that $x\in G\subset N$.

Let R be a set of real numbers and ***U***be the collection of subset of R and if ***U*** satisfies all the properties of topology then ***U*** is a topology on R and pair (R, ***U***) is called *usual topological space or standard topological space.*

Let (X,$\Im$) be a topological space and A be a subset of X. Then a point $x\in X$ is a limit point of A if each neighborhood of x contains at least one point of A other than x.A collection of the limit points of a set A is called the *derived set of A* denoted by A' or D (A).

A set A is said to be *closed* if every limiting point of A belong to the set A itself i.e., A set A is said to be *closed* if $D(A)\subset A$.Let (X,$\Im$) be a topological space. Then (X,$\Im$) is said to be *door space* if every subset of X is either open or closed.

7.8 Terminal Questions

Q.1. Define the topological space

Q.2. What do yo mean by trivial and non-trivial topologies ?

Q.3. Find four mutually non-comparable topologies for X= {a, b, c, d}.

Q.4. Write all the possible topologies for X= {a, b}.

Q.5. Prove that the finite intersection of two topologies on X is again a topology on X.

Q.6. Prove that the intersection of any number of topologies on X is again a topology on X.

Q.7. Let X= {a, b, c, d, e} and $\Im$={X, ϕ, {a}, {a, b}, {a, b, e}, {a, c, d}, {a, b, c, d}}. Find $\Im$-

neighborhood of (i) a (ii) b (iii) c (iv) d and (v) e.

Q.8. Let X= {a, b, c, d, e} and $\Im$={X, ϕ, {a}, {c, d}, {a, c, d}, {b, c, d, e}}. Show that $\Im$ is a topology for X and find $\Im$-closed subset of X.

Q.9. Let X= {a, b, c} and $\Im$={X, ϕ, {a}, {b}, {a, b}}. Show that $\Im$ is a topology for X. Find all the limit point of {a, b}.

Q.10. Let X= {a, b, c, d, e} and $\Im$={X, ϕ, {a}, {a, b}, {c, d}, {a, b, c, d}}. Find the derived set of each of the following sets: (i) {a, b} (ii) {b, d} (iii) {a, b, c} and (iv) {b, c, d}.

Answers

Q.3. $\Im_1$={X, ϕ, {a}}, $\Im_2$={X, ϕ, {b}}, $\Im_3$={X, ϕ, {c}}, $\Im_4$={X, ϕ, {d}},

Q.4. $\Im_1$={X, ϕ}, $\Im_2$={X, ϕ, {a}}, $\Im_3$={X, ϕ, {b}}, $\Im_4$={X, ϕ, {a}, {b}},

Q.8. ϕ, X, {b, c, d, e}, {a, b, e}, {b, e}, {a}.

Q.9. $A' = \{c\}$

Q.10.(i) {b, e} (ii) {c, e} (iii) {b, e} (iv) {c, d, e}.

UNIT-8: Topological Spaces-II

Structure

8.1 Introduction

In topology, the closure of a set is a fundamental concept that describes the behavior of points near the set. The closure of a set can also be defined in terms of the interior of the complement of the set. The closure of a set helps in characterizing the behavior of sequences and functions in topology, and it plays a crucial role in defining many other concepts, such as closed sets, limit points, and continuity of functions. Topology provides a framework for studying geometric properties of spaces, such as shape, size, and dimension, without being restricted to specific metrics or coordinate systems. It helps in understanding and formalizing concepts related to continuity, convergence, and connectedness, which are fundamental in analysis and calculus.

Many concepts in network theory and graph theory are closely related to topology, making it a valuable tool in understanding the structure of complex networks. The importance of topology lies in its ability to provide a unified framework for understanding the structure of spaces, making it a fundamental area of study with diverse applications across various disciplines.

8.2 Objectives

After reading this unit the learner should be able to understand about the:

- Closure of a set and properties of closure of a set
- Separated Set
- Interior points and the Interior of a Set and their properties
- Exterior of a Set
- Boundary Points or Frontier Points
- Dense Set

8.3 Closure of a Set

The concept of closure is fundamental in topology as it helps define closed sets and provides a way to understand the behavior of sets and points in a topological space.

Let $(X,\Im)$ be a topological space and A be a subsets of X. Then the closure of A, denoted by $\overline{A}$, is defined as

$$\overline{A} = \cap\{\text{all } \Im\text{-closed subsets of X containing A}\}$$

or c(A) is the smallest $\Im$-closed set of X that contains A.

Some properties of closure of a set

(i) A set A is closed set if and only if $\overline{A}=A$.

(ii) $\overline{A}$ is the smallest closed set containing A.

(iii) $\overline{\phi}=\phi$.

(iv) $\overline{A}=A\cup D(A)$.

(v) $\overline{A\cup B}=\overline{A}\cup\overline{B}$.

8.4 Separated Set

Let $(X,\Im)$ be a topological space and A, B be any non-empty subsets of X. Then A and B are said to be *separated set* if

(i) $A\cap\overline{B}=\phi$ (ii) $\overline{A}\cap B=\phi$.

Theorem.1. Let (X,$\Im$) be a topological space and A, B$\subset$X. Then (i) $\overline{\phi}=\phi$ **(ii) A$\subset$**$\overline{A}$ **(iii) A$\subset$B$\Rightarrow$**$\overline{A}\subset\overline{B}$ **(iv) A$\subset$B$\Rightarrow$** $A'\subset B'$.

Proof. Given that (X,$\Im$) is a topological space and A, B$\subset$X.

(i)Since ϕ is closed $\Rightarrow\overline{\phi}=\phi$.

(ii)Using definition of closure, $\overline{A}$ is the smallest closed set containing A. So $A\subset\overline{A}$.

(iii) Let A$\subset$B then

A$\subset$B$\subset\overline{B}$ i.e., A$\subset\overline{B}$ {using (ii)}

Since closure of any set is closed and so $\overline{B}$ is closed set containing A. Also $\overline{A}$ is the smallest closed set containing A. Consequently $\overline{A}\subset\overline{B}$.

(iv)Let x$\notin$ A'

$\Rightarrow$ x is a limit point of A

We have A$\subset$B

$\Rightarrow$ x is a limit point of B

Then x$\in$ $B'\Rightarrow A'\subset B'$.

Theorem.2. Let (X,$\Im$) be a topological space and A, B$\subset$ X. Then

(i) $\overline{A\cap B}\subset\overline{A}\cup\overline{B}$ **(ii)** $\overline{A\cap B}=\overline{A}\cup\overline{B}$.

Proof. Given that (X,$\Im$) is a topological space and A, B$\subset$X.

(i) We know that A$\cap$B$\subset$A, A$\cap$B$\subset$B

$\Rightarrow \overline{A \cap B} \subset \overline{A}, \quad \overline{A \cap B} \subset \overline{B}$

$\Rightarrow \overline{A \cap B} \subset \overline{A} \cup \overline{B}.$

(ii) We know that $A \subset A \cup B$, $B \subset A \cup B$

$\Rightarrow \overline{A} \subset \overline{A \cup B}, \quad \overline{B} \subset \overline{A \cup B}$

$\Rightarrow \overline{A} \cup \overline{B} \subset \overline{A \cup B}$ …(1)

We have $A \subset \overline{A}$, $B \subset \overline{B}$

$\Rightarrow A \cup B \subset \overline{A} \cup \overline{B}$

Since $\overline{A}$, $\overline{B}$ are closed

$\Rightarrow \overline{A} \cup \overline{B}$ is closed.

Now let $\overline{A} \cup \overline{B}$ is a closed set containing $A \cup B$. Also $\overline{A \cup B}$ is the smallest closed set containing $A \cup B$, Consequently

$\overline{A \cup B} \subset \overline{A} \cup \overline{B}$ …(2)

Hence from (1) and (2), we get $\overline{A \cup B} = \overline{A} \cup \overline{B}.$

Examples

Example.1. Let X= {a, b, c, d}, $\Im$={X, ϕ, {a}, {b, c}, {a, d}, {a, b, c}}. Find the closure of

(i) A ={b}, (ii) B={a, b} and (iii) C={b, c, d}.

Solution. Given that$X = \{a, b, c, d\}$,

$\mathfrak{I} = \{X, \phi, \{a\}, \{b, c\}, \{a, d\}, \{a, b, c\}\}$.

$\mathfrak{I}$-open sets are $X, \phi, \{a\}, \{b, c\}, \{a, d\}, \{a, b, c\}$.

$\mathfrak{I}$-closed sets are $\phi, X, \{b, c, d\}, \{a, d\}, \{b, c\}, \{d\}$.

(i) It is also given that $A = \{b\}$.

The closure of A is

$$\overline{A} = \cap\{\text{all } \mathfrak{I}\text{-closed subsets of X containing A}\}$$

$$= \cap\{X, \{b, c, d\}, \{b, c\}\}$$

$$= \{b, c\}.$$

(ii) It is also given that $B = \{a, b\}$.

The closure of B is

$$\overline{B} = \cap\{\text{all } \mathfrak{I}\text{-closed subsets of X containing B}\}$$

$$= \cap\{X\}$$

$$= X.$$

(iii) It is also given that $C = \{b, c, d\}$.

The closure of C is

$$\overline{C} = \cap\{\text{all } \mathfrak{I}\text{-closed subsets of X containing C}\}$$

$$= \cap\{X, \{b, c, d\}\}$$

$$= \{b, c, d\}.$$

Example.2. Let X= {a, b, c, d, e} and $\Im$={X, ϕ, {a}, {c, d}, {a, c, d}, {a, c, d, e}}. Find the closure of (i) A = {b, c}, (ii) B= {a, c}, (iii) C={a, b, c} and (iv) D={d}.

Solution. Given thatX= {a, b, c, d, e}

and $\Im$={X, ϕ, {a}, {c, d}, {a, c, d}, {a, c, d, e}}.

$\Im$-closed sets are ϕ, X, {b, c, d, e}, {a, b, e}, {b, e} {b}.

(i) It is also given that A = {b, c}.

$\overline{A}$ = $\cap${all $\Im$-closed subsets of X containing A}

= $\cap${X, {b, c, d, e}

= {b, c, d, e}.

(ii) It is also given that B = {a, c}.

$\overline{B}$ = $\cap${all $\Im$-closed subsets of X containing B}

= $\cap${X}

= X.

(iii) It is also given that C = {a, b, c}.

$\overline{C}$ = $\cap${all $\Im$-closed subsets of X containing C}

= $\cap${X}

= X.

(iv) It is also given that D = {d}.

$\overline{D}$ = $\cap${all $\Im$-closed subsets of X containing D}

$= \cap\{X, \{b, c, d, e\}$

$= \{b, c, d, e\}$.

8.5 Interior Points and Interior of a Set

Interior points of a set play a crucial role in topology, particularly in defining the openness of sets and understanding the structure of topological spaces. An interior point of a set is a point that has an open neighborhood entirely contained within the set.

Let $(X,\Im)$ be a topological space and A be any subset of X. A point $x\in A$ is said to be an *interior point* of A if and only if A is a neighborhood of *x*. The collection of all interior point of a set is called the*interior of A* and denoted by A^0 or int(A) and defined as

$$A^0 = \cup\{\text{all } \Im\text{-open subset of X contained in A}\}$$

Some properties of interior of a set

(i) A set A is open set if and only if $A^0=A$.

(ii) A^0 is the largest open set contained in A.

(iii) $\phi^0 = \phi$.

(iv) $X^0=X$.

(v) $A\subset B\Rightarrow A^0\subset B^0$.

(vi) $A^0\cup B^0\subset(A\cup B)^0$.

(vii) $(A\cap B)^0=A^0\cap B^0$.

(viii) $\overline{X - A}=X-A^0$.

Theorem.3. Let $(X, \Im)$ be a topological space and $A\subset X$. Then show that A is open if and only if $A^0 = A$.

Proof. Given that $(X,\Im)$ is a topological space and $A\subset X$.

Suppose that A is open, therefore A^0 is itself largest open subset of A.

$\Rightarrow A^0=A$.

Conversely, let $A^0=A$. To show that A is open.

Because A^0 is open therefore A is open (by definition).

Hence $A^0=A$.

Theorem.4. Let $(X,\Im)$ be a topological space and A, $B\subset X$. Then (i) $A\subset B\Rightarrow A^0 \subset B^0$ (ii)$(A\cap B)^0=A^0\cap B^0$ (iii)$(A^0\cup B^0)\subset(A\cup B)^0$ and (iv) $(A\cup B)^0\neq A^0\cup B^0$.

Proof. Given that $(X,\Im)$ is a topological space and A, $B\subset X$.

(i) Let $x\in A^0\Rightarrow$ A is a neighborhood of x (by definition)

Given $A\subset B\Rightarrow$ B is a neighborhood of x

$\Rightarrow$ x is interior point of B

$\Rightarrow x\in B^0$

Hence $A\subset B\Rightarrow A^0 \subset B^0$.

(ii) We have $A\cap B\subset A$

$\Rightarrow (A\cap B)^0\subset A^0$

Similarly, $(A\cap B)^0\subset B^0$

$\Rightarrow(A\cap B)^0\subset A^0\cap B^0$ …(1)

Now to show that $\Rightarrow A^0\cap B^0\subset(A\cap B)^0$

$A\supset A^0$ (A contained A^0)

$B \supset B^0$ (B contained B^0)

$\Rightarrow A \cap B \supset A^0 \cap B^0$

$\Rightarrow A^0 \cap B^0 \subset A \cap B$

$\Rightarrow (A^0 \cap B^0)^0 \subset (A \cap B)^0$

$\Rightarrow (A^0 \cap B^0) \subset (A \cap B)^0$ {because $A^0 \subset A$} ...(2)

(iii) We have $A \subset A \cup B$

$\Rightarrow A^0 \subset (A \cup B)^0$ (by definition)

Similarly, $B \subset A \cup B$

$\Rightarrow B^0 \subset (A \cup B)^0$ (by definition)

$\Rightarrow A^0 \cup B^0 \subset (A \cup B)^0$

(iv) using (iii), we have $A^0 \cup B^0 \subset (A \cup B)^0$

But here to show that $(A \cup B)^0 \not\subset A^0 \cup B^0$

We proof this using an example.

Let A =[0,1[, and B=[1, 2[

Then A^0=]0, 1[, B^0=]1, 2[

$(A \cup B)^0$=]0, 2[….(3)

$A \cup B$=[0, 2[

$A^0 \cup B^0$=]0, 1[$\cup$]1, 2[=]0, 2[-{1} …(4)

Using (3) and (4), we show that

$(A \cup B)^0 \neq A^0 \cup B^0$.

8.6 Exterior of a Set

The exterior of a set in topology is the set of all points in the topological space that do not belong to the closure of the given set.

Let $(X,\Im)$ be a topological space and A be any subset of X. The set of all those point of A which are interior to X-A is called the *exterior of A* and denoted by ext(A) or $(X-A)^0$ and defined as

$$\text{ext}(A) = (X-A)^0 \text{ or } X-\overline{A}$$

i.e., exterior of A is the complement of closure of A.

8.7 Boundary Points

Boundary points are points that are neither entirely in the interior of a set nor entirely in the exterior of the set. In other words, a point is a boundary point of a set if every neighborhood of the point contains points both inside and outside the set.

Let $(X,\Im)$ be a topological space and A be any subset of X. The boundary or frontier of the set A in $(X,\Im)$ is the set of all those points which do not belong to the interior or exterior of A and is denoted by b(A) or $F_r(A)$ and defined as

$$b(A)' = [A^0 \cup (A')^0] \text{ or } \overline{A} \cap A' - A^0 \text{ and } \overline{A} = A^0 \cup b(A).$$

8.8 Dense Set

Dense sets are important in topology because they provide a way to approximate any point in a space using points from the dense set. For example, the set of rational numbers is dense in the real numbers, as every interval contains rational numbers. Similarly, the set of real numbers is dense in the complex numbers.

Let $(X,\Im)$ be a topological space and A be any subset of X. If $\overline{A}$= X, then A is said to be dense in X.

Examples

Example.3. Let X= {a, b, c, d, e} and $\Im$={X, ϕ, {a}, {a, b}, {a, c, d}, {a, b, c, d}, {a, b, e}} and A={c, d, e}. Find the (i) closed set, (ii) closure, (iii) interior and (iv) exterior of A.

Solution. Given thatX= {a, b, c, d, e}

and $\Im$={X, ϕ, {a}, {a, b}, {a, c, d}, {a, b, c, d}, {a, b, e}}.

(i) We know that every member of $\Im$ is a open set.

X, ϕ, {a}, {a, b}, {a, c, d}, {a, b, c, d}, {a, b, e}

$\Im$-closed sets are ϕ, X, {b, c, d, e}, {c, d, e}, {b, e}, {e}, {c, d}.

(ii)The closure of A is

$\overline{A}$=$\cap${all $\Im$-closed subsets of X containing A}

= $\cap${X, {b, c, d, e}, {c, d, e}}

= {c, d, e}.

(iii)The interior of A is

A^0= $\cup${all $\Im$-open subsets of X contained in A}

= $\cup${ϕ}

$= \phi$.

(iv)The exterior of A is

ext(A) = $\cup$\{all $\Im$-open subsets of X contained in X-A\}

= $\cup\{\phi, \{a\}, \{a, b\}\}$

= \{a, b\}.

Example.4. Let X= {a, b, c} and $\Im$={X, ϕ, {a}, {b}, {a, b}}. Find the limit point, closure, interior, exterior and boundary points of the following subsets of X:

(i) A= {a, c} (ii) B= {b, c}.

Solution. Given thatX= {a, b, c}and $\Im$={X, ϕ, {a}, {b}, {a, b}}.

$\Im$-closed sets are ϕ, X, {b, c}, {a, c}, {c}.

(i) It is also given that A = {a, c}.

$\Im$-neighborhoods of a are X, {a}, {a, b}.

Here a is not limit point of A because {a} is singleton and b$\notin$A.

$\Im$-neighborhoods of b are X, {b}, {a, b}.

Here b is not limit point of A because {b} is singleton.

$\Im$-neighborhood of c is X.

Here c is limit point of A

The closure of A is

$$\overline{A} = \cap\{\text{all } \mathfrak{I}\text{-closed subsets of X containing A}\}$$

$$= \cap\{X, \{a, c\}\}$$

$$= \{a, c\}.$$

The interior of A is

$$A^0 = \cup\{\text{all } \mathfrak{I}\text{-open subsets of X contained in A}\}$$

$$= \cup\{\phi, \{a\}\}$$

$$=\{a\}$$

The exterior of A is

$$\text{ext}(A) = \cup\{\text{all } \mathfrak{I}\text{-open subsets of X contained in X-A}\}$$

$$= \cup\{\phi, \{b\}\}$$

$$=\{b\}$$

The boundary of A is

b(A) = {The set of all those elements of A which neither belong to A^0 nor to ext(A) i.e., {c}}

$$= \{c\}.$$

(i) It is also given that B = {b, c}.

$\mathfrak{I}$-neighborhoods of a are X, {a}, {a, b}.

Here a is not limit point of A because {a} is singleton.

$\mathfrak{I}$-neighborhoods of b are X, {b}, {a, b}.

Here b is not limit point of A because {b} is singleton.

$\mathfrak{I}$-neighborhoods of c is X.

Here c is limit point of B.

The closure of B is

$\overline{B} = \cap$ {all $\mathfrak{I}$-closed subsets of X containing B}

$= \cap\{X, \{b, c\}\}$

$= \{b, c\}$.

The interior of B is

$B^0 = \cup$ {all $\mathfrak{I}$-open subsets of X contained in B}

$= \cup\{\phi, \{b\}\}$

$= \{b\}$

The exterior of B is

$\text{ext}(B) = \cup\{\text{all } \mathfrak{I}\text{-open subsets of X contained in X-B}\}$

$= \cup\{\phi, \{a\}\}$

$= \{a\}$

The boundary of B is

b(B) = {The set of all those elements of A which neither belong to B^0 nor to

ext(B) i.e., {c}}

= {c}.

Example.5. Consider the following topology on X= {a, b, c, d, e} and

$\mathfrak{I}$={X, ϕ, {a}, {a, b}, {a, c, d}, {a, b, c, d}, {a, b, e}}.

Find (i) Determine the closure of the sets {a}, {b} and {c, e}.

(ii) Which set in (i) are dense in X.

Solution. Given thatX= {a, b, c, d, e}

and $\mathfrak{I}$={X, ϕ, {a}, {a, b}, {a, c, d}, {a, b, c, d}, {a, b, e}}.

$\mathfrak{I}$-closed sets are ϕ, X, {b, c, d, e}, {c, d, e}, {b, e}, {e}, {c, d}.

(i) The closure of {a}, denoted by $\overline{A_1}$

$$\overline{A_1} = \cap\{\text{all } \mathfrak{I}\text{-closed subsets of X containing } \{a\}\}$$

$= \cap\{X\}$

$= X.$

Hence $\overline{A_1} = X.$

The closure of {b}, denoted by $\overline{A_2}$

$\overline{A_2} = \cap\{$all $\Im$-closed subsets of X containing $\{b\}\}$

$= \cap\{X, \{b, c, d, e\}, \{b, e\}\}$

$= \{b, e\}.$

Hence $\overline{A_2} = \{b, e\}.$

The closure of {c, e}, denoted by $\overline{A_3}$

$\overline{A_3} = \cap\{$all $\Im$-closed subsets of X containing $\{c, e\}\}$

$= \cap\{X, \{b, c, d, e\}, \{c, d, e\}\}$

$= \{c, d, e\}.$

Hence $\overline{A_3} = \{c, d, e\}.$

(ii) We know that any set A is dense in X if $\overline{A} = X$.

Here $A_1 = \{a\}$ in the given dense set in X because $\overline{A_1} = X$.

8.9 Summary

Let $(X,\Im)$ be a topological space and A be any subset of X. Then closure of A, denoted by $\bar{A}$, defined as $\bar{A} = \cap$\{all $\Im$-closed subset of X containing A\}.

Let $(X,\Im)$ be a topological space and A, B be any non-empty subsets of X. Then A and B are said to be *separated set* if (i) $A \cap \bar{B} = \phi$ (ii) $\bar{A} \cap B = \phi$.

Let $(X,\Im)$ be a topological space and A be any subset of X. A point $x \in A$ is said to be an *interior point* of A if and only if A is a neighborhood of *x*. The collection of all interior point of the set A is called the *interior of A* and denoted by A^0 or int(A) and defined as

$$A^0 = \cup\{\text{all } \Im\text{-open subsets of X contained in A}\}$$

Let $(X,\Im)$ be a topological space and A be any subset of X. The set of all those points of A which are interior to X-A is called the *exterior of A* and denoted by ext(A) or $(X-A)^0$ and defined as ext(A) $=(X-A)^0$ or $X-\bar{A}$.

Let $(X,\Im)$ be a topological space and A be any subset of X. The boundary or frontier of a set in $(X,\Im)$ is the set of all those points which do not belong to the interior or exterior of A and is denoted by b(A) or $F_r(A)$ and defined as

$$b(A)' = [A^0 \cup (A')^0] \text{ or } \bar{A} \cap A' - A^0 \text{ and } \bar{A} = A^0 \cup b(A).$$

Let $(X,\Im)$ be a topological space and A be any subset of X. If $\bar{A} = X$, then A is said to be dense in X.

8.10 Terminal Questions

Q.1. Explain the closure of a set with their properties.

Q.2. Define the Interior points and the interior of a set.

Q.3. What do you mean by Exterior of a set and Boundary points?

Q.4. To show that $\overline{A \cap B} \neq \overline{A} \cap \overline{B}.$

Q.5. Let (X, $\Im$) be a topological space and let $A \subset X$. Then (i) If A is open, then b (A)=$\overline{A} = A.$

(ii) b(A) =ϕ if and only if A is open as well as closed.

(iii) A is closed if and only if $b(A) \subset A$.

(iv) A is open if and only if $A \cap b(A) = \phi$, i.e., if and only if $b(A) \subset A'.$

Q.6. Let (X, $\Im$) be a topological space and let A, $B \subset X$. To show that $ext(A) \supseteq ext(B)$.

Q.7. Let X= {a, b, c, d, e}, $\Im$={X, ϕ, {a}, {a, b}, {a, c, d}, {a, b, c, d}, {a, b, e}} and A={a, b, c}. Find (i) Closed set, (ii) Closure, (iii) Interior (iv) Exterior of A and (v) Boundary point.

Answer

7. (i) ϕ, X, {b, c, d, e}, {c, d, e}, {b, e}, {e}, {c, d}. (ii) X

(iii) {a, b} (iv) ϕ (v) {c}.

UNIT-9: Base ansd Sub-base

Structure

9.1 Introduction

Bases and subbases play a crucial role in defining the topology of a space and understanding its properties. In topology, a base (or basis) and a subbase are fundamental concepts used to define the topology of a space. Both bases and subbases provide a way to generate a topology on a set by specifying a smaller collection of sets that behave nicely with respect to unions and intersections. By specifying a base or subbase, one can generate the open sets of the topology without needing to explicitly list all open sets, which can be cumbersome for more complex spaces. In analysis and geometry, bases and subbases are used to define topologies on spaces such as the real numbers or metric spaces. They provide a way to study the properties of these spaces using concepts from general topology. Hence the bases and subbases are foundational concepts in topology that help define, characterize, and study the properties of topological spaces in a concise and systematic manner.

9.2 Objectives

After reading this unit the learner should be able to understand about the:

- Relative Topology and their applications
- Subspace and Base for a topology,
- Sub-bases and Local base,
- First Countable Space and Second Countable Space,
- Topologies Generated by Classes of Sets,
- Separable Space, Cover of a Space and Lindelof Space

9.3 Relative Topology

Relative topology is important in various areas of mathematics, including topology, analysis, and geometry. Relative topology allows for the study of properties of subsets of a space without considering the entire space. Hence the relative topology provides a framework for studying subsets of spaces in relation to the larger space's topology, enabling deeper insights into the structure and properties of spaces in various mathematical contexts.Let $(X,\Im)$ be a topological space and Y be a subset of X. The family v of all subsets which are intersection of Y with member of $\Im$ i.e.,

$$v=\{Y\cap \text{every member of } \Im\}$$

or

$$v=\{Y\cap A : A\in \Im\}$$

is called relative topology on Y.

9.4 Subspace

Subspaces are fundamental objects in mathematics with a wide range of applications in various areas of mathematics and its applications.Let $(X,\Im)$ be a topological space. If v is a relative topology for a subset Y of X then the corresponding topological space (Y,v) is called a subspace of $(X,\Im)$.

Examples

Example.1: Let $X=\{a,b\}$ and $\Im=\{X,\phi,\{a\},\{b\}\}$ be a topology for X. Find the relative topology of $A\subset X$, where $A=\{a\}$.

Solution: Given that $X = \{a,b\}, \Im = \{X, \phi, \{b\}\}$

And $A = \{a\}$.

We have $X \cap A = A; \phi \cap A = \phi; \{a\} \cap A = \{a\}$ and $\{b\} \cap A = \phi$

Hence, the relative topology v for A is

$$v = \{A, \phi\}.$$

Example 2: Let $X = \{1,2,3,4,5\}$ and $\Im\{X, \phi, \{1\}, \{3,4\}, \{1,3,4\}, \{2,3,4,5\}\}$ to be a topology for X. Let $A = \{1,4,5\}$ be a subset of X. Find the relative topology and subspace for A.

Solution: Given that $X = \{1,2,3,4,5\}$

$$\Im = \{X, \phi, \{1\}, \{3,4\}, \{1,3,4\}, \{2,3,4,5\}\}$$

And $A = \{1,4,5\}$

We have $X \cap A = A;$

$\phi \cap A = \phi;$

$\{1\} \cap A = \{1\};$

$\{3,4\} \cap A = \{4\};$

$\{1,3,4\} \cap A = \{1,4\}$

And $\{2,3,4,5\} \cap A = \{4,5\}$

Hence, the relative topology v for A is

$$v = \{A, \phi, \{1\}, \{4\}, \{1,4\}, \{4,5\}\}$$

And (A, v) is a subspace of $(X, \Im)$.

Example 3: Let $X = [a,b,c,d,e]$ **and** $\Im = \{X, \phi, \{a\}, \{a,b\}, \{a,c,b\}, \{a,b,c,d\}, \{a,b,e\}\}$ **be a topology for** X**.Let** $A = \{a,c,e\}$ **be a subset of** X **. Find the relative topology for** A **.**

Solution: Given that $X = \{a,b,c,d,e\}$,

$$\Im = \{X, \phi, \{a\}, \{a,b\}, \{a,c,d\}, \{a,b,c,d\}, \{a,b,e\}\}$$

and $\qquad A = \{a,c,e\}$

We have $\qquad X \cap A = A;$

$$\phi \cap A = \phi;$$

$$\{a\} \cap A = \{a\};$$

$$\{a,b\} \cap A = \{a\};$$

$$\{a,c,d\} \cap A = \{a,c\};$$

$$\{a,b,c,d\} \cap A = \{a,c\};$$

And $\qquad \{a,b,e\} \cap A = \{a,e\};$

Hence, the relavtive topology v for A is

$$v = \{A, \phi, \{a\}, \{a,c\}, \{a,e\}\}$$

Theorem 1: To show that relative topology v **is a topology for** Y **.**

Proof: let $(X, \Im)$ be a topological space and $Y \subset X$. Let v be a relative topology for Y of X.

(1) Since $Y \cap X = Y$, therefore $Y \in v$.

and $Y \cap \phi = \phi$, therefore $\phi \in v$

(2) Let $A, B \in v$ $\left[v = \{Y \cap \text{every member of } \Im\}\right]$

To show that $A \cap B \in v$

Let $A = Y \cap C; C \in \Im$

$B = Y \cap D; D \in \Im$

We have $AB = (Y \cap C) \cap (Y \cap D)$

$= Y \cap (C \cap D)$ [since $C, D, \in \Im$]

$\Rightarrow Y \cap$ every member of $\Im$

$\Rightarrow (A \cap B) \in v$

Hence , it is closed under the operation of finite intersection.

(3) Let $\{A_i : i \in I\} \in v$

To show that $U\{A_i : i \in I\} \in v$

We have

$A_i = Y \cap U_i; U_i \in \Im$

$\Rightarrow U\{A_i : i \in I\} = U\{Y \cap U_i : i \in I\}$

$Y \cap \{\cup U_i : i \in I\}$

Since $U_i \in \Im$ and $\Im$ is a topology on X

$$\Rightarrow \cup U_i \in \Im$$

$$\Rightarrow [Y \cap \{\cup U_i : i \in I\} \in v$$

Therefore, $\cup\{A_i : i \in I\} \in v$

i.e., it is closed under the operation of arbitrary union. Hence, (Y, v) is a subspace of $(X, \Im)$, which is called relative topology.

Theorem 2: Let (Y, v) be a subspace of $(X, \Im)$ and (Z, ω) be a subspace of (Y, v). Then (Z, ω) is a subspace of $(X, \Im)$.

Proof: Given that (Y, v) be a subspace of $(X, \Im)$

$$\Rightarrow \quad Y \subset X \qquad \ldots(1)$$

And (Z, ω) is a subspace of (Y, v)

$$\Rightarrow \quad X \subset Y \qquad \ldots(2)$$

From (1) and (2), we have

$$Z \subset X .$$

Now to show that (Z, ω) is a subspace of $(X, \Im)$ if and only if $\omega = \Im_z$. $\{\Im_z$ is topology for Z$\}$

Let $G \in \omega$, since (Z, ω) is a subspace of (Y, v) there exist $H \in v$ such that

$$G = H \cap Z$$

Also (Y, v) is a subspace of $(X, \Im)$ there exists $A \in \Im$ such that

$$H = A \cap Y$$

Hence, $G = H \cap Z$

$$= (A \cap Y) \cap Z$$

$$= A \cap (Y \cap Z)$$

$$= A \cap Z \qquad [\because Z \subset Y]$$

Using definition of topology, we have

$$\omega \in \Im_z$$

$$\therefore \quad \omega \subset \Im_z \qquad \ldots(3)$$

Conversely, let $N \in \Im_z$ then by the definition of $\Im_z$ thereexist $A \in \Im$ such that

$$N = A \cap Z$$

Since (Y, v) is a subspace of $(X, \Im)$, we have

$$A \cap Y \in v$$

Also (Z, ω) is a subspace of (Y, v) we have

$$(A \cap Y) \cap Z \in \omega$$

$$\Rightarrow A \cap Z \in \omega$$

But $N = A \cap Z \Rightarrow N \in \omega$

$$\Rightarrow \Im_z \subset \omega \qquad \ldots(4)$$

From equations (3) and (4), we have $\Im_z \subset \omega$.

9.5 Base for a Topology

The concept of a base for a topology is important in topology and has several applications. Bases can simplify the study of topological properties. Instead of considering all open sets in a topology, one can often work with a base and still deduce many properties of the space. This can lead to more efficient proofs and calculations.

Let $(X, \Im)$ be any topological space. Let B be a collection of subsets of X such that:

(i) $B \subseteq \Im$ or subclass of $\Im$

(ii) For each $x \in X$ and for all neighborhoods N of x there exist a member $B \in \beta$ such that

$$x \in B \subseteq N.$$

Then B is called a base for topology $\Im$.

or

Note: Let $(X, \Im)$ be any topological space. Then B is said to be a base for the topology $\Im$ on X, if for $x \in G \in \Im$ this implies thereexist $B \in \beta$ such that $x \in B \subset G$.

Examples

Example.4: Let $X = \{a,b,c\}$ and $\Im = \{X, \phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}\}$ be a topology for X. If $\beta = \{\{a\}, \{b\}, \{c\}\}$ then show that β is a base for $\Im$.

Solution: Given that $X = \{a,b,c\}$.

$$\Im = \{X, \phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}\}$$

And $\beta = \{\{a\}, \{b\}, \{c\}\}$

To show that β is a base for topology $\Im$.

(1) $\beta \subseteq \Im$

(2) First we find neighbourhood of a, b and c. We have

$$\aleph_a = \{\{a\},\{a,b\},\{a,c\},X\}$$

and $$\aleph_b = \{\{b\},\{a,b\},\{b,c\},X\}$$

$\aleph_c = \{\{c\},\{a,c\},\{b,c\},X\}$

Here $a \in \{a\} \subseteq X$ or $a \in \{a\} \subseteq \{a,b\}$ or $a \in \{a\} \subseteq \{a,c\}$

$b \in \{a\} \subseteq X$ or $b \in \{b\} \subseteq \{a,b\}$ or $b \in \{b\} \subseteq \{b,c\}$

$c \in \{c\} \subseteq X$ or $c \in \{c\} \subseteq \{a,c\}$ or $c \in \{c\} \subseteq \{b,c\}$

Hence, for all $x \in X$ there exist a member $\{x\}$ in β such that

$$x \in \{x\} \subseteq N \quad \forall \text{neighbourhood } N \text{ of } X.$$

Therefore, β is a base for topology $\Im$.

Example.5: Let $X = \{1,2,3,4\}$ and $\Im = \{X,\phi,\{1\},\{2\},\{1,.2\},\{3,4\},\{1,3,4\},\{2,3,4\}\}$ be a topology for X. Show that $\beta = \{\{1\},\{2\},\{3,4\}\}$ is a base for $\Im$.

Solution: Given that $X = \{1,2,3,4\}$

$$\Im = \{X,\phi,\{1\},\{2\},\{1,2\},\{3,4\},\{1,3,4\},\{2,3,4\}\}$$

And $$\beta = \{\{1\},\{2\},\{3,4\}\}$$

To show that β is a base for topology $\Im$.

(1) $\beta \subseteq \Im$

(2) First we find neighbourhood of $1, 2, 3$ and 4, we have

$\aleph_1 = \{\{1\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, X\}$

$\Rightarrow 1 \in \{1\} \subseteq$ each neighbourhood of 1

i.e., here $\{1\} \in \beta$

$\aleph_2 = \{\{1\}, \{1,2\}, \{2,3\}, \{2,4\}, \{2,3,4\}, \{1,2,3\}, \{1,2,4\}, X\}$

$2 \in \{2\} \subseteq$ each neighbourhood of 2

i.e., here $\{2\} \in \beta$

to find neighbourhood of 3, we have

$\Im -$ open set containing 3 are $X, \{3,4\}, \{1,3,4\}, \{2,3,4\}$

Superset of X is X.

Superset of $\{3,4\}$ is $\{3,4\}, \{1,3,4\}, \{2,3,4\}$.

Superset of $\{1,3,4\}$ is $\{1,3,4\}\ X$.

Superset of $\{2,3,4\}$ is $\{2,3,4\}, X$.

$= \aleph_3\ 4\}, \{1,3,4\}, \{2,3,4\}, X\}$

$3 \in \{3,4\} \subseteq$ each neighbourhood of 3

i.e., here $\{3,4\} \in \beta$

Similarly, $= \{\{\aleph_4\}, \{1,3,4\}, \{2,3,4\}, X\}$

$\Rightarrow \qquad 4 \in \{3,4\} \subseteq$ each neighbourhood of 3

i.e., here $\{3,4\} \in \beta$

Hence for all $x \in X$ there exist a member $\{x\}$ or $\{x, y\}$ is β such that

$$x \in \{x\} \text{ or } \{x, y\} \subseteq N \forall \text{ neighourhood } N \text{ of } X$$

Therefore β is a base for topology $\Im$

Theorem 3: Let $(X, \Im)$ be a topological space and $\beta \subseteq \Im$ then β is a base for $\Im$ iff each open set can be expressed as the union of member of β.

Proof: Assume that β is a base for $\Im$. Let G be an open set.

Let $x \in G$. Since G is open set then G is a neighborhood of x therefore using definition of base β thereexist a member $B \in \beta$ such that

$$x \in B \subset G, \forall x \in G$$

$$\Rightarrow G = \cup\{B : B \in \beta\} \text{ and } B \subset G$$

i.e., each open set is the union of member of β.

Conversely, suppose that each open set can be expressed as the union of member of β .Now to show that β is a base for $\Im$.

It is given $\beta \subseteq \Im$. Let N be any neighborhood of x then using definition of neighborhood, there exist an open set G such that

$$x \in G \subseteq N$$

$$\Rightarrow x \in \cup\{B : B \in \beta\} \subset N$$

By assumption, $x \in B \subseteq N$, for at least one $B \in \beta$. Therefore β is a base for $\Im$.

Theorem 4: Let $\{X, \Im\}$ be a topological space and β is a base for topology $\Im$. Then show that (1) for all $x \in X$ there exist $B \in \beta$ such that

(2) For all $B_1, B_2 \in \beta$ and every point $x \in B_1 \cap B_2$ there exist $B \in \beta$ such that

$$x \in B \subset B_1 \cap B_2$$

i.e., the intersection of any two member of B is a union of member of β.

Proof: Given that $\{X, \Im\}$ be topological space and β is a base for topology $\Im$.

(1) Since X is a open set then it is the neighborhood of each of its point then using definition of the base for every $x \in X$ there exist some $B \in \beta$ such that $x \in B \subset X$ i.e.,

$$X = \cup\{B : B \in \beta\}$$

(2) Let $B_1, B_2 \in \beta$, then B_1 and B_2 are open sets.

$\Rightarrow B_1 \cap B_2$ is open set.

$\Rightarrow B_1 \cap B_2$ is the neighbourhood of each of its point

$\Rightarrow$There exsit some $B \in \beta$ such that $x \in B \subset B_1 \cap B_2$ i.e., the intersection of any two member of

β is a union of member of B is a union of member of β.

9.6 Sub-bases and Local Base

Sub-bases and local bases are important concepts in topology because they provide ways to describe topologies in terms of simpler collections of sets (sub-bases) or to understand the local structure of a space around a point (local bases).

Let $(X, \Im)$ be a topological space. A family β of subsets of X is called a subbase for the topology $\Im$ if and only if finite intersections of members of β form a base for $\Im$.

Let $(X, \Im)$ be a topological space. A class β_x of open sets containing x is said to be a local base at x (or base for the neighborhood system of x) if and only if for each open set G containing x. There exist $G_x \in B_x$ with $x \in G_x \subset G$.

9.7 First Countable and Second Countable Space

First countable and second countable spaces are important concepts in topology because they provide useful properties that simplify the study and characterization of topological spaces.

A topological space $(X, \Im)$ is said to be the first countable space if and only if every point $x \in X$ has a countable local base.

A topological space $(X, \Im)$ is said to be second countable space if and only if there exist a countable base β for the topology $\Im$.

9.8 Topologies Generated by Classes of Sets

In topology, topologies can be generated by different classes of sets, such as bases, sub-bases, and local bases.Let X be a non-empty set and A be a non-empty collection of subsets of X. Then the collection A always generates a topology on X.

Examples

Example.6: Let $X=\{a,b,c,d,e\}$ and $A=\{\{a,b,c\},\{c,d\},\{d,e\}\}$. Find the topology on X generated by A.

Solution: Given that $X=\{a,b,c,d,e\}$ and $A=\{\{a,b,c,\},\{c,d\},\{d,e\}\}$

The collection of all finite intersections of sets in A is denoted by

$$\beta=\{a,b,c\},\{c,d\},\{d,e\},\{c\},\{d\},\phi,X\}$$

Using definition of base $\{X\in\beta\}$. The union of member of β gives the class

$\Im=\{X,\phi\{c\},\{d\},\{a,b,c\},\{c,d\},\{d,e\},\{a,b,c,d\},\{c,d,e\}$ which is the topology on X generated by A

Theorem 5: Prove that every discrete space is a first countable space.

Proof: Let $(X,\Im)$ be a discrete topological space. We know that in a discrete topological space every subset of X is open. Hence, if $x\in X$ then $\{x\}$ is open and contained in every open set G which contains x. Therefore $(X,\Im)$ is a first countable space.

Theorem 6: Prove that each second countable space is a first countable space.

Proof: Let $(X, \mathfrak{I})$ be a second countable space and β be a countable base for $\mathfrak{I}$. Let $\mathfrak{I}$ be the subfamily of β which contains x. Then $\mathfrak{I}$ is countable because β is countable. Now to show that $\mathfrak{I}$ is the local base at x.

Let N_x is the neighbouhood of x and an open set $x \in G$ i.e.,

$$\Rightarrow x \in G \subset N_x$$

$$\Rightarrow x \in N_x \in \mathfrak{I}$$

$$\Rightarrow \cup\{N_i : N_i \in \beta\}$$

$$\Rightarrow x \in \text{some } N_i \subset G \subset N_x \in \mathfrak{I}$$

$$\Rightarrow x \in G$$

$\Rightarrow \mathfrak{I}$ is countable local base at x. Since x is arbitrary.

The given space is first countable space.

9.9 Separable Space

A separable space is a topological space that contains a countable, dense subset.Separable spaces are often used as a convenient assumption in various theorems and proofs. Many important spaces studied in topology, such as separable metric spaces, are separable.

Let $(X, \mathfrak{I})$ be a topological space. Then $(X, \mathfrak{I})$ is said to be separable space if and only if there exist a finite countable dense subsets of X.

Note: Let $(X, \Im)$ be a topological space. Then $(X, \Im)$ is said to be separable space iff there exist a finite countable subsets A of X such that $\bar{A} = X$.

9.10 Cover of A Space

In topology, a cover of a topological space X is a collection of subsets of X whose union contains X as a subset. Let $(X, \Im)$ be a topological space and $A \subset X$. A family A of subsets of X is a cover for $(X, \Im)$ if and only if

$$\cup\{v : v \in \wedge\} = X$$

And if $\beta \subset A$ such that β is also a cover for X then β is a subcover of A.

9.11 Lindelöf Space

A Lindelöf space is a topological space in which every open cover has a countable subcover. Lindelöf spaces are a generalization of countably compact spaces, which are spaces in which every countable open cover has a finite subcover. Countably compact spaces are important in their own right, and Lindelöf spaces provide a broader class of spaces with similar properties. Let $(X, \Im)$ be a topological space. Then $(X, \Im)$ is said to be Lindelöf space if and only if every open cover of X has a countable subcover.

Theorem 7: Prove that every second countable space is separable.

Proof: Let $(X, \Im)$ be a second countable space.

To show that $(X, \Im)$ is separable space.

Since the given space is second countable, so there exist a countable base β for $\Im$. We take a point b from each member B of β. Let the collection of each point is denoted by A then A is countable because β is countable.

Now to show that A is dense in X. Let $x \in X$ and N is any neighbourhood of x. Since β is base then

$$x \in B \subset N$$

Using definition of $A, b \in A$ such that

$$b \in B \subset N$$

Thus, N contains a point of this set A other than x i.e., x is limit point A.

Since x is arbitrary point than all the points of set A are limit point, i.e.,

$$\bar{A} = X$$

Hence, A is countable dense subset of X. Therefore $(X, \Im)$ is separable.

Theorem 8: Every second countable space is Lindelof.

or

Every open cover of a second countable space is reducible to a countable sub-cover.

Proof: Let $(X, \Im)$ be a second countable space and β be a countable base for $\Im$. Let C be an open cover of X.

To show that C has a countable subcover, i.e., each member of C is expressible as a union of member of β. Suppose A is the set of all those members of β which are actually required in expressing members of C as union of members of β.

Therefore A is countable and is a cover of X i.e., A is a countable open copver of X. For each $A_i \in A$ choose a $C_i \in C$ such that

$$A_i \subset C_i$$

i.e., the collection of these C_i is also a countable open cover of X. So A has a countable subcover. Hence, $(X, \Im)$ is a Lindelof space.

9.12 Summary

Let $(X, \Im)$ be a topological space and Y be a subset of X. The family v of all subsets which are intersection of Y with member of $\Im$ i.e.,

$$v = \{Y \cap \text{every member of } \Im\} \text{ or } v = \{Y \cap A : A \in \Im\}$$

is called relative topology on Y.

Let $(X, \Im)$ be a topological space. If v is a relative topology for a subset Y of X then the corresponding topological space (Y, v) is called a subspace of $(X, \Im)$.

Let $(X, \Im)$ be any topological space. Then B is said to be a base for the topology $\Im$ on X, if for $x \in G \in \Im$ this implies thereexist $B \in \beta$ such that $x \in B \subset G$

Let $(X, \Im)$ be a topological space. A family β of subsets of X is called a subbase for the topology $\Im$ if and only if finite intersections of members of β form a base for $\Im$.

Let $(X, \Im)$ be a topological space. A class β_x of open sets containing x is said to be a local base at x (or base for the neighbourhood system of x) if and only if for each open set G containing x. There exist $G_x \in B_x$ with $x \in G_x \subset G$.

A topological space $(X, \Im)$ is said to be the first countable space if and only if every point $x \in X$

has a countable local base.

A topological space $(X, \Im)$ is said to be second countable space if and only if there exist a countable base β for the topology $\Im$.

Let X be a non-empty set and A be a non-empty collection of subset of X. Then the collection A always generates a topology on X.

Let $(X, \Im)$ be a topological space. Then $(X, \Im)$ is said to be separable space iff there exist a finite countable subset A of X such that $\bar{A} = X$.

Let $(X, \Im)$ be a topological space and $A \subset X$. A family A of subset of X is a cover for $(X, \Im)$ if and only if $\cup\{v : v \in \wedge\} = X$ and if $\beta \subset A$ such that β is also a cover for X then β is a subcover of A.

Let $(X, \Im)$ be a topological space. Then $(X, \Im)$ is said to be lindelof space if and only if every open cover of X has a countable subcover.

9.13 Terminal Questions

Q.1. Explain the relative topology and subspace with example.

Q.2. Show that every subspace of a discrete space is also discrete.

Q.3.Let Y be a subspace of X if U is open in Y and Y is open in X then U is open in X.

Q.4. Let $X = \{a, b, c, d, e\}$ and $\Im = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, d, e\}, \{a, b, d, e\}\}$ be a topology for X. Find the relative topology v for the subset $A = \{a, b, c\}$ of X.

Q.5. Let $X=\{a,b,c,d,e\}$ and $\Im=\{X,\phi,\{a\},\{c,d\},\{a,c,d\},\{b,c,d,e\}\}$ be a topology for X. Let $A\,\{a,d,e\}$ be a subset of X. Find the relative topology for A.

Q.6. Prove that every finite space is also first countable.

Q.7. Let $X=\{a,b,c,d\}$ and $\Im=\{\phi,X\,\{a\},\{b\},\{a,b\},\{c,d\},\{a,c,d\},\{b,c,d\}$ be a topology for X. Show that $\beta=\{\{a\},\{b\},\{c,d\}\}$ is a base for $\Im$.

Q.8. Let $X=\{a,b,c,d,e\}$ and $A=\{\{a,b\},\{b,c\},\{a,d,e\}\}$. Find the topology on X generated by A.

Answers:

4. $v=\{A,\phi,\{a\},\{b\},\{a,b\}\}$.

5. $v=\{A,\phi,\{a\},\{d\},\{a,d\},\{d,e\}\}$.

8. $\Im=\{X,\phi,\{a\},\{b\},\{a,b\},\{b,c\},\{a,b,c\},\{a,d,e\},\{a,b,d,e\}\}$.

UNIT-10: Continuous Maps and Homeomorphism

Structure

10.1 Introduction

Continuous maps and homeomorphisms are fundamental in topology for understanding the relationships between various topological spaces. Homeomorphisms are particularly significant as they identify spaces that are "topologically equivalent" or "topologically the same," indicating they share identical topological properties despite potentially differing geometric characteristics. In topology, homeomorphic spaces are often regarded as equivalent because many key topological properties, such as connectedness, compactness, and continuity, remain unchanged under homeomorphisms. By utilizing continuous maps and homeomorphisms, it becomes possible to compare and categorize different topological spaces based solely on their topological properties, independent of their specific geometric or metric attributes.

Hence these concepts provide a foundational understanding of the structure and behavior of spaces in topology.

10.2 Objectives

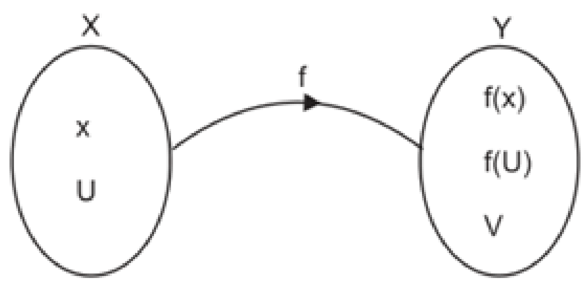
After reading this unit the learner should be able to understand about the:

- Continuous Function
- Open Mapping and Closed Mapping
- Bicontinuous Mapping, Bijective Mapping
- Sequential Continuity
- The pasting Lemma
- Homeomorphism

10.3 Continuous Function

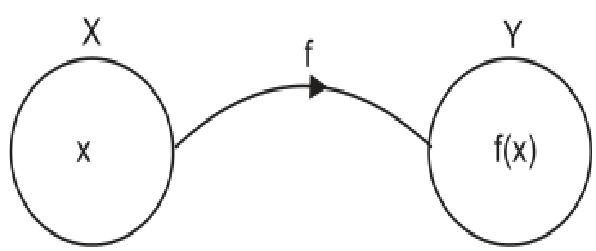
The term "topology" is used in two distinct ways: to refer to the mathematical discipline itself, and to describe a family of sets with specific properties that are used to define a topological space—an essential concept in topology. Continuous functions are the most important mappings between topological spaces, as they are used to define homeomorphisms that establish the equivalence of two spaces.

Let $(X,\Im)$ and (Y,v) be two topological spaces. A function $f:(X,\Im)\to(Y,v)$ is said to be continuous at the point $x\in X$ if thereexist a neighborhood U_x such that $f(U)\subset V$, where V is the neighborhood of $f(x)$.



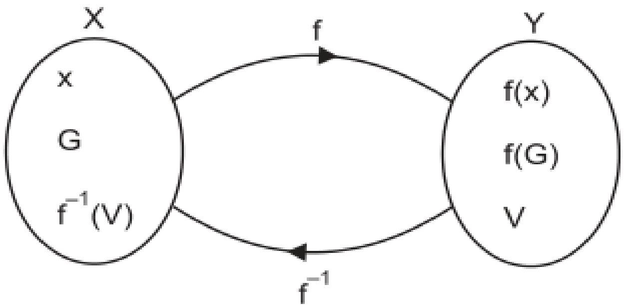
or

Suppose $(X,\Im)$ and (Y,v) are two topological spaces; let $f:X\to Y$ be a function. Then f is said to be continuous if it continuous at each point of X .



Theorem 1: Let $f:(X,\Im)\to(Y,v)$ be a mapping. Then f is continuous if and only if the inverse image of every open set in Y is open in X .

Proof: Let $f:(X,\Im)\to(Y,v)$ be a continuous. Let V is any open set in Y.



To show that $f^{-1}(V)$ is open set in X.

(1) If $f^{-1}(V)=\phi \Rightarrow$ it is open set

(2) If $f^{-1}(V)\neq\phi$ then there exist $x\in f^{-1}(V)\Rightarrow f(x)=V$

Since it is given f is continuous then there exist an open set G in X such that

$$x\in G \text{ and } f(G)\subset V \Rightarrow G\subset f^{-1}(V)$$

i.e., $$x\in G\subset f^{-1}(V)$$

Thus, $f^{-1}(V)$ is a neighbourhood of each of its points and so it is open in X.

Conversely, let the inverse image of all open set in Y are open set in X. To show that f is continuous at x. Let $x\in X$ and G is an open set in Y.

$$\Rightarrow f^{-1}(G) \text{ is open set in } X \text{ such that } x\in f^{-1}(G)$$

If $f^{-1}(G)=M$, then M is an open set in X continuous x such that

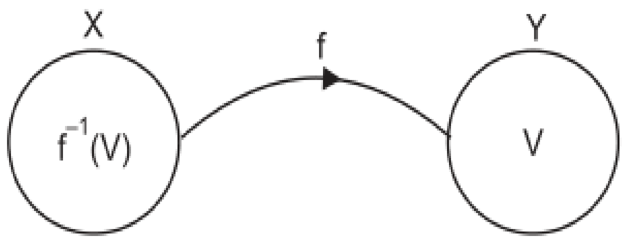
$$f(M)\subset G$$

$$\Rightarrow f \text{ is continuous at } x\in X$$

Since x is arbitrary then f is continuous on X.

Theorem 2: Let $f:(X,\Im)\to(Y,v)$ be a mapping. Then f is continuous if and only if the inverse image of every closed set in Y is closed in X.

Proof: Let $f:(X,\Im)\to(Y,v)$ be a continuous function and V is any closed set in Y



To show that $f^{-1}(V)$ is closed set in X. Since V is closed set in Y then $Y\sim V$ is open set in Y. Also it is given mapping in continuous

$\Rightarrow f^{-1}(Y\sim V)$ is open set in X

$\Rightarrow f^{-1}(Y)\sim f^{-1}(V)$ is open set X

$\Rightarrow X\sim f^{-1}(V)$ is open set X

$\Rightarrow f^{-1}(V)$ is closed set in X.

Conversely, let the inverse image of closed set in Y is closed in X to show that f is continuous. Let G is any open set in Y.

To show that $f^{-1}(G)$ is open set in X

$\Rightarrow Y\sim G$ is closed set in Y

$\Rightarrow f^{-1}(Y\sim G)$ is closed set in X

$$\Rightarrow f^{-1}(Y) \sim f^{-1}(G) \text{ is closed set in } X$$

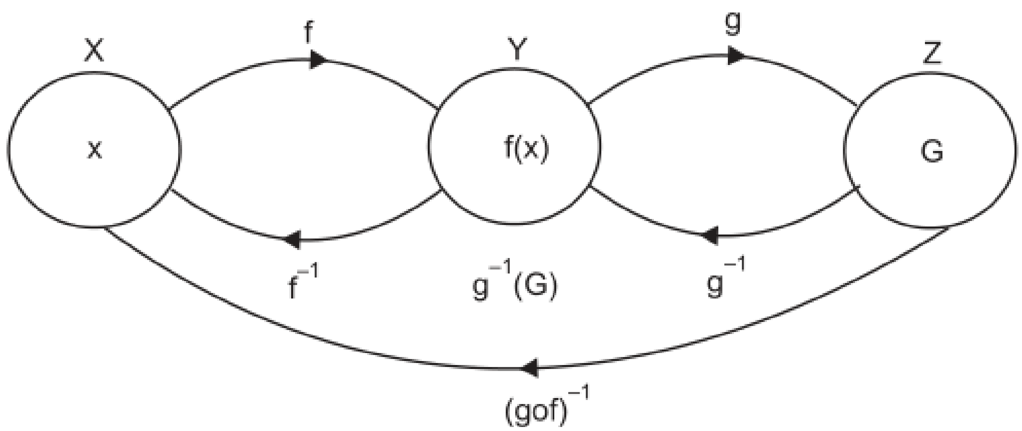
$$\Rightarrow X \sim f^{-1}(G) \text{ is closed set in } X$$

$$\Rightarrow f^{-1}(G) \text{ is open set in } X$$

Hence, f is continuous.

Theorem 3: Let X, Y and Z are topological spaces. The mapping $f : X \to Y$ and mapping $g : Y \to Z$ are continuous then $gof : X \to Z$ is also continuous.

Proof: Given that X, Y and Z are topological spaces.



Let G is open set in Z. Since it is given mapping $g : Y \to Z$ is continuous.

$$\Rightarrow g^{-1}(G) \text{ is open set in } X .$$

Also it is given $f : X \to Y$ is continuous.

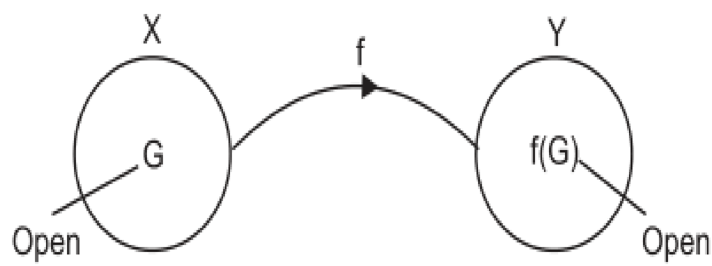
$$\Rightarrow f^{-1}\left[g^{-1}(G)\right] \text{ is open set in } X$$

$$\Rightarrow (gof)^{-1}(G) \text{ is open set in } X \ \{\text{for every open set } G \text{ of } Z\}$$

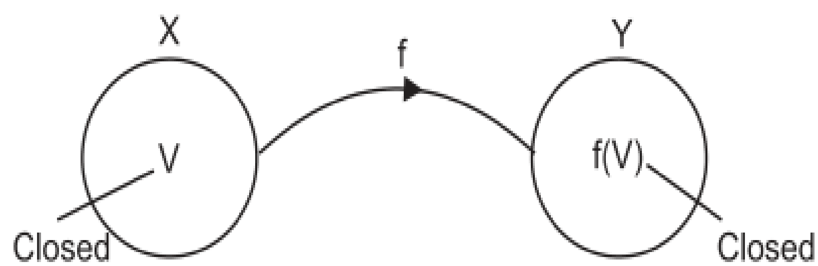
$\Rightarrow (gof)$ is continuous from X to Z.

10.4 Open Mapping and Closed Mapping

Let $f:(X,\Im)\to(Y,v)$ be a mapping f is said to be open mapping iff images of every open set in X are open in Y.

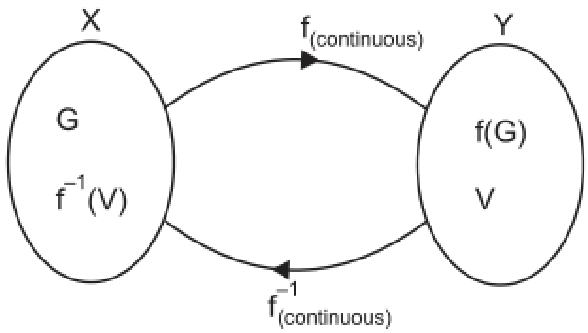


Let $f:(X,\Im)\to(Y,v)$ be a mapping. Then mapping f is said to be closed mapping iff images of every closed set in X are closed in Y.



10.5 Bicontinuous Mapping

Let $(X,\Im)$ and (Y,v) be any two topological spaces. A mapping f is said to be bicountinuous mapping if both f and f^{-1} are continuous mapping.



10.6 Bijective Mapping

Let $(X,\Im)$ and (Y,v) be any two topoligical spaces. A mapping f is said to be bijective mapping if f is one-one onto mapping.

Note: Continuity of f^{-1} is the same as open mapping or closed mapping.

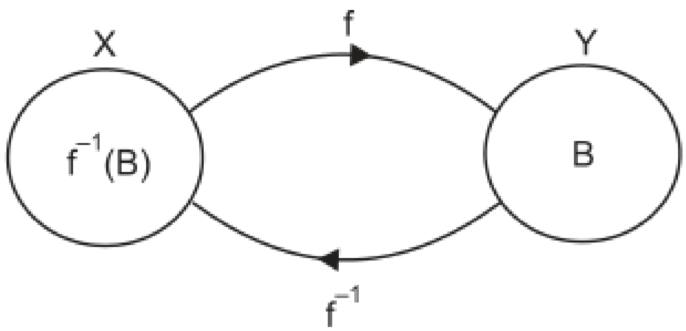
10.7 Sequential Continuity

Let $(X,\Im)$ and (Y,v) be any two topological spaces. A mapping $f: X \to Y$ is said to be squentially continuous at a point $x_0 \in X$ if and only if for every sequence $\{x_n\}$ in X converging to x_0, the sequence $\{f(x_n)\}$ in Y converges to $f(x_0)$.

i.e., $$x_n \to x_0 \Rightarrow f(x_n) \to f(x_0).$$

Theorem 4: Let $f: X \to Y$ be a mapping then f is continuous if and only if $\left[f^{-1}(B)\right]^{\circ} \supset \left[f^{-1}(B^{\circ})\right]$, $\forall B \subset Y$.

Proof: Let $f: X \to Y$ be a continuous mapping and $B \subset Y$



To show that $\left[f^{-1}(B)\right]^{\circ} \supset f^{-1}\left(B^{\circ}\right)$

We know that by definition of interior, B° is a open set in Y .

Since f is continuous this implies $f^{-1}\left(B^{\circ}\right)$ is open set in X . By definition of interior, if B° is a open set then $B^{\circ} = B$, i.e.,

$$\left[f^{-1}(B)\right]^{\circ} = f^{-1}\left(B^{\circ}\right) \tag{1}$$

Also by definition, we have

$$B \supset B^{\circ} \Rightarrow f^{-1}(B) \supset f^{-1}\left(B^{\circ}\right)$$

If $A \subset B$ then $A^{\circ} \subset B^{\circ}$

$$\Rightarrow \left[f^{-1}(B)\right]^{\circ} \supset \left[f^{-1}\left(B^{\circ}\right)\right]^{\circ} \tag{2}$$

From equations (1) and (2), we have

$$\left[f^{-1}(B)\right]^{\circ} \supset f^{-1}\left(B^{\circ}\right) \tag{3}$$

`Conversely, let $\left[f^{-1}(B)\right]^{\circ} \supset f^{-1}\left(B^{\circ}\right)$ (4)

To show that f is continuous mapping.

Let H be any open set in Y then to show that $f^{-1}(H)$ is also open in X. We know that if H is an open set then $H^{\circ} = H$

From equation (4), we have

$$\Rightarrow \left[f^{-1}(H) \right]^{\circ} \supset f^{-1}(H) \quad (5)$$

By definition of interior, we have

$$H \supset H^{\circ}$$

$$\Rightarrow f^{-1}(H) \supset \left[f^{-1}(H) \right]^{\circ} \quad (6)$$

From equations (5) and (6) , we have

$$\left[f^{-1}(H) \right]^{\circ} = f^{-1}(H)$$

i.e., $f^{-1}(H)$ is open in X.

Because we know that B° is open set if $B^{\circ} = B$ .Hence, f is continuous mapping.

Theorem 5: Let $f : X \rightarrow Y$ be a mapping then f is continuous if and only if

$$\overline{f^{-1}(B)} \subset f^{-1}\left(\overline{B}\right), \forall B \subset Y.$$

Proof: Suppose f is continuous mapping and B is closed set in Y, therefore $f^{-1}\left(\overline{B}\right)$ is closed set in X. By definition of closure if B is closed set then $\overline{B} = B$ i.e.,

$$\overline{\left[f^{-1}\left(\overline{B}\right) \right]} = f^{-1}\left(\overline{B}\right) \quad (1)$$

Also by definition, we have

$$B \subset \overline{B} \Rightarrow f^{-1}(B) \subset f^{-1}\left(\overline{B}\right)$$

If $A \subset B$ then $\overline{A} \subset \overline{B}$

$$\Rightarrow \overline{f^{-1}(B)} \subset \overline{\left[f^{-1}\left(\overline{B}\right) \right]} \quad (2)$$

From equations (1) and (2), we have $\overline{f^{-1}(B)} \subset f^{-1}\left(\overline{B}\right)$ (3)

Conversely, let $\overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$ (4)

To show that f is continuous mapping.

Let B is any closed set in Y then to show that $f^{-1}(B)$ is also closed set in X. We know that if B is any closed set then $\bar{B} = B$.

It is given that $\overline{f^{-1}(B)} \subset f^{-1}(\bar{B}) = f^{-1}(B)$ for $B = \bar{B}$ (5)

By definition, we have

$$B \subset \bar{B} \Rightarrow f^{-1}(B) \subset \overline{f^{-1}(B)} \tag{6}$$

From equations (5) and (6), we have

$$f^{-1}(B) = \overline{f^{-1}(B)}$$

i.e., $f^{-1}(B)$ is closed in X. Hence, f is continuous mapping.

Theorem 6: Let $(X, \mathfrak{I})$ and (Y, v) be tow topological spaces, then a mapping $f : X \to Y$ is open if and only $f(A^\circ) \subset [f(A)]^\circ \; \forall A \subset X$.

Proof: Suppose $(X, \mathfrak{I})$ and (Y, v) are two topological spaces. Let A be a subset of X and $f : X \to Y$ be an open mapping.

Since A° is a $\mathfrak{I}$-open subset and f is an open mapping so $f(A^\circ)$ is a v-open subset of Y.

By definition of interior, we have

$$A^\circ \subset A$$

$$\Rightarrow f(A^\circ) \subset f(A)$$

Thus, $f(A^\circ)$ is a v-open set contained in $f(A)$.

Therefore, $f(A^\circ) \subset [f(A)]^\circ$

Conversely, $f(A^\circ) \subset [f(A)]^\circ$ and G be any open set in X. Then we have

$$f(G) = f(G^\circ) \subset [f(G)]^\circ \text{ because } G \text{ is open } iff \; G^\circ = G.$$

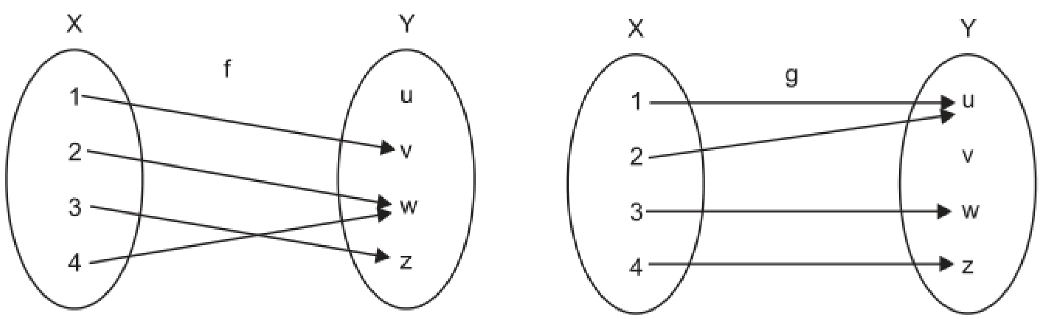
Hence, $f(G)$ is an v-open subset of Y i.e., f is open mapping.

Examples

Example.1: Consider the following topologies on X $=\{1,2,3,4\}$ and $Y=\{u,v,w,z\}$ respectively.

$$\Im=\{X,\phi,\{1\},\{1,2\},\{1,2,3\}\} \textbf{ and } \nu=\{Y,\phi,\{u\},\{v\},\{u,v\},\{v,w,z\}\}$$

Also consider the function $f:X\to Y$ and $g:X\to Y$ defined by the diagrams below:



To show that $(1)\, f$ is continuous and $(2)\, g$ is not continuous.

Solution: (1) We have $f^{-1}\{Y\}=X$, $f^{-1}\{\phi\}=\phi$, $f^{-1}\{u\}=\phi$

$$f^{-1}\{v\}=\{1\},\ f^{-1}\{u,v\}=\{1\}, \text{and } f^{-1}\{v,w,z\}=\{1,2,3,4\}=X$$

Hence, f is continuous because inverse of each member of ν on Y is a member of topology $\Im$ on X .

(2) We have $f^{-1}\{Y\}=X$, $f^{-1}\{\phi\}$, $f^{-1}\{v\}=\{\phi\}$,

$$f^{-1}\{u\}=\{1,2\} \text{ and } f^{-1}\{u,v\}=\{1,2\}.$$

But $f^{-1}\{u,v,w\}=\{1,3\}$ which is not open set of X i.e., not belongs to $\Im$.

Hence, g is not continuous because inverse of each member of v on Y is not a member of the topology $\Im$ on X.

Example 2: let (Y,v) be a topological space. Let $f:X\to Y$ be a mapping, where X is a non-empty set.

(a) What is the smalles topology for X which makes f continuous?

(b) Is it always possible to assign a topology for X so that f is continuous?

Solution: (a) Suppose $\Im=\{f^{-1}(\mathrm{H}):H\in v\}$

If $\Im'$ is any topology for X then f is $\Im'$-v-continuous if and only if $\Im'\subset\Im$. Thus, we have to find the smallest topology for X containing $\Im$.

(i) We have $f^{-1}(\phi)=\phi$

$\therefore$ $\phi\in\Im$

Also $f^{-1}\{Y\}=X$

$\therefore$ $X\in\Im$

(ii) Let A and B two any member of $\Im$. Then there exist $v-$open subsets G and H such that

$A=f^{-1}(G)$ and $B=f^{-1}(H)$

Then $A\cap B=f^{-1}(G)\cap f^{-1}(H)$

$$=f^{-1}(G\cap H)=\Im$$

(iii) Let $A_i\in\Im,\forall i\in\wedge$. Then

$A_i = f^{-1}(H_i) \forall i \in \wedge$, where $H_i \in v$

$$\cup\{A_i : i \in \wedge\} = \cup\{f^{-1}(H_i) : i \in \wedge\}$$

$$= f^{-1}\left[\cup\{H_i : i \in \wedge\}\right] \in \Im$$

$$\left[\because \cup\{H_i : i \in \wedge\} \in v\right]$$

(c) Yes, it is alsways possible to assign a topology for X so that f is continuous. The discrete topology D is the required topology because for each $v-$open set H, $f^{-1}(H)$ is D-open.

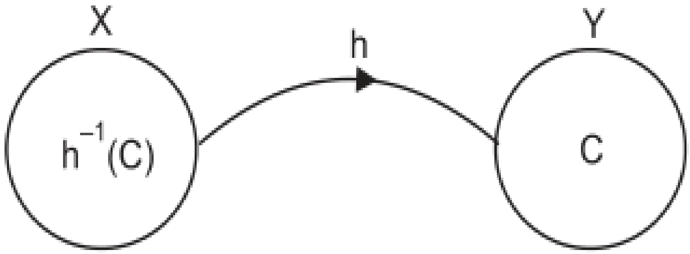
10.8 The Pasting Lemma

Theorem: Let $(X, \Im)$ be the topological space and $X = A \cup B$, where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ are continuous mapping into a topological space (Y, v). If $f(x) = g(x)$ for all $x \in A \cap B$, then f and g combine to give a continuous mapping $h : X \to Y$, defined by setting $h(x) = f(x)$ for $x \in A$ and $h(x) = g(x)$ for $x \in B$, is continuous.

Proof: Suppose that the topological space $(X, \Im)$ is the union of two closed subsets A and B. Let $f : A \to Y$ is continuous then $f^{-1}(G)$ is closed for each closed subset of G of Y. Let $g : B \to Y$ is continuous then $g^{-1}(H)$ is closed for each closed subset H of Y. Also it is given $f(x) = g(x), \quad \forall x \in A \cap B$. To show that $h : X \to Y$ is continuous mapping. Suppose C is a closed subset of Y, i.e., to show that $h^{-1}[C]$ is also a closed subset of X.

Suppose $C = G \cup T \cup H$, where G is the closed subset of Y such that $h(x) \in G \Rightarrow x \in A$. Also H is the closed subset of Y such that

$$h(x) \in H \Rightarrow x \in B.$$



And T is t

$$h(x) \in T \Rightarrow x \in A \cap B$$

We have $\quad h^{-1}[C] = h^{-1}[G \cup T \cup H]$

$$= h^{-1}[G] \cup h^{-1}[T] \cup h^{-1}[H]$$

$$= f^{-1}[G] \cup f^{-1}[T] \cup g^{-1}[H]$$

For $\quad f(x) = h(x)$ for all $x = A$

And $\quad g(x) = h(x)$ for all $x \in B$

Then $\quad h(x) = f(x) = g(x)$ for all $x \in A \cap B$.

Now $f^{-1}[G], f^{-1}[T]$ and $g^{-1}[H]$, are all closed subsets for f and g are continuous mappings. Since a finite union of closed subsets is closed therefore $h^{-1}(C]$ is a closed set wherever C is closed in Y.

Hence, $h: X \rightarrow Y$ is a continuous mapping.

10.9 Homeomorphism

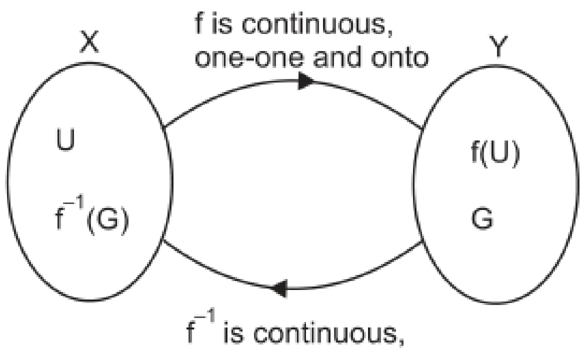
Homeomorphisms are of particular significance in this context, as they are defined as continuous functions with continuous inverses. For instance, the function $y = x^3$ is a homeomorphism on the real line. Let $(X, \Im)$ and (Y, v) be two topological spaces. A mapping $f:(X, \Im) \to (Y, v)$ is said to be homeomorphism if and only if

(1) f is one-one, and onto.

(2) f and f^{-1} both are continuous

or

A mapping $f:(X, \Im) \to (Y, v)$ is said to be homeomorphism if and only if f is bijective and bicontinuous.



Theorem 7: Prove that a homeomophism is an equivalence relation in a collection of all topological spaces.

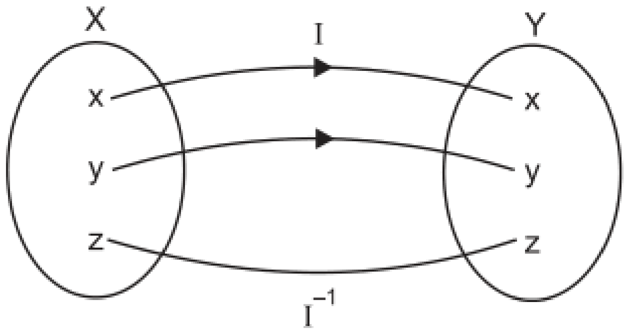
Proof: We know that homeomorphism is a mapping which is reflexive, symmetric and transitive if it is an equivalence relation.

(1) Reflexive: Let an identity mapping

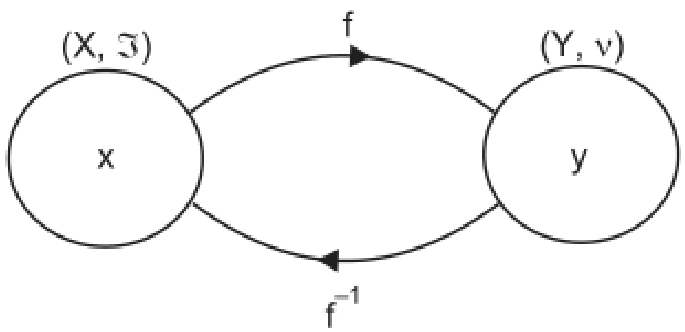
$I : X \to X$ such that

$I(x) = x$ for all $x \in X$.

Here I is one-one onto. I and I^{-1} is continuous. Hence it is homeomorphism, i.e., the relation of homeomorphism is reflexive relation.



(2) Symmetry: Let $(X, \mathfrak{I})$ and (y, v) are two topological spaces.



Since f is homeomorphism $f: X \to Y$ then f is one-one onto, f and f^{-1} are continuous. i.e.,

$$f^{-1}: Y \to X$$

$\Rightarrow f^{-1}$ is one-one onto because f is one-one onto. Also f^{-1} is continuous i.e., $\left(f^{-1}\right)^{-1} = f$ is continuous. This implies f^{-1} is homeomorphism.

Hence, the relation of homeomorphism is symmetric relation.

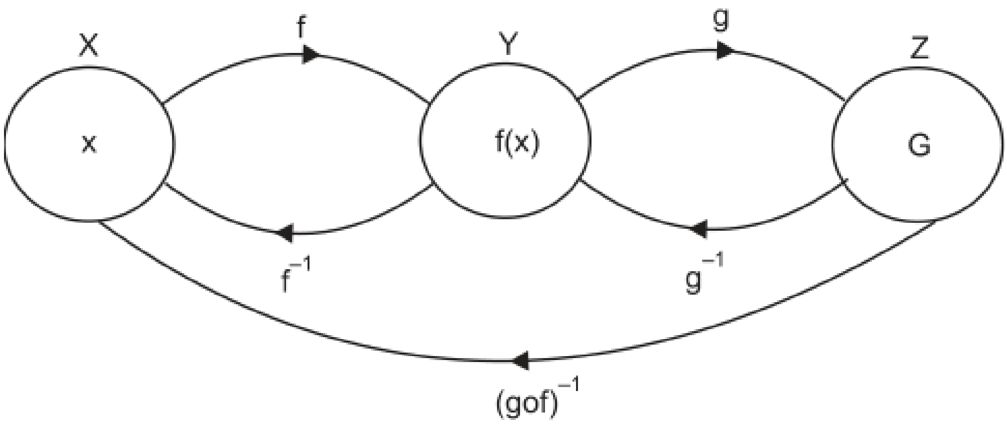
(3) Transitivity: Let $(X, \mathfrak{I}), (Y, v)$ and (Z, W) are three topological spaces.

Let $f: X \to Y$ and $g: Y \to Z$ are the corresponding homeomorphism, then we have to show that the mapping $gof: X \to Z$ is homeomorphism. Now we have:

(1) Since f is one-one onto and g is one-one onto then gof is also one-one onto.

(2) Since $f\ X \to Y$ and $g: Y \to Z$ are continuous then gof is also continuous.

(3) Since f is continuous then implies f^{-1} is continuous.



And g is continuous then implies g^{-1} is continuous. Thus the mapping $f^{-1}\text{o}g^{-1} = (gof)^{-1}$ is also a continuous mapping. Hence, gof is homeomorphism, i.e., the relation of homeomorphism is Transitivity.

Theorem 8: Let $\{X, \Im\}$ and (Y, v) be two topological spaces and let the mapping $f: X \to Y$ be one-one onto. Then f is homeomorphism if and only if $f\left(\overline{A}\right) = \overline{\left[f(A)\right]},\ \forall A \subset X$.

Proof: Suppose $(X, \Im)$ and (Y, v) are two topological spaces. Let a mapping $f: X \to Y$ be one-one, onto and

$$f\left(\overline{A}\right) = \overline{\left[f(A)\right]}, \quad \forall A \subset X. \qquad (1)$$

To show that f is homeomorphism

By (1), we have

$$f\left(\bar{A}\right) \subset \overline{\left[f\left(A\right)\right]}$$

This implies f is continuous

Now let H be any closed set then $\bar{H} = H$

Using above conditions, we have

$$f\left(\bar{F}\right) = f\left(F\right) = \overline{f\left(F\right)}$$

This implies $f\left(F\right)$ is $v-$closed. Thus, f is closed mapping. Here f is closed and continuous so f is homeomorphism.

Conversely, let f is homeomorphism and to show that

$$f\left(\bar{A}\right) = \overline{\left[f\left(A\right)\right]}, \quad \forall A \subset X$$

Suppose A is any subset of X and given f is continuous

Then $$f\left(\bar{A}\right) \subset \overline{\left[f\left(A\right)\right]}$$

Also $$A \subset \bar{A} \quad \Rightarrow \quad f\left(A\right) \subset f\left(\bar{A}\right)$$

Since f is closed so $\overline{\left[f\left(A\right)\right]} = \left(A\right)$

Thus, we have $\overline{\left[f\left(A\right)\right]} \subset f\left(\bar{A}\right)$

Hence, $f\left(\bar{A}\right) = \overline{\left[f\left(A\right)\right]}, \quad \forall A \subset X$.

Examples

Example.3: Let $X = \{a, b, c\}$ **and** $Y = \{1, 2, 3\}$. **Let** $\Im = \{X, \phi, \{a, b\}, \{c\}\}$ **and** $\nu = \{Y, \phi, \{1\}, \{2, 3\}\}$. **Suppose** $f(a) = 1,\ f(b) = 2$ **and** $f(c) = 3$.

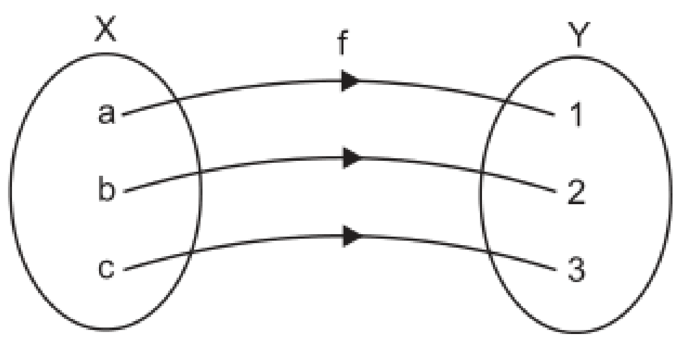
(1) **Is** $f : X \to Y$ **continuous?**

(2) **If** $f : X \to Y$ **is homeomorphism?**

Solution: Given that $X = \{a, b, c\}$ and $Y = \{1, 2, 3\}$.

Also given $\Im = \{X, \phi, \{a, b\}, \{c\}\}$ and $\nu = \{Y, \phi, \{1\}, \{2, 3\}\}$.

(1)Here f is one-one mapping from X to Y. But $f : X \to Y$ is not continuous. Since $\{1\}$ is ν -open set but $f^{-1}\{1\} = \{a\}$ is not $\Im -$open set.



(2) From (i), $f : X \to Y$ is not continuous. Hence, $f : X \to Y$ is not homeomorphism.

10.10 Summary

Let $(X, \Im)$ and (Y, ν) are two topological spaces. A function $f : (X, \Im) \to (Y, \nu)$ is said to be continuous at the point $x \in X$ if there exist a neighborhood U_x such that $f(U) \subset V$, where V is the neighborhood of $f(x)$.

Let $f:(X,\Im)\rightarrow(Y,v)$ be a mapping f is said to be open mapping iff images of every open set in X are open in Y.

Let $(X,\Im)$ and (Y,v) be any two topological spaces. A mapping f is said to be bicountinuous mapping if both f and f^{-1} are continuous mapping.

Let $(X,\Im)$ and (Y,v) be any two topoligical spaces. A mapping f is said to be bijective mapping if f is one-one onto mapping.

Let $(X,\Im)$ and (Y,v) be any two topological spaces. A mapping $f:X\rightarrow Y$ is said to be squentially continuous at a point $x_0\in X$ if and only if for every sequence $\{x_n\}$ in X converging to x_0, the sequence $\{f(x_n)\}$ in Y converges to $f(x_0)$.

i.e., $x_n\rightarrow x_0\Rightarrow f(x_n)\rightarrow f(x_0)$.

Let $(X,\Im)$ and (Y,v) be two topological spaces. A mapping $f:(X,\Im\rightarrow(Y,v)$ is said to be homeomorphism if and only if

(i) f is one-one and onto (ii) f and f^{-1} both are continuous.

10.11 Terminal Questions

Q.1. Explain the continuous and homeomorphism mapping.

Q.2. What do you mean by Open and Closed mapping.

Q.3. Define continuous and homeomorphism mapping.

Q.4. A mapping $f:X\rightarrow Y$ is continuous mapping if and only if $f(\overline{A})\subset\overline{f(A)}, \forall A\subset X$.

Q.5. Give an example of a one-one continuous mapping which is not a homeomorphism.

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Master of Science/ Master of Arts

PGMM-109N
MAMM-109N
Topology

Block

4 Separation Axioms on Topological Spaces

Block-4

Separation Axioms on Topological Spaces

In this block we deal with the basic concept of separation axioms such as $T_0 -$space, $T_1 -$space, $T_2 -$space, regular space, $T_3 -$space, $T_{3/2} -$space, normal-space, $T_4 -$space, and their properties. Urysohn's lemma, Teitze extension theorem and statement of Urysohn's metrization theorem, Connected Set, Disconnected Set, Connectedness on the Real Line, components, Maximal Connected Set, Locally Connected Space and Totally Disconnected Set, Cover, Open Cover, Compact Space, Compact Set, Finite Intersection Property, Locally Compact Space, Lindelof Space, Bolzano Weierstrass Property, Sequentially Compact, Uniformly Continuous, Lebesgue Covering Lemma, Heine-Borel Theorem, Product Topology, Projection Mappings are be discussed here. Compactness is an important property because it ensures that certain properties hold in a space. There are different equivalent characterizations of compactness in terms of open sets, closed sets, and continuous functions, which make compactness a versatile and important concept in topology. Connectedness is a fundamental concept in topology that describes the property of a space being in one piece, without being able to be split into two or more disjoint nonempty open sets. Some symbols are defined below with their name and notations: $T_1 =$ Frechet space $T_2 =$ Housdorff space

$T_3 =$ Regular + T_1 space $T_4 =$ Normal + T_1 space

$T_{3\frac{1}{2}}$ = Tychonoff space or completely regular $T_1 -$space $T_5 =$ Completely normal $T_1 -$space

$R =$ Regular space $N =$ Normal space

CN = Completely normal space $CR =$ Completely regular space

C_1 = First countable space C_2 = Second countable space

$L =$ Lindelof space

UNIT-11: Separation Axioms-I

Structure

11.1 Introduction

Separation axioms are properties that describe the level of "separation" or "disconnectedness" between points and sets in a topological space. These axioms help to classify and distinguish different types of topological spaces based on their separation properties. In mathematics, particularly in topology and functional analysis, a To -space is a type of topological space that satisfies the To separation axiom. To spaces are considered the weakest separation axiom in topology, weaker than T_1 (or "Kolmogorov") spaces, which satisfy the T_1 separation axiom: for any two distinct points, each point has a neighborhood not containing the other point.In topology, a T_1 space, also known as a Fréchet space (is a topological space) in which every singleton set (a set containing only one point) is a closed set. The T_1 separation axiom is a step stronger than the T_0 separation axiom. T_1 spaces are important in topology because they are strong enough to ensure many desirable properties, yet they are still relatively general. Most commonly studied topological spaces, such as metric spaces, are T_1 spaces. In topology, a T_2 space, also known as a Hausdorff space, is a topological space in which any two distinct points have disjoint neighborhoods. The T_2 separation axiom is stronger than the T_1 separation axiom. Hausdorff spaces are important in topology because they provide a setting in which limits of sequences and continuity of functions are well-behaved. Most spaces encountered in analysis and geometry, such as Euclidean spaces, are Hausdorff spaces.

11.2 Objectives

After reading this unit the learner should be able to understand about the:

- T_0 –Space with their properties and applications
- T_1 –Space or Frechet Space with theor properties and applications
- Co-finite Topology
- T_2 –Space (Hausdorff Space) with theor properties and applications

11.3 T_0 -Space

To-space are important in understanding the properties of topological spaces and their relationships with other separation axioms.The To axiom states that for any two distinct points in the space, there exists an open set containing one point but not the other. In other words, To spaces can distinguish between different points based on open sets.

Let $(X,\Im)$ be a topological space. Then $(X,\Im)$ is said to be To-space if and only if for distinct points x_1 and x_2 in X there exist a $\Im-$open set G such that

$$x_1 \in G \text{ and } x_2 \notin G$$

or

$$x_2 \in G \text{ and } x_1 \notin G$$

Examples

Example.1. Let $X = \{a,b,c\}$, $\Im = \{X,\phi\{a\},\{b\},\{a,b\}\}$. To show that $(X,\Im)$ is a T_0- space.

Solution: Given that $$X = \{a,b,c\}$$

and $$\Im = \{X,\phi,\{a\},\{b\},\{a,b\}\}$$

Using definition of T_0-space, for distinct element a and b there exists a $\Im-$open set $\{a\}$ or $\{b\}$ such that

$$a \in \{a\} \text{ and } b \notin \{a\}$$

or

$$a \notin \{b\} \text{ and } b \in \{b\}$$

Hence, $(X,\Im)$ is a T_0-space.

Example 2: To show that every discrete space $(X,\Im)$ is a T_0-space.

Solution: Suppose $X=\{a,\mathrm{b}\}$ and $\Im=\{X,\phi,\{a\},\{b\}\}$

For if $a,b\in X$ are two distinct points

(1) Then there exists a $\Im-$open set $\{a\}$ such that

$$a\notin\{a\} \text{but } b\in\{a\}$$

(2) Then there exists a $\Im-$open set $\{b\}$ such that

$$a\notin\{b\} \text{but } b\in\{b\}$$

Hence every discrete space $(X,\Im)$ is a T_0-space.

Example 3: To show that an indiscrete space $(X,\Im)$ is not a T_0-space.

Solution: Suppose $X=\{a,b\}$ and $\Im=\{X,\phi\}$.

Here $(X,\Im)$ is an indiscrete space.

For if $a,b\in X$ and $a\neq b$ then there exists no $\Im-$open set which contains one of a and b and does not contain the other, i.e., here only one non-empty $\Im-$open set is X which contains both open and closed. Hence, an indiscrete space is not a T_0-space.

Theorem 1: Prove that every subspace of a T_0-space is a T_0-space and hence the property is hereditary.

Proof: Let $(X,\Im)$ be a T_0-space and (Y,v) is a subspace of $(X,\Im)$. To show that (Y,v) is a T_0-space.

Since $Y \subset X, x_1$ and x_2 are also distinct point of Y.

It is given $\{X, \Im)$ is a T_0-space then there exists an open set $G \in \Im$ such that

$$x_1 \in G, x_2 \notin Y$$

Now $x_1 \in G$ and $x_1 \in Y$

$$\Rightarrow x_1 \in Y \cap G \in U$$

Such that $x_1 \in Y \cap G, x_2 \notin Y \cap G$

This implies (Y, v) is a T_0-space.

Hence, the property of being a T_0-space is a hereditary propert.

Theorem 2: Let $(X, \Im)$ is (Y, v) is a subspace of $(X, \Im)$. If $f: X \to Y$, f is one-one onto and open mapping then (Y, v) is an open space or T_0-space.

or

The property of a space of being a T_0-space is presented by one to one, onto, open mapping and hence is a Topological property.

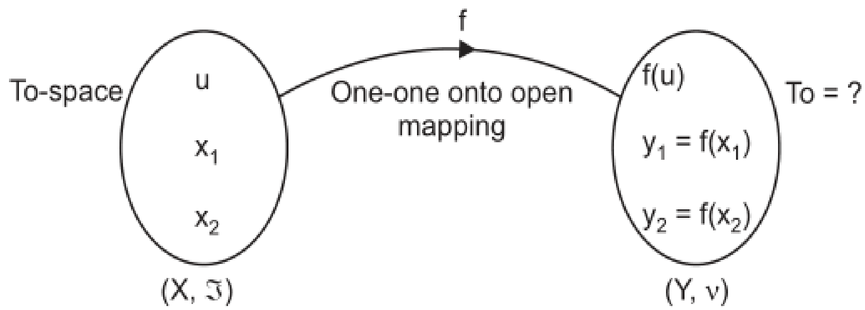
Proof: Let $(X, \Im)$ be a T_0-space and let f be one-one open mapping of $(X, \Im)$ onto another topological space (Y, v).

To show that (Y, v) is also to T_0-space.

Let $y_1, y_2 \in Y$ where $y_1 \neq y_2$

$\Rightarrow$ there exist $x_1, x_2 \in X$ such that

$$f\left(x_1\right)=y_1 . f\left(x_2\right)=\left(y_2\right)$$



It is given $(X, \Im)$ is a T_0 -space

$\Rightarrow$ There exist a neighbourhood u of x_1 which does not contains x_2.

$\Rightarrow$ $f(u)$ is a neighbourhood in y (since it is given that it is an open mapping)

i.e., $y_1 \in f(u)$ which does not contains y_2

Hence, (Y, v) is also a T_0 – space.

Theorem 3: A topological space $(X, \Im)$ is a T_0 space if for any distinct arbitrary points x, y of X, the closure of $\{x\}$ and $\{y\}$ are distinct, i.e., $\{\bar{x}\} \neq \{\bar{y}\}$.

Proof: Let $(X, \Im)$ be a T_0 – space. Let $x, y \in X$ such that $x \neq y$. To show that $\{\bar{x}\} \neq \{\bar{y}\}$. Since X is a T_0 – space, then there exist $G \in \Im$ such that

$$x \in G, y \notin G$$

$$\Rightarrow x \notin X \sim G \text{ and } \Rightarrow y \in X \sim G$$

i.e., $X \sim G$ is closed set

using definition of closure, we have

$$\bar{A} = \cap \qquad \{\text{of all closed set containing A}\}$$

We have $\{\bar{y}\} = \cap\{F : F \text{ is a closed set y } \in \text{F}\}$

Also $X \sim G$ is a closed set containing y

$$\{y\} \subset X \sim G \Rightarrow x \notin X \sim G \text{ so that}$$

$$x \notin \{\bar{y}\}$$

Obviously $\{x\} \subset \{\bar{x}\}$

This impies $\qquad x \in \{\bar{x}\}$

Using (1) and (2) we have $\qquad \{\bar{x}\} \neq \{\bar{y}\}$

Conversely, let $\{\bar{x}\} \neq \{\bar{y}\}$ such that $x \neq y$ and

$$\{\bar{x}\} \neq \{\bar{y}\}$$

To show that $(X, \mathfrak{I})$ is a $T_0 -$space

Given that $\{\bar{x}\} \neq \{\bar{y}\}$ then there exists one point $p \in X$ such that

$$p \in \{\bar{x}\} \text{ and } p \in \{\bar{y}\}$$

We have $x \notin \{\overline{y}\}$.

But if $x \in \{\overline{y}\}$. then

$$\{x\} \subset \{\overline{y}\}$$

$$\Rightarrow \{\overline{x}\} \subset \{\overline{\overline{y}}\} = \{\overline{y}\}$$

$$\Rightarrow \{\overline{x}\} \subset \{\overline{y}\}$$

i.e., $p \in \{\overline{y}\}$ which is contradiction

Using equation (4), we have

$$x \in \{\overline{y}\} \Rightarrow x \in X \sim \{\overline{y}\}$$

$$y \in \{\overline{y}\} \Rightarrow y \notin X \sim \{\overline{y}\}$$

$$\Rightarrow X \sim \{\overline{y}\} \text{ is open for } \{\overline{y}\} \text{ is closed}$$

Then $x \sim \{\overline{y}\}$ is open set such that

$$x \in X \sim \{\overline{y}\} \text{ and } y \notin X \sim \{\overline{y}\}$$

i.e., space is T_0 -space.

Theorem 4: If $(X, \mathfrak{I})$ is a T_0 -space and $\mathfrak{I}_1$ is finer than $\mathfrak{I}$, then $(x, \mathfrak{I}_1)$ is also a T_0 -space.

Proof: Suppose $(X, \mathfrak{I})$ is a T_0 -space then for any two distinct points x_1, x_2 in X, there exists a $\mathfrak{I}-$ open set G such that

$$x_1 \in G \text{ and } x_1 \not\in G$$

Since $\Im_1$ is finer than $\Im$. So every $\Im -$ open set is also $\Im_1 -$ open set. Hence, G is a $\Im_1$-open set which contains x_1 but not x_2.

Thus, the space $(X, \Im_1)$ is also a T_0 -space.

11.4 T_1 –Space or Frechet Space

T_1 spaces are important in topology for their separation properties, their compatibility with analysis, their connection with Hausdorff spaces, their applications in topological dynamics, and their role in algebraic topology. T_1 spaces are well-suited for analysis and related fields. Properties like continuity, convergence, and limits are well-behaved in T_1 spaces, making them valuable in functional analysis and other areas of mathematics.

Let $(X, \Im)$ be a topological space. Then $(X, \Im)$ is said to be a $T_1 -$ space if for each distinct pair x, y then there exist two sets G and H such that

$$x \in G \text{ but } y \notin G$$

and

$$y \in H \text{ but } x \not\in H$$

Note: A topological space $(X, \Im)$ is said to be $T_1 -$ space if each singleton is closed.

Examples

Example.4. Let $X = \{a, b\}$ and $\Im = \{X, \phi, \{a\}, \{b\}\}$. To show that $(X, \Im)$ is a $T_1 -$ space.

Solution: Given that $X = \{a, b\}$ and $\Im = \{X, \phi, \{a\}, \{b\}\}$

$\Im -$ open sets are $\phi, X, \{b\}, \{a\}$.

$\Im -$ closed sets are $X, \phi, \{a\}, \{b\}$

Here each singleton is closed so the given space $(X, \Im)$vis a $T_1 -$ space

Another Solution: Let a and b are two distinct points (pair) of X there exist two open set $\{a\}$ and $\{b\}$ such that

$$a \in \{a\}, b \notin \{a\}$$

and $$b \in \{b\}, \mathrm{a} \notin \{b\}$$

Hence, $\{X, \Im)$ is a T_1-space.

Example 5: Let $X = \{1, 2, 3\}$ and $\Im = \{X, \phi, \{1\}, \{2\}, \{1, 2\}\}$.To show that $(X, \Im)$is a T_1-space.

Solution: Given That $X = \{1, 2, 3\}$

And $$\Im = \{X, \phi, \{1\}, \{2\}, \{1, 2\}\}$$

Here 1 and 2 are two distinct pair of X then there exist two open set $\{1\}$ and $\{2\}$ such that

$$1 \in \{1\}, 2 \notin \{1\}$$

and $$2 \in \{2\}, 1 \notin \{2\}$$

Hence, $(X, \Im)$is a T_1-space

11.5 Co-Finite Topology

The co-finite topology is also known as the finite complement topology. The co-finite topology is used in functional analysis, particularly in the study of the space of continuous functions. It provides a useful examples for understanding convergence and continuity in function spaces. The co-finite topology is one of the simplest examples of a non-trivial topology. It is easy to understand and provide a basic model for studying more complex topological spaces. Hence the co-finite topology is an important example in topology that helps to illustrate key concepts and has applications in various areas of mathematics, including number theory and functional analysis.

Let $A, B, C, ...,\subset X$ and $(X, \Im)$ be a topological space if $\Im$ be the collection of all subset of X whose complement is finite then $(X, \Im)$ is known as co-finite topology.

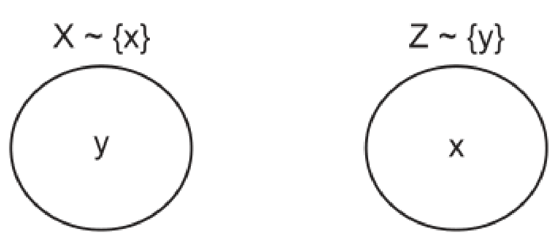
Theorem 5: Prove that a topological space $(X, \Im)$ is a T_1-space if and only if each singleton subset $\{x\}$ of X is closed.

Proof: Let $\{x\}$ is closed, $\forall x \notin X$

To show that the space is T_1-space

Let x, y be two distinct point of X .

$\Rightarrow \{x\}$ and $\{y\}$ are closed.



$\Rightarrow X \sim \{x\}$ is open set which does not contain x and $X \sim \{y\}$ is open set which does not contain y. $\Rightarrow$There exist a neighbourhood $X \sim \{x\}$ and $X \sim \{y\}$ are open sets such that

$$x \in X \sim \{y\} \text{ and } y \in X \sim \{x\}$$

$$x \notin X \sim \{x\} \text{ and } y \notin X \sim \{y\}$$

Hence, given space is $T_1 -$ space

Conversely, let the given space is $T_1 -$ space To show that $\{x\}$ is closed.

i.e., $X \sim \{x\}$ is open set.

Let $y \in X \sim \{x\} \Rightarrow y \neq x$

$\Rightarrow$There exist a open neighbourhood U_y of y which does not contain x

$\Rightarrow U_y \subset X \sim \{x\}$

$\Rightarrow U\left\{U_y : y \in X \sim \{x\}\right\} = X \sim \{x\}$

$\Rightarrow X \sim \{x\}$ is union of open set.

$\Rightarrow X \sim (x\}$ is open set

$\Rightarrow \{x\}$ is closed.

Theorem 6: Every subspace of a T_1-space is T_1-space.

Proof: Let $(X, \Im)$ be a T_1-space and (Y, v) be a subspace of $(X, \Im)$.

To show that (Y,v) is also a T_1-space

Let $x, y \in Y$ be arbitrary such that $x \neq y$.

Then $x, y \in X$ such that $\{y \subset x\}$

It is given that $X-$space is T_1.

This implies there exist an open set G and H of $\Im$ such that

$$x \in G, y \notin G$$

And $$x \notin H, y \in H$$

We have

$$x \in G \text{ and } x \in Y \Rightarrow x \in G \cap Y \text{ and } x \notin G \cap Y$$

$$y \in H \text{ and } y \in Y \Rightarrow y \in H \cap Y \text{ and } x \notin H \cap Y$$

Suppose $G \cap Y = P, H \cap Y = Q$ and we know that because

$$G, H \in \Im \Rightarrow G \cap Y, H \cap Y \in v$$

$$\Rightarrow P, Q \in v$$

Now we take a pair of distinct point $x, y \in Y$

$\Rightarrow$ there exist P and Q open sets of v such that

$$x \in P, y \in Q, x \notin P, y \notin Q$$

i.e., (y,v) is a T_1-space.

Theorem 7: Every T_1-space is T_0 -space but converse is not true.

Proof: Let $(X,\Im)$ be a T_1-space. Then there exist two distinct elements x_1, x_2 such that

$$x_1 \in U, x_2 \notin U$$

$$x_1 \in V, x_2 \notin V$$

Hence, there exists a neighbourhood U of x, which does not contain x_2. Therefore the given space is T_0-space. Conversely, let $X = \{a, b\}$ and $\Im = \{\mathrm{X}, \phi, \{a\}\}$.

If $a \neq b \Rightarrow$ there exists a neighbourhood $\{a\}$ of X such that

$$a \notin \{a\} \text{ and } b \notin \{a\}$$

i.e., the given space is T_0 -space.

Because each singleton is not a closed set therefore the space is not T_1-space or conversely, let $a, b \in X$.

$$\Rightarrow \text{There exists } \{a\} \text{ and } X \in \Im \text{ such that}$$

$$a \in \{a\}, b \notin \{a\}$$

And $$b \in X, a \in X$$

i.e., the space is not T_1 space.

Thus, every T_1-space is T_0 -space but converse is not true.

Theorem 8: Every finite T_1-space is a discrete space.

Proof: Let $(X,\Im)$ be a finite T_1-space and $A \subset X$..

To show that $(X,\Im)$ is a discrete space.

Let $A \subset X, X$ is finite

$\Rightarrow A$ is a finite set

Since X is $T_1 -$ space $\Rightarrow \{x\}$ is closed $\forall x \in X$

$A = \cup \{\{x\} : x \in A\}$

$=$ a finite union of closed set

$\Rightarrow A$ is closed set (2)

We know that a space X is T_1 if and only if every finite subset of X is closed

Also $X \sim A$ is finite set

$\Rightarrow X \sim A$ is closed set

$\Rightarrow A$ is open set.

Thus, we proved that every subset A of X is both open and closed. Hence, $(X, \Im)$ is a discrete space.

Theorem 9: A topological space $(X, \Im)$ is a T_1-space if and only if $\Im$ -contains the co-finite topology on X .

Proof:

Let $(X, \Im)$ be a T_1-space.

To show that $\Im$ contains co-finite topology on X. i.e., to show that $\Im$ contains subset A of X such that $X \sim A$ is finite.

Since it is given X is $T_1 -$ space

$\Rightarrow \{x\}$ is closed $\forall x \in X$

$\Rightarrow X \sim \{x\}$ is open subset of X

$\Rightarrow X \sim \{x\} \in \Im$

Thus, $X \sim \{x\} \in \Im \Rightarrow X \sim \left[X \sim \{x\}\right]$

$= \{x\}$ (finite)

This is true for all $x \in X$.

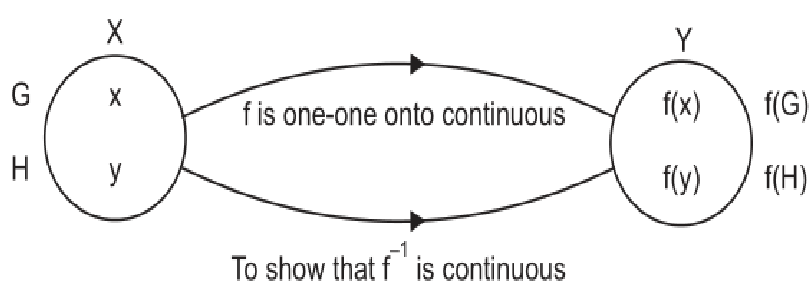
Using definition of confinite, $\Im -$ contains co-finite topology of X .

Theorem 10: Prove that a homeomorphism image of $T_1 -$ space is T_1.

Proof: Let $(X, \Im)$ be a $T_1 -$ space and $f : (X, \Im) \to (Y, v)$ be a homeomorphism.

To show that Y is T_1-space.

Sine it is given X is T_1-space



$\Rightarrow$ There exist G and H open set of such that

$x \in i, y \in H, x \notin H, y \notin G$

Because $x \neq y \Rightarrow f(x) \neq f(y)$

Since mapping is one-one onto (homeomorphism)

$\Rightarrow$ it is open mapping

$\Rightarrow f(G)$ and $f(H)$ are open in Y

$\Rightarrow f(X) \in f(G)$ and $f(x) \notin f(H)$

and $\Rightarrow f(y) \in f(H)$ and $f(y) \notin f(G)$

Hence, the space y is T_1-space.

11.6 T_2 –Space (Hausdorff Space)

Hausdorff spaces are important in topology for their separation properties, their applications in analysis and algebraic topology, and their role in understanding the structure of topological vector spaces. Hausdorff spaces satisfy the T_2 separation axiom, which states that for any two distinct points, there exist disjoint open sets containing each point. This separation property is crucial in distinguishing points and sets in a topological space.

Let $(X, \Im)$ be a topological space. Then $(X, \Im)$ is said to be a T_2 – space if for each distinct pair of element x and y there exist neighborhood N and M such that

$$x \in N, y \in M \text{ and } N \cap M = \phi.$$

Examples

Example.6: Let $X = \{1,2,3\}$ and $\Im = \{X, \phi, \{1,2\}, \{3\}\}$. Then show that $(X, \Im)$ is not a Hausdorff space.

Solution: Given that $X = \{1,2,3\}$

And $\qquad\qquad\qquad \Im = \{X, \phi, \{1,2\}, \{3\}\}$

For a,b distinct elements of X there are no disjoint neighbourhoods.

Hence, the given $(X, \Im)$ space is not a Hausdorff space.

Example 7: Show that an indiscrete space consisting of at least two point is not a Hausdorff space.

Solution: Let $\Im$ be an indiscrete topology on X consisting of at least two point. Then we have

$$\Im = \{X, \phi\}$$

This show that there exist no pair of non-empty disjoint open set.

Hence, $(X, \Im)$ is not a T_2 $-$space.

Theorem 11: Every discrete space is a Hausdorff space.

Proof: Let $(X, \Im)$ be a topological space and $x, y \in X$ be arbitrary such that $x \neq y.$ Using definition of discrete space, we have

$$\{x\} \text{ and } \{y\} \text{ are open set}$$

Obviously $\qquad\qquad \{x\} \cap \{y\} = \phi$

Hence, there exist disjoint open set $\{x\}$ and $\{x\}$ containing x and y respectively.

Thus, $(X, \Im)$ is Hausdorff space.

Theorem 12: Each singleton set in a Hausdorff space is closed.

Proof: Let $(X, \Im)$ be a Hausdorff space.

Since X is T_2 -space $\Rightarrow X$ is T_1-space $\Rightarrow \{x\}$ is closed for $x \in X$

Hence, each singleton set in a Hausdorff space is closed.

Theorem 13: Prove that every subspace of a Hausdorff space is T_2 .

Proof: Let $(X, \Im)$ be a Hausdorff space and (Y, v) be a subcpace of $(X, \Im)$

Let $y_1, y_2 \in Y$ such that $y_1 \neq y_2$

Since $(X, \Im)$ is a T_2-space.

This implies there exist disjoint neighbourhoods N_1 and N_2 of y_1 and y_2 respectively.

Now using the definition of neighbourhood, there exists $\Im-$open sets G_1 and G_2 such that

$$y_1 \in G_1 \text{ and } y_2 \in G_2$$

And $G_1 \cap Y$ and $G_2 \cap Y$ are disjoint $v-$open subsets. Hence, (Y, v) is also a T_2 -space.

Theorem 14: Every finite Hausdorff space is discrete.

Proof: Let $(X, \Im)$ be a finite T_2-space.

To show that $(X, \Im)$ be a discrete space.

Since X is T_2-space $\Rightarrow X$ is T_1space

$$\Rightarrow \{x\} \text{ is closed subset of } X, \forall x \in X$$

Let $A = \{a_1, a_2, ..., a_n\}$ be a finite subset of X.

Then $A = \{a_1\} \cup \{a_2\} \cup .... \cup \{a_n\}$

$=$ finite union of closed set

$=$ a closed set

Thus, every finite subset X is closed (1)

Because X is finite and $A \subset X$

$\Rightarrow X \sim A$ is finite

$\Rightarrow X \sim A$ is closed set (by (1))

$\Rightarrow A$ is open

Hence, every subset of X is closed as well as open i.e., X is a discrete space.

Theorem 15: Every T_2-space is a T_1-space but converse is not true.

Proof: Let $(X, \Im)$ be a T_2-space.

Let $x, y \in X$ such that $x \neq y$

This implies disjoint open set G and H of $\Im$ such that

$$x \neq G, y \in H, x \notin H, y \notin G$$

And $G \cap H = \phi$

Hence, given $x, y \in X$ such that $z \neq y$

This implies there exists G and H of $\Im$ such that

$$x \in G, y \notin G \quad \text{and} \quad y \in H, x \notin H$$

i.e., given space is T_1 – space.

Conversely, we prove this, in two parts. If $\Im$ is a co-finite topology on an infinite set X, then $(X, \Im)$ is T_1 space but not a T_2 -space. Let $\Im$ be a co-finite topology on X. To show that $(X, \Im)$ is a T_1 – space let $x \in X$ then $\{x\}$ is a finite set so that $X \cap \{x\}$ is $\Im$ – open set (using definition of topology).

Now let $x, y \in X$ such that $x \neq y$

We take $G = X \sim \{x\}$ and $H = X \sim \{y\}$, then G and H are open subsets of X.

Also $\quad x \notin H, y \in G, x \notin G, y \notin H$

i.e., $(X, \Im)$ is a T_1 – space.

Let $G, H \in \Im$ then by the definition of co-finite topology $X \sim G$ and $X \sim H$ are finite subsets of X.

We know that there does not exist any pair disjoint open set (suppose not).

But if we let G and H are disjoint open sets so that

$$G \cap H = \phi$$

Taking complement of both sides, we get

$$(G \cap H)' = \phi'$$

$$G' \cap H' = X$$

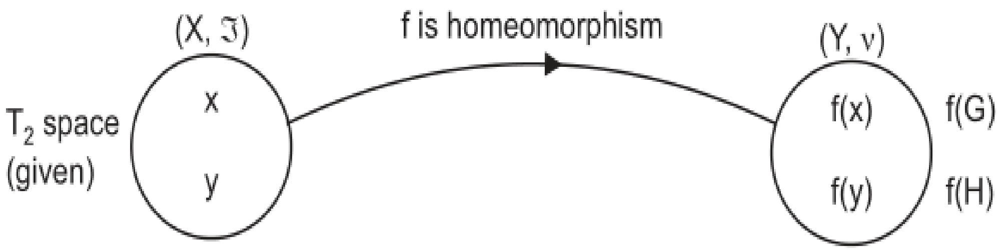
i.e., $\qquad (X \sim G) \cup (X \sim H) = X$

i.e., finite union of finite sets = an infinite set. This is impossible.

Hence there exist no pair of disjoint open sets. This implies $(X,\Im)$ is not a Hausdorff space. Therefore, every T_2-space is a T_1-space but converse is not true.

Theorem 16: Prove that a homeomorphic image of T_2-space is T_2.

Proof: Let $(X,\Im)$ and (Y,v) are two topological spaces let $(X,\Im)$ be a T_2 -space and to show that (Y,v) is also a T_2-space.



It is given that X is T_2 -space; $x \neq y$.

This implies there exists open sets Gand H such that

$$x \in G, y \in H$$

$$x \notin H, y \notin G \quad \text{and} \quad G \cap H = \phi \qquad (1)$$

Let $f(x), f(y) \in Y$ and $f(x) \neq f(y)$. The mapping is one-one

This implies there exist $f(G)$ and $f(H\}$ such that

$$f(x) \in f(G) \text{ and } f(y) \in f(H)$$

$$f(x) \notin f(H) \text{ and } f(y) \notin f(G)$$

And $$f(G) \cap f(H) = f(G \cap H)$$

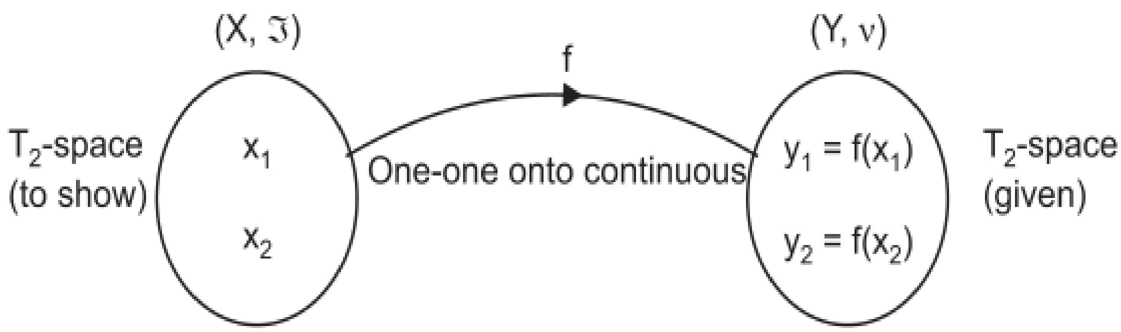
$$= f(\phi) \qquad \{\text{using (1)}\}$$

$$= \phi$$

This implies (Y, v) is T_2-space.

Theorem 17:Let $(X, \Im)$ be a topological space and (Y, v) be a T_2-space. Let $f: X \to Y$ be a one-one continuous mapping then X is also a T_2 -space.

Proof: We have



Let x_1, x_2 be any two distinct points of X

Since f is one-one $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

Let $y_1, y_2 \in Y$ $y_1 = f(x_1)$ and $y_2 = f(x_2)$ (by continuous mapping)

$$\Rightarrow x_1 = f^{-1}(y_1) \text{ and } x_1 = f^{-1}(y_2)$$

Since $y_1, y_2 \in Y$ such that $y_1 \neq y_2$

It is given that Y is T_2-space.

This implies there exist open sets G and H such that

$$y_1 \in G, y_2 \in H \text{ and } G \cap H = \phi$$

Since f is continuous, $f^{-1}(G)$ and $f^{-1}(H)$ are open.

Now $$f^{-1}(G) \cap f^{-1}(H) = f^{-1}(G \cap H) = f^{-1}(\phi) = \phi$$

And $$y_1 \in G \Rightarrow f^{-1}(y_1) \in f^{-1}(G) \Rightarrow x_1 \in f^{-1}(G)$$

$$y_2 \in H \Rightarrow f^{-1}(y_2) \in f^{-1}(H) \Rightarrow x_2 \in f^{-1}(H)$$

It is shown that every pair of disjoint points $x_1, x_2 \in X$.

This implies there exist open sets $f^{-1}(G)$ and $f^{-1}(H)$ such that

$$x_1 \in f^{-1}(G) \text{ and } x_2 \in f^{-1}(H)$$

$$f^{-1}(G) \cap f^{-1}(H) = \phi$$

This implies given space is T_2 (Hausdorff) space

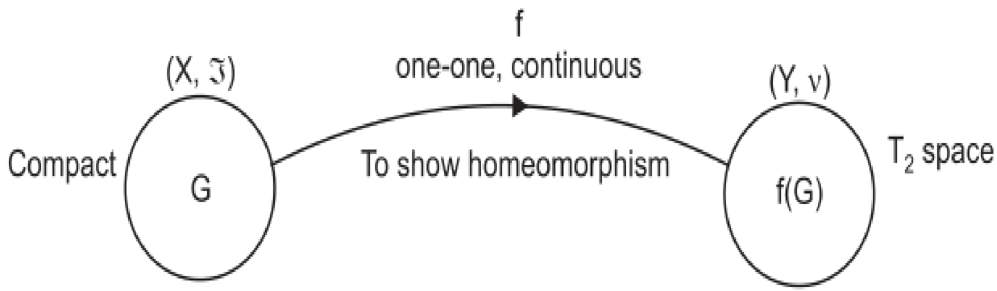
Hence, $(X, \Im)$ is a $T_2 -$space.

Theorem 18 : Prove that a one-one continuous mapping of a compact set onto Hausdorff space is a Homeomorphism.

Proof: Let $(X, \Im)$ be a compact space and (Y, v) is a $T_2 -$space and let $f : X \rightarrow Y$ be a one-one onto continuous mapping

To show that f is homeomorphism i.e., to show f is closed mapping.

Let G be any closed set of X



To show that f is homeomorphism i.e., we need to show that f is closed mapping.

Let G be any closed set of X

To show that $f(G)$ is closed set in Y.

(1) If $G = \phi$ then $f(G)$ is also null set, i.e., it is closed

(2) If $G \neq \phi$, since G is a closed subset of a compact set $X \Rightarrow G$ is compact.

We know that every closed subset of a compact set is compact.

And we know that every continuous subset of a compact set is compact.

$$\Rightarrow f(G) \text{ is a compact subset of } Y$$

It is given Y is T_2 -space.

We know that every compact subcet of a T_2 -space is closed.

This implies $f(G)$ is closed. Hence, f is homeomorphism.

11.5 Summary

Let $(X,\Im)$ be a topological space. Then $(X,\Im)$ is said to be To-space if and only if for distinct points x_1 and x_2 in X there exist a $\Im-$open set G such that

$$x_1 \in G \text{ and } x_2 \notin G \quad \text{or} \qquad x_2 \in G \text{ and } x_1 \notin G$$

Let $(X,\Im)$ be a topological space. Then $(X,\Im)$ is said to be T_1-space if for each distinct pair x, y then there exist two sets G and H such that

$$x \in G \text{ but } y \notin G$$

and $\quad y \in H$ but $x \not\in H$

A topological space $(X,\Im)$ is said to be T_1-space if each singleton is closed.

Let $A, B, C, ..., \subset X$ and $(X,\Im)$ be a topological space if $\Im$ be the collection of all subset of X whose complement is finite then $(X,\Im)$ is known as co-finite topology.

Let $(X,\Im)$ be a topological space. Then $(X,\Im)$ is said to be a T_2-space if for each distinct pair of elements x and y there exist neighborhood N and M such that

$$x \in N, y \in M$$

and $N \cap M = \phi$.

11.6 Terminal Questions

Q.1. Explain the T_0 and T_1 spaces with examples.

Q.2. Define the T_2 space with examples.

Q.3. To show that for a space X is T_1 if and only if every finite subset of X is closed

Q.4. A finite subset of a T_1-space has no limit point.

Q.5. If $(X, \Im)$ is a T_1-space and $\Im_1 \geq \Im$, then show that $(X, \Im_1)$ also a T_1-space.

Q.6. Prove that every finite Hausdorff space is discrete.

UNIT-12: Separation Axioms-II

Structure

12.1 Introduction

In topology, regular spaces and T_3 spaces are specific types of topological spaces that adhere to certain separation axioms, ensuring they behave predictably when it comes to separating points and closed sets. These spaces are significant in topology because they establish a context in which numerous foundational results in topology and analysis are valid. Their properties make them especially suitable for examining concepts like continuity, convergence, and other fundamental aspects of topology.

In topology, normal spaces, T_4 spaces, and completely normal spaces are distinguished types of topological spaces that conform to specific separation axioms, guaranteeing consistent behavior in separating points and closed sets. These spaces hold great significance in topology as they establish the foundation for many results and concepts in the field. They are particularly instrumental in studying the structure and properties of topological spaces.

12.2 Objectives

After reading this unit the learner should be able to understand about the:

- Regular Space and T_3 -Space
- Completely Regular Space
- $T_{3/2}$ –Space or Tychonoff Space
- Normal Space, T_4 –Space and Completely Normal Space
- T_5 –Space, Urysohn's Lemma and Urysohn Metrization Theorem
- Tietze-Extension Theorem

12.3 Regular Space

Regular spaces play a role in theoretical computer science, particularly in the study of computability and complexity. They provide a framework for understanding topological aspects of computation. Regularity is a fundamental property that helps us to understand the structure of topological spaces. It allows us to distinguish points and closed sets using open neighborhoods, providing a clearer picture of the space's internal arrangement. Regular spaces are well-suited for analysis and related fields. Properties like continuity, convergence, and limits are well-behaved in regular spaces, making them useful in functional analysis and other branches of mathematics.

Hence the regular spaces are essential in topology for their foundational role, their connection to metric spaces, their compatibility with analysis, their relationship with compactness, and their applications in theoretical computer science.

Let $(X, \mathfrak{I})$ be a topological space. Then $(X, \mathfrak{I})$ is said to be a regular space if given an element $x \in X$ and closed set $F \subset X$ such that

$$x \notin F$$

There exist disjoint open sets $G, H \subset X$ such that $x \in G, F \subset H$

or

Let $(X, \mathfrak{I})$ be a topological space. The $(X, \mathfrak{I})$ is said to be regular space if and only if for every closed set F and every point $p \notin F$, there exist $\mathfrak{I}-$ open sets G and H such that

$$p \in G, F \subset H \text{ and } G \subset H = \phi.$$

12.4 T_3 -Space

T_3 spaces are used in the study of topological dynamics, which deals with the behavior of continuous mappings on topological spaces. T_3 spaces provide a suitable setting for studying the dynamics of such mappings. T_3 spaces are used in the study of topological dynamics, which deals with the behavior of continuous mappings on topological spaces. T_3 spaces provide a suitable setting for studying the dynamics of such mappings.Hence the T_3 spaces are essential in topology for their fundamental properties, their generalization of metric spaces, their compatibility with analysis, their connection with compactness, and their applications in topological dynamics.

Let $(X,\Im)$ be a topological space. Then $(X,\Im)$ is said to be T_3-space if following conditions are satisfied:

(i) Given space is T_1.

(ii) For all $x \in X$ every neighborhood U of x there exist neighborhood V of x such that

$\bar{V} \subset U$.

Note: T_3-space T_1 – space + Regular space.

12.5 Completely Regular Space

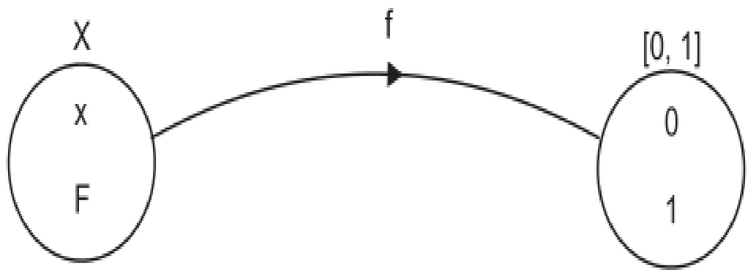
Completely regular spaces are important in topology for their generalization of T_2 spaces, their compatibility with analysis, their connection with normal spaces, their applications in functional analysis, and their role in algebraic topology. Completely regular spaces play a role in algebraic topology, particularly in the study of homotopy theory and homology theory. They provide a framework for understanding the topological properties of spaces in relation to their algebraic structures.

Let $(X,\Im)$ be a topological space, then $(X,\Im)$ is said to be completely regular space if it satisfies the following condition:

If F is a closed subset of X and $x \in X \sim F$ then there exist a continuous mapping

$f : X \to [0,1]$ such that

$f(x) = 0$ and $f(F) = 1$



12.6 $T_{3/2}$ –Space or Tychonoff Space

$T_{3/2}$ spaces generalize the separation properties of regular spaces (T_3 spaces) by adding the T_0 separation axiom. This additional property allows for finer distinctions between points and closed sets, leading to a more refined understanding of topological spaces.$T_{3/2}$ spaces are used in functional analysis to study topological vector spaces and other structures. Their properties make them useful for understanding the behavior of linear operators and function spaces. Hence the $T_{3/2}$ spaces are important in topology for their generalization of regular spaces, their compatibility with analysis, their connection with compact spaces, their applications in functional analysis, and their role in algebraic topology.

A completely regular T_1-space is known as tychonoff or tichonoy or $T_{3\frac{1}{2}}$ space.

12.7 Normal Space

Normality extends the separation properties of Hausdorff (T2) spaces by ensuring that any two disjoint closed sets can be separated by disjoint open neighborhoods. This property is essential for many topological constructions and arguments. Normal spaces play a role in the study of topological dynamics, which deals with the behavior of continuous mappings on topological spaces. Normal spaces provide a suitable setting for studying the dynamics of such mappings. Hence the normal spaces are essential in topology for their structural understanding, compatibility with analysis, connection with compactness, extension of separation axioms, and applications in topological dynamics.

Let $(X,\Im)$ be a topological space. Then $(X,\Im)$ is said to be a normal space if and only if for every pair of disjoint closed sets $F_1,\ F_2 \subset H$ there exist $\Im-$open sets G and H such that

$$F_1 \subset G, F_2 \subset H$$

and $$G \cap H = \phi.$$

12.8 T_4–Space and Completely Normal Space

T_4 spaces are used in the study of topological dynamics, which deals with the behavior of continuous mappings on topological spaces. T_4 spaces provide a suitable setting for studying the dynamics of such mappings. Hence the T_4 spaces are important in topology for their fundamental properties, their generalization of T_2 spaces, their compatibility with analysis, their connection with compactness, and their applications in topological dynamics.

A normal T_1-space is known as T_4-space.

Let $(X,\Im)$ be a topological space. Then $(X,\Im)$ is said to be completely normal space if there exist two separated sets A and B of X such that

$$A \subset G, B \subset H \text{ and } G \cap H = \phi.$$

Note: Two sets A and B are separated if

$$A = \phi, B \neq \phi, \bar{A} \cap B = \phi, A \cap \bar{B} = \phi.$$

12.9 T_5–Space

In topology, a T_5 space, also known as a perfectly normal space, is a topological space that satisfies the T_4 separation axiom and is also completely regular T_5 spaces are important in topology for their generalization of T_4 spaces, their compatibility with analysis, their applications in topological dynamics, and their connection with compactness.

A completely normal T_1-space is known as T_5-space.

Examples

Example.1: Let $(X,\Im)$ be a topological space. Let $X=\{a,b,c\}$ and $\Im=\{X,\phi,\{a\},\{b,c\}\}$. To show that $(X,\Im)$ is normal and regular space but not T_3 and T_4-space.

Solution: Given that

$$X=\{a,b,c\} \text{ and } \Im=\{X,\phi,\{a\},\{b,c\}\}$$

$\Im$ -open sets are $x,\phi,\{a\},\{b,c\}$

$\Im$ -closed sets are $\phi,X,\{b,c\},\{a\}$

Let a pair of distinct closed sets

$$\{a\},\{b,c\}\in X$$

This implies there exist disjoint open sets $\{a\},\{b,c\}\in X$ such that

$$\overset{open}{\{a\}}\subset\overset{closed}{\{a\}},\overset{open}{\{b,c\}}\subset\overset{closed}{\{b,c\}}$$

and $$\{a\}\cap\{b,c\}=\phi$$

Hence, given space is normal.

Now let a pair $a\in X$ and closed set $\{b,c\}\in X$ such that $a\notin\{b,c\}$.

This implies there exist disjoint open sets $\{a\}$ and $\{b,c\} \subset X$ such that

$$a \in \{a\}, \{b,c\} \subset \{b,c\}$$

And $$\{a\} \cap \{b,c\} = \phi$$

Hence, given space is also regular space.

Now consider a pair of distinct elements $b,c \in X$ then the only open set containing either of the element $b_1 c$ are X and $\{b,c\}$ such that

$$b \in X, C \in \{b,c\}, c \in X, b \in \{b,c\}$$

Hence, given space is not T_1-space

Because every singleton is not closed therefore the given space is not T_1 space.

Hence, the given space is not T_3 and T_4 space.

Therorem.1: Every $T_3 -$ space is T_2 -space

Proof: We know that a regular T_1-space is called a T_3-space.

Let $(X, \Im)$ be a $T_3 -$ space

Let x, y be any two distinct points of x using definition of $T_3 -$ space.

This implies X is also a T_1-space and so $\{x\}$ is a closed set.

Also $$y \notin \{x\}$$

Since X is a regular space.

This implies there exist open sets G and H such that

$\{x\} \subset G, y \in H$

And $G \cap H = \phi$

Also $\{x\} \subset G \Rightarrow x \in G$

$\Rightarrow x, y$ belong respectively two disjoint open set G and H.

$\Rightarrow x \in G, y \in H, x \notin H, y \notin G$

And $G \cap H = \phi$

i.e., given space is $T_2 -$space.

Theorem.2: If $(X, \Im)$ is a $T_4 -$space then it is also $t_3 -$ space, i.e., If $(X, \Im)$ is a normal space then it is also regular space.

Proof: Let $(X, \Im)$ be a T_4 -space, i.e.,

(1) X is T_1-space and

(2) X is normal space.

To show that $(X, \Im)$ is T_3 -space i.e.,

(1) X is T_1-space and

(2) X is regular space

It is given that X is T_4 -space $\Rightarrow x$ is $T_1 -$space

$\Rightarrow \{x\}$ is closed in X

X is normal space $\Rightarrow$ given a pair of distinct closed set $\{x\}$ and $F \subset X$ such that there exist disjoint open set G and H of X such that

$$\{x\} \subset G, F \subset H$$

i.e., given a closed set F and $x \in X$ such that $x \notin F$ there exist disjoint open sets G and H of X such that

$$x \in G, F \subset H$$

Thus, space is regular space

Hence, if $(X, \Im)$ is a T_4 -space then it is also T_3-space.

Theorem.3: Let $(X, \Im)$ be a topological space. If $(X, \Im)$ is a T_4 -space then it is also T_2 – space.

Proof: Let $(X, \Im)$ be a T_4 -space i.e.,

(1) X is T_1-space and

(2) X is normal space

T_0 show that X is T_2 – space.

Let $x, y \in X$ be arbitrary such that $x \neq y$ because X is T_1-space

$\Rightarrow \{x\}$ and $\{y\}$ are disjoint closed set in X

Also X is normal space

$\Rightarrow$ given a pair of disjoint closed set $\{x\}, \{y\} \subset X$

There exist disjoint open sets $G, H \in \Im$ such that

$$\{x\} \subset G, \{y\} \subset H$$

i.e., $x \in G, y \in H$

Given $x, y \in X$ such that $x \neq y$

$\Rightarrow$ there exist disjoint open sets G and H such that

$$x \in G, y \in H$$

i.e., space is T_2 $-$space. Hence, T_4 $-$space is also T_2 $-$space.

Theorem.4: A topological space X is said to be regular space if and only if for every $x \in X$ and every neighbourhood U of x there exists a neighbourhood H of x such that

$$\bar{H} \subset U$$

Proof: Let $(X, \Im)$ be a regular space.

To show that a neighbourhood N of x there exists a neighborhood M of x such that

$$\bar{M} \subset N$$

Since x is regular space $\Rightarrow$ a closed set F and a element $x \in X$ such that $x \notin F$ there exist disjoint open sets G and H of X such that

$$x \in G, F \subset H \text{ and } G \cap H = \phi$$

$x \in G \Rightarrow G$ is a neighbourhood of x.

$$G \cap H = \phi \Rightarrow G \subset X \sim H$$

$$\Rightarrow \bar{G} \subset \overline{X \sim H} \qquad \text{(using closure property)}$$

$$\Rightarrow \bar{G} \subset X \sim H \qquad \left(\bar{A} = A, \text{ if } A \text{ is closed}\right)$$

Here $X \sim H$ is closed because H is open

$$\Rightarrow \bar{G} \subset X \sim H \qquad (1)$$

Now we have $\qquad F \subset H$

$$X \sim F \supset X \sim H$$

$$X \sim H \subset X \sim F \qquad (2)$$

Using (1) and (2), we have

$$\bar{G} \subset X \sim H \subset X \sim F$$

$$\Rightarrow \bar{G} \subset X \sim F$$

$\because F$ is closed $\Rightarrow X \sim F$ is open

i.e., a neighbourhood $X \sim F$ of x there exist a neighborhood G of x such that

$$x \in G \subset \bar{G} \subset X \sim F$$

$$\Rightarrow \bar{G} \subset X \sim F$$

Theorem.5: Prove that every closed subspace of a normal space is normal.

Proof: Let $(X, \mathfrak{I})$ be a normal space and (Y, v) be a closed subspace of $(X, \mathfrak{I})$.

To show that (Y, v) is also normal space.

Let U and Vare disjoint v closed subset of $Y \qquad (1)$

This implies there exist closed subsets N and M of X such that

$$U = N \cap Y \qquad \text{using relative topology}$$

$$V = M \cap Y$$

It is given Y is closed $\Rightarrow U$ and V are disjoint closed subsets of X .

It is given also X is normal space $\Rightarrow$There exist open sets G and H of X such that

$$U \subset G, V \subset H \text{ and } G \cap H = \phi$$

Using (1) and (2)

$$U \subset G \cap Y, V \subset H \cap Y$$

And $$(G \cap Y) \cap (H \cap Y) = (G \cap H) \cap Y$$

$$= \phi \cap Y \qquad (G \cap H = \phi)$$

$$= \phi$$

Also $$U \subset Y, V \subset G \Rightarrow U \subset Y \cap G \in v$$

$$V \subset Y, V \subset H \Rightarrow V \subset Y \cap H$$

And $$(Y \cap G) \cap (Y \cap H) = \phi, U \cap V = \phi$$

i.e., (Y, v) is normal space.

Hence, every closed subspace of a normal space is normal.

Theorem.6: Prove that a topological space X is normal if and only if for every closed set F and open set G containin F there exist an open set V such that

$$F \subset V \text{ and } \bar{V} \subset G$$

Proof: Let X be a normal space and let F be any closed set and G be an open set such that

$$F \subset G$$

$$\Rightarrow X \sim G \text{ is closed set}$$

$$F \cap (X \sim G) = \phi$$

Thus, $X \sim G$ and F are disjoint closed substs of X .

It is also given X is normal $\Rightarrow$there exist two open sets U and V such that

$$(X \sim G) \subset U, F \subset V \text{ and } U \cap V = \phi$$

So that $$V \subset X \sim U \text{ but } V \subset X \sim U \Rightarrow \bar{V} \subset \overline{(X \sim U)}$$

$$\Rightarrow \bar{V} \subset X \sim U \{X \sim U \text{ is closed}\} \quad (1)$$

We know that the closure of a closed set is closed.

$$\Rightarrow x \sim G \subset U \Rightarrow X \sim (X \sim G) X \sim U \quad (2)$$

$$\Rightarrow G \supset X \sim U$$

Using equations (1) and (2), we get $V \subset G$.

This implies there exists an open set V such that

$$F \subset V \text{ and } \bar{V} \subset G$$

Conversely, let the above conditions hold.

Let A and B be closed subsets of X and $A \cap B = \phi$

To show that space is normal we have $A \cap B = \phi \Rightarrow A \subset X \sim B \Rightarrow$ closed set A is contained in open set $X \sim B$.

It is given that there exists an open set V such that

$$A \subset V \text{ and } \bar{V} \subset X \sim B \Rightarrow B \subset X \sim \bar{V}$$

Also $$V \cap \left(X \sim \overline{V} \right) = \phi$$

Thus, V and $X \sim \overline{V}$ are two disjoint open sets such that

$$A \subset V \text{ and } X \sim \overline{V}$$

Hence the given space is normal.

Theorem.7: Prove the every completely normal space is normal and hence, T_5-space is a T_4-space.

Proof: Let X be a completely normal space.

To show that X is also normal space.

Let A and B be any two closed subsets of X such that

$$A \cap B = \phi$$

Since A and B are closed, we have

$$\overline{A} = A \text{ and } \overline{B} = B$$

And $$\overline{A} \cap B = \phi, A \cap \overline{B} = \phi$$

$$\Rightarrow A, B \text{ are separated subsets of } Y$$

Using completely normality

There exist open sets $A \subset G, B \subset H$ and

$$G \cap H = \phi$$

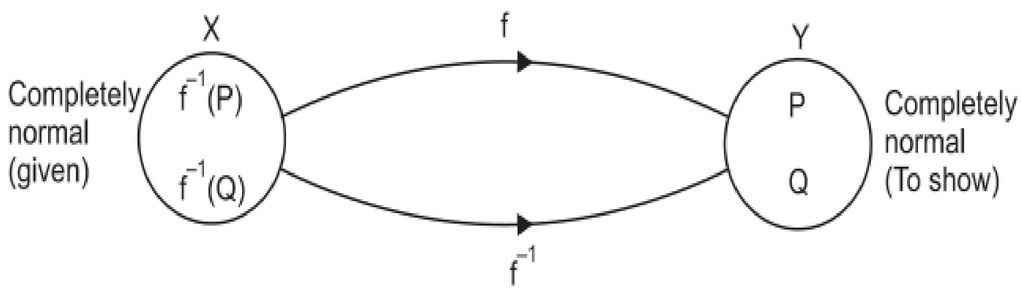
$$\Rightarrow X \text{ is normal space.}$$

Also we know that $T_5 -$space is completely normal $T_1 -$space and T_4 is a normal T_1-space hence,

T_5-space is a T_4-space.

Theorem.8: Let $f: X \to Y$ be a homeomorphism where XY are topological space and X is completely normal space. Then show that Y is also a completely normal space.

Proof: Let P and Q are subsets of Y such that P and Q are separated set, i.e., $\bar{P} \cap Q = \phi, P \cap \bar{Q} = \phi$ Given $f: X \to Y$ be a homeomorphism



$$\Rightarrow f^{-1}(P) \text{ and } f^{-1}(Q) \text{ are subset of } X$$

Since mapping is continuous

$$\Rightarrow \overline{\left[f^{-1}(P)\right]} \subset f^{-1}\left(\bar{P}\right) \text{ and } \overline{\left[f^{-1}(Q)\right]} \subset f^{-1}\left(\bar{Q}\right)$$

Now $$\overline{\left[f^{-1}(P)\right]} \cap f^{-1}\left(\bar{Q}\right) \subset f^{-1}\left(\bar{P}\right) \cap f^{-1}(Q)$$

$$= f^{-1}\left(\bar{P} \cap Q\right)$$

$$= f^{-1}(\phi)$$

$$= \phi$$

Similarly $$f^{-1}(P) \cap \overline{\left[f^{-1}(Q)\right]} = \phi$$

Hence, it is given X is completely normal

$\Rightarrow$ there exist open sets G and H of X such that

$$f^{-1}(P) \subset G, f^{-1}(Q) \subset H \text{ and } G \cap H = \phi$$

$$f^{-1}(P) \subset G \Rightarrow P \subset f(G)$$

$$f^{-1}(Q) \subset H \Rightarrow Q \subset f(H)$$

$$\text{and } f(G \cap H) = \phi$$

For two separated set P and Q of Y.

This implies there exist $f(G)$ and $f(H)$ open sets such that

$$P \subset f(G), \quad Q \subset F(H)$$

And $$f(G) \cap f(H) = \phi$$

Hence, the space Y is completely normal.

12.10 Urysohn's Lemma

Urysohn's Lemma is widely used in topology and related areas of mathematics. It is a key tool in the proof of many important theorems, including the Tietze Extension Theorem and the Stone-Weierstrass Theorem. Urysohn's Lemma is also essential in the study of topological vector spaces, functional analysis, and other branches of mathematics where understanding the structure of topological spaces is crucial.

Let F_1, F_2 **be any pair of disjoint closed sets in a normal space** X**. Then there exists a**

continuous mapping $f : X \to [0,1]$ such that $f(x) = 0$ for $x \in F_1$ and $f(x) = 1$ for $x \in F_2$.

Proof: Let X be a normal space and let F_1 and F_2 be any two disjoint closed sets in X. Then $F_1 \cap F_2 = \phi$

$$\Rightarrow F_1 \subset F_2{}' \text{ which is open set}$$

$\Rightarrow$ there exists an open set $G_{1/2}$ such that

$$f_1 \subset G_{1/2} \subset \bar{G}'_{1/2} \subset F_2'$$

Here $G_{1/2}$ and F_2' are open sets containing the closed set F_1 and $G_{1/2}$ respectively as same way there exist open sets $G_{1/4}$ and $G_{3/4}$ such that

$$F \subset G_{1/4} \subset \bar{G}_{1/4} \subset G_{1/2} \subset \bar{G}_{1/2} \subset G_{3/4} \subset \bar{G}_{3/4} \subset F_2'$$

Counting in this manner, for each rational number in $]0,1[$ of the form

$$r = \frac{m}{2^n} \qquad (\text{where } n = 1, 2, \ldots, m = 1, 3, \ldots., 2^{n-1})$$

We obtain an open set of the form G_r such tha

$$r < s \Rightarrow F_1 \subset G_r \subset \bar{G}_r \subset G_s \subset \bar{G}_s \subset F_2' \qquad (1)$$

Let we denote the set of all such rational numbers r by D. Now we define a function

$$f(x) = \begin{cases} 1, & x \in F_2 \\ \inf \{r : r \in D_1 x \in G_r\} & x \notin F_2 \ i.e., x \in F_2 \end{cases}$$

If $r \in F_1$, then $x \in G_r$ for all $r \in D$ by (1)

Using definition of f, we have

$$f(x) = \inf. D = 0$$

Thus $\qquad f(x) = 0$ wherever $x \in F_1$

And $\qquad f(x) = 1$ for $x \in F_2$

And $\qquad 0 \le f(x) \le 1, \ \forall x \in X$

It remains to show that f is continuous.

Here $[0,1]$ is a topological space with its relative topology. Clearly all intervals of the form $[0,a[$ and $[b,1[$ where $0 < a < 1$ and $0 < b < 1$.

Here $0 \le f(x) < a$ if and only if $x \in G_r$

$$\Rightarrow f^{-1}\{[0,a[\} = \{x \in G : 0 \le f(x) < a\}$$

$$= U\{G_r : r \in D, r < a\}$$

Thus, G_r is an open set and we know that union of open sets is open. Thus, the inverse image of an open set is open. Similarly, $f^{-1}\,]b,1]$ is also an open set in X. Hence f is continuous mapping. This proves the theorem.

12.11 Urysohn Metrization Theorem

The Urysohn Metrization Theorem is a fundamental result in topology that provides a characterization of metrizable topological spaces. It states that a topological space is metrizable if and only if it is regular (T_3) and has a countable basis. Hence the Urysohn Metrization Theorem is important in topology for its characterization of metrizable spaces and its applications in analysis, topology design, and compactness considerations.

Every regular space X with a countable basis is metrizable.

Note:1. A topological space $(X,\Im)$ is known as second countable space if there exists a countable base for the topology $\Im$.

2. Every second countable normal space is metrizable.

3. If X is a second countable normal space then there exists a homeomorphism $f : X \to R^{\infty}$ and so X is metrizable.

12.12 Tietze-Extension Theorem

The Tietze Extension Theorem is a result in topology that provides conditions under which a continuous function defined on a closed subset of a topological space can be extended to a continuous function defined on the entire space. Hence the Tietze Extension Theorem is an important result in topology and analysis, providing a powerful tool for extending functions and characterizing normal spaces.

A topological space $(X,\Im)$ is normal if and only if for every real valued continuous mapping f of a closed subset F of x into the closed interval $[a,b]$ there exists a real valued continuous mapping g of X into $[a,b]$ such that $g/F = f$ i.e., g is a continuous extension of f over X.

Proof: Suppose for every real valued continuous mapping f of a closed subset F of X into $[a,b]$, there exist a continuous extension of f over X.

To show that $(X,\Im)$ is normal space

Let F_1 and F_2 be two closed subsets of X such that

$F_1 \cap F_2 = \phi$ and let $[a,b]$ be any closed interval

We define a mapping

$$f : F_1 \cup F_2 \to [a,b]$$

Such that

$$f(x) = a \text{ if } x \in F_1$$

$$f(x) = b \text{ if } x \in F_2$$

Let H be any closed subset of $[a,b]$ then

$$f^{-1}(H) = \begin{cases} F_1 & \text{if } a \in H \text{ and } b \notin H \\ F_2 & \text{if } b \in H \text{ and } a \notin H \\ F_1 \cup F_2 & \text{if } a \in H \text{ and } b \in H \\ \phi & \text{if } a \notin H \text{ and } b \notin H \end{cases}$$

Thus, the function f is continuous

By hypothesis, there exists a continuous extension, namely, g of f over X, i.e., $g : X \to [a,b]$ such that

$$g(x) = \begin{cases} a & \text{if } x \in F_1 \\ b & \text{if } x \in F_2 \end{cases}$$

This implies g satisfied all the conditions of Urusohn lemma, hence $(X,\Im)$ is a normal space.

Conversely. Let $(X,\Im)$ be a normal space and let f be a real valued continuous mapping of the closed set F into the closed interval $[a,b]$. For numerical convenience, we define a function

$$f_0 : F \to [-1,1]$$

By setting $$f_0(x) = f(x) \ \forall x \in F \qquad (1)$$

Let $$G_0 = f_0^{-1}\left[\left[-1,-\frac{1}{3}\right]\right], H_0 = f_0^{-1}\left[\left[\frac{1}{3},1\right]\right]$$

Since $\left[-1,-\frac{1}{3}\right]$ and $\left[\frac{1}{3},1\right]$ are closed in $[-1,1]$ and f_0 is continuous. If follows that G_0 and H_0 are closed in F and so also closed in X.

$$G_0 \cap H_0 = \left\{ f_0^{-1}\left[\left[-1,\frac{1}{3}\right]\right] \cap f_0^{-1}\left[\left[-\frac{1}{3},-1\right]\right]\right\}$$

$$= \left[\left\{ f_0^{-1}\left[\left[-1,\frac{-1}{2}\right]\right]\right\} \cap \left[\frac{1}{3},1\right]\right]$$

$$= f_0^{-1}\{\phi\}$$

$$= \phi$$

Thus, G_0 and H_0 are disjoint closed subsets of X.

Since X is normal.

By Urysohn's lemma, there exists a continuous mapping

$$g_0 : X \to \left[-\frac{1}{3},\frac{1}{3}\right]$$

Such that $$g_0[G_0] = \left\{-\frac{1}{3}\right\} \text{ and } G_0(H_0) = \left\{\frac{1}{3}\right\}$$

Now again we define a mapping

$$f_1 : F \to \left[-\frac{2}{3},\frac{2}{3}\right]$$

By setting

$$f_1(x) = f_0(x) - g_0(x)$$

Since f_0, g_0 are continuous, f_1 is also a continuous mapping.

Now let $$G_1 = f_1^{-1}\left[\left[-\frac{2}{3}, -\frac{1}{3}\left(\frac{2}{3}\right)\right]\right]$$

$$H_1 = f_1^{-1}\left[\left[\frac{1}{3}\left(\frac{2}{3}\right), \frac{2}{3}\right]\right]$$

This implies G_1, H_1 are disjoint closed sets of X .

By Uryshan's lemma, there exist a continuous mapping

$$g_1 X - \left[-\frac{1}{3}\left(\frac{2}{3}\right), \frac{1}{3}\left(\frac{2}{3}\right)\right]$$

Such that

$$g_1[G_1] = \left\{-\frac{1}{3}\left(\frac{2}{3}\right)\right\}$$

And $$G_1(H_1) = \left\{\frac{1}{3}\left(\frac{2}{3}\right)\right\}$$

Now again define a mapping

$$f_2 : F \to \left[-\left(\frac{2}{3}\right)^2 . \left(\frac{2}{3}\right)^2\right]$$

By setting $$f_2(x) = f_1(x) - g_1(x)$$

$$= f_0(x) - g_0(x) - g_1(x) \quad \forall x \in F$$

Observe as before that f_2 is continuous. proceeding this process in same way:

$$g_n : X \to \left[-\frac{1}{3}\left(\frac{2}{3}\right)^n, \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^n \right] \quad \forall n = 0,1,2,....,m-1$$

We define a mapping

$$f_m : F \to \left[-\left(\frac{2}{3}\right)^m, \left(\frac{2}{3}\right)^m \right]$$

By setting $\qquad f_m(x) = f_0(x)$

$$= \sum_{n=0}^{m-1} g_n(x) \quad \forall x \in F$$

Set $\qquad G_m = f_m^{-1}\left[\left[-\left(\frac{2}{3}\right)^m, -\frac{1}{3}\left(\frac{2}{3}\right)^m \right]\right]$

And $\qquad H_m = f_m^{-1}\left[\left[\frac{1}{3}\left(\frac{2}{3}\right)^m, \left(\frac{2}{3}\right)^m \right]\right]$

Since $\qquad H_m = f_m^{-1}\left[\left[\frac{1}{3}\left(\frac{2}{3}\right)^m, \left(\frac{2}{3}\right)^m \right]\right]$ and $\left[\frac{1}{3}\left(\frac{2}{3}\right)^m, \left(\frac{2}{3}\right)^m \right]$

Are disjoint closed subsets of $[-1,1]$ and f_m is a continuous mapping

Now

$$g(x) = \sum_{n=01}^{\infty} g_n(x) \quad \forall x \in X$$

And show that g is continuous extension of f over X are have

$$\left|g(x)\right| = \left|\sum_{n=0}^{\infty} g_n(x)\right| \leq \sum_{n=0}^{\infty}\left|g_n(x)\right| \leq \sum_{n=0}^{\infty}\frac{1}{3}\left(\frac{2}{3}\right)^n = \frac{\frac{1}{3}}{1-\frac{2}{3}} = 1.$$

So by weierstress's $M-$test the series $\sum_{n=0}^{\infty} g_n(x)$ converges uniformly and abolutely over X and since each $g_n(x)$ is continuous if follow that g is a continuous mapping of X into $[-1,1]$.

Now $\qquad \left|f_m(x)\right| \leq \left(\frac{2}{3}\right)^m$ which $\to 0$ as $m \to \infty$

Since $\qquad f_m(x) = f_0(x) = \sum_{n=0}^{m-1} g_n(x) \forall x \in F.$

We have $\qquad \lim_{m\to\infty} f_m(x) = f_0(x) - \lim_{m\to\infty}\sum_{n=0}^{m-1} g_n(x)$

Hence, $\qquad 0 = f_0(x) - g(x) \quad \forall x \in F$

i.e., $\qquad g(x) = f_0(x) - f(x) \quad \forall x \in F$

Hence, g is continuous extension of f over X .

12. 13 Summary

Let $(X,\Im)$ be a topological space. Then $(X,\Im)$ is said to be a regular space if given an element $x \in X$ and closed set $F \subset X$ such that $x \notin F$ there exist disjoint open sets $G, H \subset X$ such that $x \in G, F \subset H$.

Let $(X, \Im)$ be a topological space. Then $(X, \Im)$ is said to be T_3-space if following conditions are satisfied: (i) Given space is T_1. (ii) For all $x \in X$ every neighborhood U of x there exist neighborhood v of x such that $\overline{V} \subset U$.

Let $(X, \Im)$ be a topological space, then $(X, \Im)$ is said to be completely regular space if it satisfies the following condition:

If F is a closed subset of X and $x \in X \sim F$ then there exist a continuous mapping

$f : X \to [0,1]$ such that $f(x) = 0$ and $f(F) = 1$.

A completely regular $T_1 -$space is known as tychonoff or tichonoy or $T_{3\frac{1}{2}}$ space

Let $(X, \Im)$ be a topological space. Than $(X, \Im)$ is said to be a normal space if and only if for every pair of disjoint closed sets $F_1, F_2 \subset X$. This implies there exist $\Im -$open sets G and H such that $F_1 \subset G,\ F_2 \subset H$ and $G \cap H = \phi$.

A normal $T_1 -$space is known as T_4-space.

Let $(X, \Im)$ be a topological space. Then $(X, \Im)$ is said to be a completely normal space if there exist two separated sets A and B of X such that $A \subset G, B \subset H$ and $G \cap H = \phi$.

A completely normal $T_1 -$space is known as $T_5 -$space.

Every regular space X with a countable basis is metrizable.

A topological space $(X, \Im)$ is normal if and only if for every real valued continuous mapping f of a closed subset F of x into the closed interval $[a,b]$ there exists a real valued continuous mapping g of X into $[a,b]$ such that $g / F = f$ i.e., g is a continuous extension of f over X.

12.14 Terminal Questions

Q.1. Write a short note for regular and normal space.

Q.2. What do you mean by $T_3$and T_4spaces?

Q.3. Explain the Simpson's 3/8 rule.

Q.4. State and prove the Urysohn Metrization Theorem.

Q.5. Give a counter example to show that a regular space is not necessarily a T_1-space.

Q.6. Prove that every indiscrete space is regular.

Q.7. If $\Im = \{X, \phi, \{a,\}\{a,b\}, \{a,c\}, \{a,b,c\}$ is a topology on $X = \{a,b,c,d\}$, then prove that $\{X, \Im)$ is a normal topological space.

Q.8. Show that every disceret topological space is a T_4-space.

Q.9. Prove that the property of being a T_5-space is a hereditary property.

Q.10. Prove that a regular lindelof space is normal.

Answer

5. $X = \{a,b,c\}$ and $\Im = \{X, \phi, \{a,b\}, \{c\}\}$.

UNIT-13: Connectedness

Structure

13.1 Introduction

Connectedness is a key concept in topology with important implications in various areas of mathematics and science. It helps to classify and understand the structure of spaces, and it forms the basis for many theorems and results in topology and related fields. In this unit we shall discuss another important property of topological spaces known as connectedness. This unit deals with connected and disconnectd sets, connectedness on the real line, components, maximal connected set, locally connected space and totally disconnected set.

Connectedness can also be characterized in terms of paths. A space is path-connected if, for any two points in the space, there exists a continuous path (a continuous map from the unit interval [0, 1] to the space) that connects the two points. Hence the connectedness is an important property because it captures the idea of "wholeness" or "integrity" of a space. Intuitively, a connected space cannot be broken apart into pieces that are not somehow "linked" or "connected" to each other.

13.2 Objectives

After reading this unit the learner should be able to understand about the:

- Separated Sets
- Connected Set and Disconnected Set
- Connectedness on the Real Line
- Components
- Maximal Connected Set and Locally Connected Space
- Totally Disconnected Set.

13.3 Separated Set

In topology, the concept of separated sets refers to sets that can be distinguished in a certain way by open sets. There are several types of separation axioms that define different levels of separation between sets.

Two subsets A and B of a topological space $(X, \Im)$ are said to be separated if any only if

$$A \cap \bar{B} = \phi \text{ and } \bar{A} \cap B = \phi.$$

13.4 Connected Set and Disconnected Set

Connectedness and disconnectedness are important concepts in topology that describe how a topological space can splitted into different parts. Connectedness and disconnectedness are related to separation axioms in topology. For example, a space is disconnected if and only if it violates the T_1 separation axiom, meaning there exist two points that cannot be separated by open sets. Connectedness and disconnectedness are fundamental concepts in topology with wide-ranging applications in various fields, including topology, analysis, geometry, and image processing. They provide a framework for understanding the structure and properties of topological spaces and geometric objects.

A set X is said to be connected if there does not exist any non-empty proper subset of X which is both open and closed.

Let $(X, \Im)$ be a topological space then X is said to be disconnected if and only if there exist two disjoint non-empty subsets A and B of X such that

(i) $\bar{A} \cap B = \phi$ and $A \cap \bar{B} = \phi$ (ii) $A \cup B = X$

Note: 1. If X is not the union of two separated subsets of X then X is said to be connected set.

2. A set X is said to be disconnected if there exists a non-empty proper subset of X which is both open and closed.

3. A set X is said to be disconnected if there exist two separated sets A and B such that $A \cup B = X$

Examples

Example.1: If $X = \{a,b,,c,d\}$ and $\Im = \{X,\phi,\{a\},\{b,c\},\{a,b,c\}\}$. Then show that the topological space is connected.

Solution: Given that $X = \{a,b,c,d\}$ and $\Im = \{X,\phi,\{a\},\{b,c\},\{a,b,c\}\}$

$\Im -$ open sets are $X,\phi,\{a\},\{b,c\},\{a,b,c\}$

$\Im -$ closed sets are $\phi, X\ \{b,c,d\},\{a,d\},\{d\}$

This implies there does not exist any non-empty proper subset of X which is both open and closed in X

Hence, $(X,\Im)$ is conneted.

Example 2: If $X = \{a,b,c\}$ and $\Im = \{X,\phi,\{a,c\},\{b\}$. Then show that the topological space $\{X,\Im\}$ is disconneceted.

Solution: Given that $X = \{a,b,c\}$ and $\Im = \{X,\phi,\{a,c\},\{b\}$.

$\Im -$ open sets are $X,\phi,\{a,c\},\{b\}$

$\Im -$ closed sets are $\phi, X,\{b\},\{a,c\}$

This implies $\{b\}$ is non-empty proper subset of X which is both open and closed in X. Hence, $\{X,\Im\}$ is disconnected.

Example 3: let (R,v) be the usual topological space. Let $A=(1,2), B=(2,3)$ and $C=[2,3)$. Show that A and B are separated but A and C are not separated.

Solution: Given that (R,v) is an usual topological space. Also given

$$A=(1,2), B=(2,3) \text{ and } C=(2,3)$$

Here A and B are seperated because

$$A\cap \bar{B}=\phi \text{ and } \bar{A}\cap B=\phi \qquad \left\{\because \bar{A}=[1,2] \text{ and} \bar{B}=[2,3]\right\}$$

i.e., $\bar{A}\cap B=[1,2]\cap(2,3)=\phi$

$A\cap \bar{B}=(1,2)\cap[2,3]=\phi$

$A\cap B=(1,2)\cap(2,3)=\phi$

But A and C are not separated because

$\bar{A}\cap C=[1,2]\cap[2,3)=\{2\}\neq\phi$

Hence, A and B are seperated but A and C are not seperated.

Example 4: If $X=\{a,b,c,d\}$ and $\Im=\{X,\phi,\{b\},\{b,c\},\{b,c,d\}\}$. Then show that $(X,\Im)$ is connected.

Solution: Given that $X=\{a,b,c,d\}$ and $\Im=\{X,\phi,\{b\},\{b,c\},\{b,c,d\}\}$.

$\Im$-open sets are $X,\phi\{b,\{b,c\},\{b,c,d\}$

$\Im-$closed sets are $\phi, X,\{a,c,d\},\{a,b\},\{a\}$.

This implies there exists no non-emprt proper subset of X which is both open and closed in X.

Hence, $(X, \Im)$ is not disconnected i.e., $(X, \Im)$ is connected.

Theorem 1: Two closed subsets of a topological space are separated if and only if they are disjoint.

Proof: Suppose A and B are closed subsets of a topological space.

Since A and B are closed, then

$$\overline{A} = A \text{ and } \overline{B} = B$$

(1) Let A and B are seperated sets, i.e.,

$\overline{A} = B = \phi, A \cap \overline{B} = \phi$

Using equation (1), we have

$$\overline{A} = B = A \cap B = \phi$$

And $$A \cap \overline{B} = A \cap B = \phi$$

(because A and B are closed, i.e., $\overline{A} = A$ and $\overline{B} = B$)

i.e., A and B are disjoint sets.

(2) Now we let A and B are disjoint set i.e.,

$A \cap B = \phi$

Using equation (1), we have

$$A \cap B = \overline{A} \cap B = \phi$$

And $$A \cap B = A \cap \overline{B} = \phi$$

(because A and B are closed, i.e., $\overline{A} = A$ and $\overline{B} = B$)

$$\Rightarrow \bar{A} \cap B = \phi, A \cap \bar{B} = \phi$$

i.e., A and B are separated sets.

Theorem 2: Two open subsets of a topological space are separated if and only if they are disjoint.

Proof: We know that two separated sets are always disjoint.

Hence, we need to prove that two open, disjoint subsets are separated.

Suppose A and B are any two open and disjoint subsets then

$$A \cap B = \phi$$

Suppose if possible A and B are not separated. Then either

$$A \cap \bar{B} \neq \phi \text{ or } \bar{A} \cap B \neq \phi$$

If $A \cap \bar{B} \neq \phi$ then there exists a point $x \in X$ such that

$$x \in A \text{ and } x \in \bar{B}$$

Since A is an open set it is a neighborhood of x. Again $x \in \bar{B} \Rightarrow x$ is a limit point of B and every neighbourhood of x must contain at least one point B. This implies that

$$A \cap B \neq \phi \qquad \text{which is contradiction}$$

Hence, A and B are seperated.

13.5 Connectedness on the Real Line

Connectedness on the real line is intimately related to the structure of intervals and plays a crucial role in understanding continuity and the behavior of functions on the real line.

A subset E of the real line R containing at least two points is connected if and only if E is an interval.

Finite interval $\qquad (a,b),(a,b),(a,b),(a,b).$

Infinite interval $\qquad (-\infty,\infty),(-\infty,a),(a,\infty),(a,\infty).$

Examples

Example.5: Let $X=\{a,b,c,d,e\}$ and $\Im=\{X,\phi,\{a\},\{c,d\},\{a,c,d\},\{b,c,d,e\}\}$.

Show that $Y=\{b,d,e\}$ is connected.

Solution: Given that $X=\{a,b,c,d,e\}$ and

$$\Im=\{X,\phi,\{a\},\{c,d\},\{a,c,d\},\{b,c,d,e\}$$

$\Im-$open sets are $X,\phi,\{a\},\{c,d\},\{a,c,d\},\{b,c,d,e\}$

$\Im-$ closed sets are $\phi,X,\{b,c,d,e\},\{a,b,e\},\{b,e\},\{a\}$

This implies $\{a\}$ and $\{b,c,d,e\}$ is non-empty proper subsets of X which is both open and closed in X.

Hence, $(X,\Im)$ is a disconnected space.

Now $v\{Y\cap$ every member of $\Im\}$

$$=\{\phi,\{b,d,e\},\{d\}\}$$

v closed sets are $Y,\phi,\{b,e\}$

This implies there does not exist any non-empty proper subsets of Y which is both open and closed in Y .Hence, (Y, v) is connected.

Theorem 3: Every indiscrete space is connected.

Proof: Let $(X, \Im)$ be an indiscrete space.

$\Im -$ open sets are X, ϕ

$\Im$ -open sets are ϕ, X

This implies there does not exist any non-empty proper subset of X which is both open and closed in X .

Hence, $(X, \Im)$ is connecetd, i.e., every indiscrete space is connected.

Theorem 4: Every discrete space is disconnected if the space contains more than one points.

Proof: Let $(X, \Im)$ be a discrete space and X contains more than one element.

Let $X = \{a, b\}$ and $\Im = \{X, \phi, \{a\}, \{b\}$ be a discrete topology on X .

$\Im -$ open sets are $X, \phi, \{a\}, \{b\}$

$\Im -$ closed sets are $\phi, X \{b\}, \{a\}$.

The implies $\{a\}$ and $\{b\}$ are two non-empty proper subsets of X which is both open and closed in X .

Hence $\{X, \Im)$ is disconnected i.e., every discrete space is disconnected if the space contains more than one points.

Theorem.5: Prove that a topological space is disconnected if and only if any one of following statements holds good:

(1) X **is the union of two non-empty disjoint closed sets.**

(2) X **is the union of two non-empty disjoint open sets.**

Proof: Suppose X is disconnected

This implies a non-empty proper subset A of X which is both open and closed.

$\Rightarrow X \sim A$ is both open and closed in X .

$\Rightarrow A \cup X \sim A = X$ and $a \cap X \sim A = \phi$

Hence, X is the union of two non-empty open sets which are disjoint.

Now let $A \cup B = X$ and $A \cup B = \phi$

Where A are B are non-empty open sets to show that X is disconnected

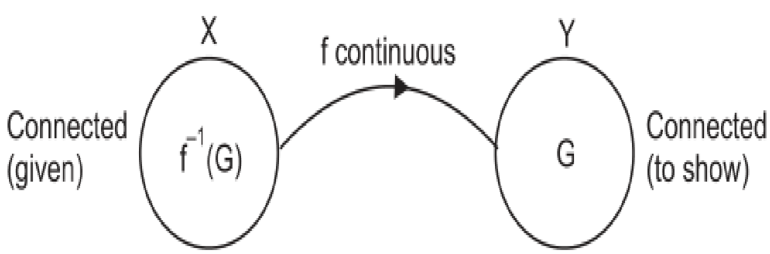
Let $A = X \sim B \Rightarrow A$ is closed

B is non-empty $\Rightarrow A$ is proper subset of X which is both open and closed.

Hence, space is disconnected.Similarly, we can prove that by taking A and B closed set.

**Theorem.6: A continuous image of a connected set is connected in a topological space. **

Proof: Suppose $f : \xrightarrow{onto} y$ is a continuous mapping. To show that if X is connected then Y is also connected.



We will prove this theorem by contradiction method. Suppose Y is disconnected and s X is connected. This implies there exists a non-empty proper subset G of Y which is both opoen and closed.

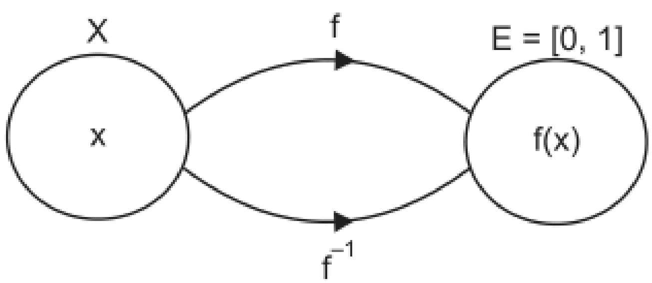
Since f is continuous $\Rightarrow f^{-1}(G)$ is both open and closed in X also f is one-one onto

$\Rightarrow f^{-1}(G)$ is also non-empty proper subset of X .

Hence $f^{-1}(G)$ is non-empty proper subset of X which is both open and closed. Therefore the set X is disconnected, which is contradiction. Hence, if X is connected then Y is also connected.

Theorem.7: Prove that a topological space is disconnected if and only if there exist a continuous mapping of X onto the discrete two point space $(0,1)$.

Proof: Let $(X,\Im)$ be a topological space . $E=(0,1)$ is a discrete space, i.e., E is disconnected space. Let X is disconnected. To show that there exists a continuous mapping $f:X\rightarrow E$.



Given X is disconnected this implies $X=A\cup B,$ where $A\cup B=\phi$ and A and B are open set

Let $f\ X\rightarrow E$ such that

$$f(x)=0,\ f(x)=1 \text{ if } x\in E$$

$$\Rightarrow f^{-1}(0)=A \text{ and } f^{-1}(1)=B$$

(Given A and B are open set then there exist $f^{-1}(0)\ ff^{-1}(1)$ are open set)

Hence, mapping is continuous.

Conversely, let there exists a continuous mapping

$$f : X \to E$$

To show that X is disconnected.

We prove this part by contradiction method suppose if possible X is connected set we know that continuous mapping of a connected set is connected therefore E is connected which is contradiction.

Because given E is disconnected.

Theorem.8: Let $(X, \Im)$ be a topological space and Y is a subset of X if Y is connected then $\overline{Y}$ is connected.

Proof: Let Y be a connected subset of a topological space $(X, \Im)$. To show that $\overline{Y}$ is also connected. Suppose if $\overline{Y}$ is disconnected then there exists non empty set A, B of X such that

$$\overline{A} \cap B = \phi, A \cap \overline{B} = \phi \qquad (1)$$

And $\quad \overline{Y} = A \cup B$ then $Y \subset \overline{Y}$

$$\Rightarrow y \subset A \cup B \qquad (Y \text{ is connected})$$

$$Y \subset A \text{ or } Y \subset B$$

We have $\quad Y \subset A \Rightarrow Y \subset \overline{A}$

$$\Rightarrow \overline{Y} \subset B \subset \overline{A} \cap B = \phi \qquad \text{using (1)}$$

$$\Rightarrow \overline{Y} \cap B = \phi \qquad (2)$$

Now $\quad \overline{Y} = A \cup B \Rightarrow B \subset \overline{Y}$

$$\Rightarrow A \cap B \subset \bar{Y} \cap B$$

$$\Rightarrow B \subset \bar{Y} \cap B$$

$$\Rightarrow B \subset \phi \qquad \text{using (2)}$$

Which is contradiction.

Because A and B are separated set therefore there are non-empty sets.

Similarly $Y \subset B \Rightarrow A = \phi$ again a contradiction thus $\bar{Y}$ is connected.

Hence, if Y is connected then $\bar{Y}$ is connected.

Theorem.9: Let $(X, \Im)$ be a topological space. Then X is disconnected if and only if there exist a non-empty proper subset of X which is both open and closed.

Proof: Let $(X, \Im)$ be a topological space and also X is disconnected. This implies there exist a non-empty disjoint open subsets G and H of X such that

$$G \cup H = X$$

And $$G \cap H = \phi$$

To show that there exist a non-empty proper subset of X which is both open and closed.

Given that $$G \cap H = \phi \Rightarrow G = X \sim H \qquad \text{(if disjoint)}$$

It is also given that H is open $\Rightarrow G$ is closed.

Also G is a subset of X . Since H is non-empty therefore G is also proper subset of X .

This implies G is non-empty proper subset of X which is either both open and closed.

Conversely, suppose there exist a non-empty subset A of X which is both open and closed.

To show that X is disconnected.

Since A is a non-empty and closed set.

$\Rightarrow X \sim A$ is a non-empty and open set.

Also given that A is a non-empty proper open subset of X then we have

$$A \cup X \sim A = X$$

This implies X is the union of two non-empty disjoint subset of A and $X \sim A$

Hence, X is disconnected.

Theorem.10: Let $(X, \Im)$ be a topological space. Let A be a connected subset of X and $A \subset B \subset \bar{A}$. Prove that B is connected and hence deduce that $\bar{A}$ is connected.

Proof: Let $(X, \Im)$ be a topological space, Let $A, B \subset X$ such that $A \subset B \subset \bar{A}$ and A is connected. To show that B is connected.

We will prove this theorem by contradiction method.

If possible B is disconnected.

This implies there exist seperated sets G and H such that

That $\qquad G \cup H = B \qquad$ and

$$G \cap \bar{H} = \phi, \bar{G} \cap H = \phi \qquad (1)$$

Since $B = G \cup H$

It is given that $\qquad A \subset B \Rightarrow A \subset G \cup H$

Then $\qquad A \subset G \Rightarrow \bar{A} \subset \bar{G}$

$\Rightarrow \bar{A} \cap H \subset \bar{G} \cap H \qquad$ (Operating $\cap H$ both sides)

Using (1), we have

$$\bar{A} \cap H \subset \phi \Rightarrow \bar{A} \cap H = \phi \qquad (2)$$

Now we have $B \subset \bar{A} \Rightarrow G \cup H \subset \bar{A}$

$$\Rightarrow H \subset G \cup H \subset \bar{A} \quad \left(B = G \cup H, G \subset B \text{ and } H \subset B\right)$$

$$\Rightarrow H \subset \bar{A}$$

$$\Rightarrow H \cap H \subset \bar{A} \cap H \qquad \text{(Operating } \cap H \text{ both side)}$$

$$\Rightarrow H \subset \bar{\phi} \qquad \text{(Using 2)}$$

Or $$H = \phi$$

Because G and H ar separated set. Therefore they are non-empty set. Hence, B is connected which is a contradiction. Since we prove above if A is connected and $A \subset B$ the B is connected it is given $B \subset \bar{A}$. Hence $\bar{A}$ is also connected.

Theorem.11: Let $(X, \Im)$ be a topological space and E be a connected subset of X such that $E \subset A \cup B$, where A and B are separated sets then $E \subset A$ or $E \subset B$ i.e., cannot intersect both A and B.

Proof: Let $(X, \Im)$ be a topological space and E be a connected subset of X such that $E \subset A \cup B$, where A and B are separated sets then

$$\bar{A} \cap B = \phi, A \cap \bar{B} = \phi \qquad \ldots(1)$$

It is also given that $E \subset A \cup B$

$$\Rightarrow E = E \cap (A \cup B)$$

$$= (E \cap A) \cup (E \cap B) \qquad \ldots(2)$$

Suppose $E \cap A$ and $E \cap B$ is empty sets. ….(3)

We will prove this by contradicition.

If possible the set are non-empty, i.e.,

$$E \cap A \neq \phi, E \cap B \neq \phi$$

We have $(E \cap A) \cap \left(\overline{E \cap B}\right) \subset (E \cap A) \cap \left(\bar{E} \cap \bar{B}\right) \left\{\overline{A \cap B} \subset \bar{A} \cap \bar{B}\right\}$ .....(4)

$$\Rightarrow \left(E \cap \bar{E}\right) \cap \left(A \cap \bar{B}\right) \subset \left(E \cap \bar{E}\right) \cap \phi = \phi \qquad \text{using equation (1)}$$

Using (3) we have

$$(E \cap A) \cap \left(\overline{E \cap B}\right) = \phi$$

Similarly, $\left(\overline{E \cap A}\right) \cap (E \cap B) = \phi$ ...(5)

This implies $(E \cap A)$ and $(E \cap B)$ are seperated sets.

Using equation (2) we have

$$(E \cap A) \cup (E \cap B) = E$$

This implies E is the union of two separated sets $(E \cap A)$ and $(E \cap B)$.

Therefore E is disconnected, which is contradiction because it is given E is connected.

Let $E \cap A = \phi$ then by (3)

Equation (2) implies

$$E = (E \cap A) \cup (E \cap B)$$

$$= \phi \cup (E \cap B)$$

$$= E \cap B$$

$$= E \subset B$$

i.e., $E \subset B$ and if $E \cap A = \phi$

Let $E \cap B = \phi$ Then by (3)

Equation (2) implies

$$E = (E \cap A) \cup (E \cap B)$$

$$= (E \cap A) \cup \phi$$

$$= E \cap A$$

$$\Rightarrow E \subset A$$

i.e., $E \subset A$ and if $E \cap B = \phi$

Hence, either $E \subset A$ or $E \cap B$

13.6 Components

Components are a fundamental concept in topology that help us to understand the structure of a space by partitioning it into maximally connected subsets. They have important applications in various fields, including graph theory, image processing, and the characterization of topological spaces. The number and nature of components can provide important information about the topological properties of a space.

For example, the number of components can help to distinguish between different types of spaces, such as those that are connected, disconnected, or have more complex structures.

Let $(X, \mathfrak{I})$ be a topological space. A component of the space X is a maximal connected subspace of $(X, \mathfrak{I})$.

13.7 Maximal Connected Set

Maximal connected sets are important in topology for their role in defining components, understanding the structure of spaces, and their applications in analysis and geometry. They provide a foundational concept for studying connectivity in topological spaces.

Let $(X, \Im)$ be a topological space and $A \subset X$. Then the set A said to be maximal connected subset of X if:

(i) A is connected.

(ii) A is not a proper subset of any connected subset of X.

Note: 1. Every indiscret space has only one component (the space itself)

2. Each connected subset of X which is both and closed is a component of X.

13.8 Locally Connected Space

Locally connected spaces are important in topology for their local structure, their relationship with path components, their applications in analysis, and their role in defining important classes of spaces such as manifolds.Many important spaces in mathematics, such as topological manifolds, are locally connected. Locally connectedness is a key property in the definition and study of these spaces.

Let $(X, \Im)$ be a topological space. Then $(X, \Im)$ is said to be locally connected at a point x if and only if every open neighbourhood of x contains a connected open neighbourhood of x.The space is said to be locally connected if and only if it is locally connected at each of its point.

13.9 Totally Disconnected Set

Totally disconnected sets are important in topology for their role in understanding the structure of topological spaces, their applications in fractal geometry, and their connection to dimension theory and compactness. A totally disconnected set is a set in which every subset with more than one point can be divided into two disjoint nonempty subsets such that no point of the set is an interior point of both subsets.

A topological space is totally disconnected if given any pair of distinct points $x, y \in X$ then there exist a disconnection $G \cup H$ of X with

$$x \in G, y \in H \text{ and } G \cup H = H .$$

Examples

Example.6:Let $X = \{a, b, c, d, e\}$ **and** $\Im = \{X, \phi, \{a\}, \{b, c\}, \{a, b, c\}, \{b, c, d, e\})$ **be a topology on** X **. Find all the component of** X **.**

Solution: Given that $X = \{a, b, c, d, e\}$ and

$$\Im = \{X, \phi, \{a\}, \{b, c\}, \{a, b, c\}, \{b, c, d, e\}\}$$

Here $\{a\}$ and $\{b, c, d, e\}$ are disjoint and their union is X . Also these two sets are both open and closed in X . Hence, components of X are $\{a\}, \{b, c, d, e\}$.

Note: Any other connected subset of X in above example such as $\{b, d, e\}$ is subset of one of the components. $A = \{b, d, e\}$. The relative topology on $A, v = \{A, \phi, \{d\}\}$.

Hence, A is connected since A and ϕ are the only subsets of A both open and closed in the relative topology.

Example 7: Every discrete space $(X, \Im)$ **is locally connected.**

Solution: Let $(X, \Im)$ be a discrete topological space for every $x \in X, \{x\}$ is a connected $\Im$ -

neighbouhood of $\Im$. Also evidently every $\Im$-neighbourhood of x contains $\{x\}$. Hence, $\{x,\Im\}$ is locally connected.

Example 8: Let $X=\{a,b,c\}$ and $\Im=\{X,\phi,\{a\},\{b\},\{c\},\{a,b\},\{b,c\},\{a,c\}\}$ be a discrete topology on X. t show that the space $(X,\Im)$ is locally connected.

Solution: Given that $X=\{a,b,c\}$

And $\Im=\{X,\phi,\{a\},\{b\},Pc\},\{a,b\},\{b,c\},\{a,c\}\}$

Let $A=\{a,b\}$.

To find relative topology on A.

We have $v\{A\cap$ every member of $\Im\}$

$$=\{A,\phi,\{a\},\{b\}\}$$

Here A and ϕ are only subset of A both open and closed in relative topology.

Hence, A is connected.

Since for every $x\in X$, $\{x\}$ is a connected $\Im-$neighbourhood of x. Also every $\Im-$neighbourhood of x contains $\{x\}$.

Hence, $\{X,\Im)$ is locally connected.Similarly, for every $a,b\in X$, $\{a\}$ and $\{b\}$ is a connected $v-$neighbourhood of x. Also every $\Im-$neighbourhood of a and b contains $\{a\}$ and $\{b\}$ respectively.

Theorem.12: Every component of a locally connected space is open.

Proof: Let $(X,\Im)$ be a locally connected space and C be a component of X. To show that C is an open set.

Let x be an element of C. Since X is locally connected, there must exist a connected open set G_x which contains x. Since C is a component.

We have

$$x \in G_x \subset C$$

Obviously $$C = \cup\{G_x : x \in C\}$$

Thus, C being a union of open set, is open set.

Theorem.13: Prove that the image of a locally connected space under continuous and open mapping is locally connected.

Proof: Let f be a continuous and open mapping from $X \to Y$, where X and Y are topological spaces and X is locally connected.

To show that Y is also locally connected.

Let $y \in f(x)$ and v be any open neighbourhood of y in $f(x)$.

This implies there exists $x \in X$ such that $y = f(x)$ since mapping f is continuous.

This implies $f^{-1}(v)$ is open set in X and $x \in f^{-1}(v) \Rightarrow f^{-1}(v)$ open neighbourhood of x. It is given that X is locally connected.

This implies there exist open set u such that

$$x \in U \subset f^{-1}(v)$$

And U is connected using definition.

We know that by a theorem continuous image of connected set is connected.

This implies $f(u)$ is connected set.

Also it is given f mapping is open mapping.

This implies $f(u)$ is open set in Y.

This implies $y \in f(u) \subset v$

Hence, $f(X) = Y$ is locally connected at point y. Therefore, $f(X) = Y$ is locally connected.

13.10 Summary

Two subsets A and B of a topological space $(X, \mathfrak{I})$ are said to be separated if any only if

$$A \cap \bar{B} = \phi \text{ and } \bar{A} \cap B = \phi.$$

A set X is said to be connected if does not exist any non-empty proper subset of X which is both open and closed.

Let $(X, \mathfrak{I})$ be a topological space then X is said to be disconnected if and only if there exist two disjoint non-empty subsets A and B of X such that

(i) $\bar{A} \cap B = \phi$ and $A \cap \bar{B} = \phi$ (ii) $A \cup B = X$

A subset E of the real line R containing at least two points is connected if and only if E is an interval.

Let $(X, \mathfrak{I})$ be a topological space. A component of the space X is a maximal connected subspace of $(X, \mathfrak{I})$.

Let $(X, \mathfrak{I})$ be a topological space and $A \subset X$. Then the set A is said to be maximal connected subset of X if:

(i) A is connected

(ii) A is not a proper subset of any connected subset of X .

Let $(X, \Im)$ be a topological space. Then $(X, \Im)$ is said to be locally connected at a point x if and only if every open neighborhood of x contains connected open neighborhood of x .The space is said to be locally connected if and only if it is locally connected at each of its point.

A topological space is totally disconnected if given any pair of distinct points $x, y \in X$ then there exists a disconnection $G \cup H$ of X with $x \in G, y \in H$ and $G \cup H = H$.

13.11 Terminal Questions

Q.1. Explain the connected and disconnected set.

Q.2. What do you mean by locally connected and totally disconnected set.

Q.3. To show that the closure of a connected set is connected.

Q.4. Let $X = \{a, b, c\}$ and $\Im - \{X, \phi, \{a, b\}, \{c\}\}$. Prove that $\{X, \Im)$ is disconnected.

Q.5. Show that $\{X, \Im$ is a connected space if $X = \{a, b, c, d\}$ and $\Im = \{X, \phi, \{a, b\}\}$.

UNIT-14: Compactness

Structure

14.1 Introduction

Compactness is a fundamental concept in topology that captures the idea of a space being "nicely bounded" or "finite in a sense." A topological space is said to be compact if every open cover of the space has a finite subcover. Compactness is a useful property because it ensures that certain properties hold in a space. For example, in a compact space, every sequence has a convergent subsequence. This property is known as the Bolzano-Weierstrass theorem. Compactness also allows for the extension of certain theorems from analysis to more general topological spaces. This unit deals with compactness, compact sets, basic properties of compactness, finite intersection property,locally compact space, Bolzano weietrass property, sequentially compact, countably compact sets, uniformaly continuous, Lebesgue covering lemma, Heine-Borel theorem, compactness and one point compactification, cartesian product of two sets, projection mapping, embedding and product topology.

14.2 Objectives

After reading this unit the learner should be able to understand about the:

- Cover and subcover of X
- Open Cover and Finite Cover
- Compact Space and Compact Set
- Finite Intersection Property
- Locally Compact Space and Lindelof Space
- Bolzano Weiertrass Property
- Sequentially Compact and Uniformly Continuous
- Lebesgue Covering Lemma and Heine-Borel Theorem
- Product Topology and Projection Mappings

14.3 Cover and Subcover of X

Covers and subcovers are important in topology for their role in defining and characterizing compactness, their applications in studying topological properties, their use in partitioning unity, and their applications in differential geometry and complex analysis.

Let $(X, \Im)$ be a topological space. Let A be a subset of X. A family $\mathcal{A}$ of subsets of X is said to be cover for $(X, \Im)$ if only if

$$U(v : v \in \mathcal{A}) = X$$

Also if $\mathcal{B} \subset \mathcal{A}$ such that $\mathcal{B}$ is also a cover for X then $\mathcal{B}$ is a subcover of $\mathcal{A}$.

14.4 Open Cover and Finite Cover

In topology, an open cover of a topological space X is a collection of open sets whose union contains X. A finite cover is an open cover that consists of only finitely many open sets. Open covers and finite covers are crucial in the study of compact spaces. A topological space is compact if every open cover has a finite subcover. Finite covers are particularly useful in proving compactness because they allow for a more manageable number of sets to work with.

An open cover of A is a family $\{v : v \in \mathcal{A}\}$ of $\Im -$ open subsets of X such that each point in X belongs to at least one number of the class $\{v : v \in \mathcal{A}\}$ i.e., $A \subseteq \{v : v \in \mathcal{A}\}$.

A cover of a topological space $(X, \Im)$ is said to be finite cover if it has only a finite number of member.

14.5 Compact Space and Compact Set

Compact spaces and compact sets are fundamental concepts in topology with wide-ranging applications in mathematics, including analysis, geometry, and topology. In topology, a compact space is a topological space in which every open cover has a finite subcover. Compactness is a fundamental concept in topology with many important properties and applications. A compact set is a subset of a topological space that is itself a compact space when endowed with the subspace topology.

Compact spaces are often connected, but there exist compact spaces that are not connected (e.g., the disjoint union of two compact spaces).

Let $(X, \Im)$ be a topological space. Then $(X, \Im)$ is said to be compact if and only if every open cover of X has a finite subcover.

(i) $U\{G_i : i \in I\} = X$ (open cover)

(ii) $U\{G_i : i \in (1, 2, \ldots\ldots, N)\} = X$ (finite subcover)

Let $(X, \Im)$ be a topological space. A set A of X is said to be compact if every $\Im$ -open cover of A has a finite subcover.

14.6 Finite Intersection Property

A collection of subsets of X is said to have finite intersection property if and only if the intersection of members of each finite sub-collection is non-empty.

14.7 Locally Compact Space and Lindelof Space

A locally compact space is a topological space in which every point has a compact neighborhood. Locally compact spaces generalize the notion of compactness to allow for spaces that are compact "around each point." This property is useful in many areas of mathematics, including functional analysis and algebraic topology. A Lindelöf space is a topological space in which every open cover has a countable subcover. These spaces are important in topology because they satisfy a "countable compactness" property that is useful for proving certain theorems.

Let $(X,\Im)$be a topological space. Then $(X,\Im)$is said to be locally compact space if and only if every point of the set has at least one neighbourhood whose closure is compact.

Let $(X,\Im)$be a topological space. Then $(X,\Im)$is said to be lindelof space if every open cover of X has a countable cover.

14.8 Bolzano Weiertrass Property

Let $(X,\Im)$be a topological space. Then $(X,\Im)$ is said to have the Bolzano-Weiertrass property if every infinite subset of X has a limit point.

Any space with Bolzano-weiertrass property is called Frechet compact space.

14.9 Sequentially Compact

Sequentially compact spaces are important in topology and analysis for their properties related to convergence of sequences. They provide a way to ensure the existence of limits for sequences in a space and have applications in various areas of mathematics.

Let (X,d)be a metric space. Then $(X,d)$is said to be sequentially compact if every sequence in X has a convergent sub-sequence. For example, the set of all real number in$(0,1)$is not sequentially compact. For the sequence $\left(\frac{1}{2},\frac{1}{3},.....\right)$in $(0,1)$ converges to a $0 \notin (0,1)$on the otherhand $[0,1]$is sequentially compact.

Note: A sequence $\{x_n \notin X : n \in N\}$ is said to be convergent if $x_n \to x_0 \ \forall x \in X$.

Examples

Example 1: Let $X=\{1,2,3,4\}$ **and** $\Im = \{X,\phi,\{1\},\{4\},\{1,4\},\{2,3\},\{1,2,3\},\{2,3,4\}\}$

be a topology for X .

The collections $C_1 = \{\{1\},\{4\},\{2,3\}\}$

$$C_2 = \{\{1,4\},\{2,3\}\}$$

and $C_3 = \{\{1\},\{2,3,4\}\}$

Are open cover of X. Since these covers have finite number of members. So they are finite covers. But $C = \{\{1,2\},\{3,4\}\}$ is a cover of X which is not open.

The collection $\{\{1\}.\{1,3\},\{1,2,3\}\}$ is not a cover of X since the union of theses members of this collection is not equal to X .

If $A = \{1,2,3\}$ is a subset of X then the collection $\{\{1\},\{2,3\},\{1,2,3\}\}$ is an open cover of A. Here C_1 is a refinement of C_2 and C_3 both.

Theorem 1: Prove that in a topological space $\{X,\Im\}$, every closed subspace of a compact space is compact.

Proof: Let $\{X,\Im\}$ be a topological space and X be compact and (Y,v) is a subspace of $(X,\Im)$. To show that Y is compact.

Let $\{G_i : i \in I\}$ is an open cover for Y

$$\Rightarrow U\{G_i : i \in \{1,2,3,......,N\}\} = Y$$

Since each G_i is the open set of Y

i.e., $G_i \in v$

Using definition we have $G_i = \{Y \cap \text{every member of } \Im\}$

i.e., $G_i = Y \cap H_i \in v \qquad (H_i \in \Im)$

It is given that Y is closed then $X \sim Y$ is open set in X.

$\Rightarrow X \sim Y$ and all $H_i \in \Im$ is an open cover for X. Since it is given X is compact, this implies it has a finite subcover for X .

$$\Rightarrow U\{H_1, H_2, ...H_n\} = X$$

$$\Rightarrow U\{H_1 \cap Y, H_2 \cap Y, ...H_n \cap Y\} = X \cap Y$$

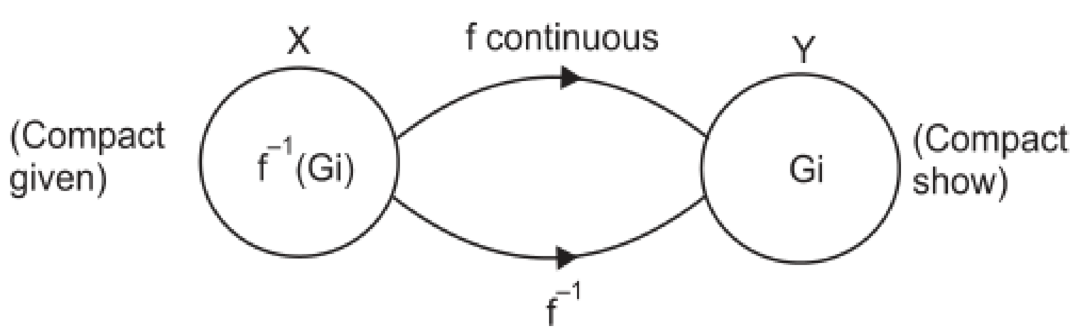
$$\Rightarrow U\{G_1, G_2, ...G_n\} = Y \qquad \text{(using definition)}$$

$$\Rightarrow U\{G_i : i \in (1, 2, 3, ...N)\} = Y$$

Hence, (Y, v) is a compact space.

Theorem 2: Prove that the continuous image of a compact space is compact in a topological space $(X, \Im)$.

Proof: Let $(X, \Im)$ and (Y, v) be two topological spaces. A mapping $f : X \to Y$ which is continuous and X is compact.



To show that Yis compact space

Let $\{G_i : i \in I\}$ is an open cover for y

$$\Rightarrow \{G_i : i \in I\} = Y \qquad (1)$$

It is given mapping continuous

$$\Rightarrow \{f^{-1}(G_i) : i \in I\} \text{ are open set in } X$$

And they form are open cover for X .

It is given X is compact $\Rightarrow$ it has a finite subcover.

$$\Rightarrow U\{f^{-1}(G_i) : i \in (1,2,3,...N)\} = X$$

$$\Rightarrow f^{-1}\{U(G_i) : i \in (1,2,3,...N)\} = X$$

$$\Rightarrow U\{(G_i) : i \in (1,2,3,...N)\} = f(X)$$

$$= Y$$

i.e., Y has a finite subcover

Hence, Y is compact space.

Theorem 3: Prove that a topological space $(X, \Im)$ is a compact if and only if every collection of closed subsets of X with finite intersection property has a non-empty intersection.

Proof: (i) Let X be compact and $\Im = \{F_i : i \in I\}$ be the collection of closed subset of X with finite intersection property.

To show that $\cap\{F_i : i \in I\} \neq \phi$

Suppose if possible $\cap\{F_i : i \in I\} = \phi$

$$\Rightarrow X \sim \cap\{F_i : i \in I\} = X \sim \phi \quad \text{(Complement taking)}$$

$$\Rightarrow \cap\{X \sim F_i : i \in I\} = X$$

$\Rightarrow$ it is open cover for X.

It is given X is compact $\Rightarrow$ it has a finite subcover.

This implies $\cup\{X \sim F_i : i \in (1, 2, ..., N)\} = X$

(Taking complement and using Demorgan law)

$$\Rightarrow X \sim X \sim \cap\{F_i : i \in (1, 2, ..., N)\} = X \sim X$$

$$\Rightarrow \cap\{F_i : i \in (1, 2, ..., N)\} = \phi$$

$\Rightarrow$ it is contradiction because X has finite intersection property

$$\Rightarrow \cap\{F_i : i \in (1, 2, ...N)\} \neq \phi.$$

(ii) Let every collection of closed subset of x with finite intersection property has a non-empty intersection.

To show that X is compact.

Let $X = \cup\{G_i : i \in I\}$ (G_i is open set)

$$x \sim X = X \sim \cup\{G_i : i \in I\} \text{ (Taking complement)}$$

$$\phi = \cap\{X \sim G_i : i \in I\}$$

$$\Rightarrow \cap\left\{X \sim G_i : i \in I\right\} = \phi$$

$$\Rightarrow \cap\left\{X \sim G_i : i \in (1, 2, ..., N)\right\} = \phi$$

$$\Rightarrow X \sim \cup\left\{G_i : i \in (1, 2, ..., N)\right\} = X \sim \phi$$

$$\Rightarrow \cup\left\{G_i : i \in (1, 2, ..., N)\right\} = X$$

$\Rightarrow$ it is finite subscover for X .

Hence, X is compact.

Theorem 4: If $(X, \Im)$ is compact, $\Im$ ' is coarser than $\Im$ then show that $(X, \Im)$ is also compact.

Proof: Let $(X, \Im)$ be a compact topological space and let $\Im'$ is coarser then $\Im$ so that $\Im' \subset \Im$.

To show that $(X, \Im')$ is compact.

Let $\{G_i : i \in I\}$ be a $\Im'$ open cover for X

Then $\{G_i : i \in I\}$ be a $\Im$ open for X for $\Im' \subset J$. Also X is compact.

Hence, $\{G_i : i \in \{1, 2, 3, .., N\}$ is reducible to finite subcover which is also $\Im'$ open. So $(X, \Im')$

Compact.

Theorem 5: Prove that every compact topological space is locally compact? Is the converse true.

Proof: Let X be a compact. To show that it is locally compact.

We know that X is both open and closed therefore it has the neighbourhood of each of its point. This implies $\bar{X} = X$ i.e., X is locally compact.

But converse is not necessary true.

Let $(X, \Im)$ be a discrete topological space where X is infinite therefore X is not compact because the collection of all singleton sets is an open cover for X but it has no finite subcover.

Whereas this set is locally compact because let $x \in X$. This emplies $\{x\}$ is the neighbouhood of x .

And we know that in a discrete space each member of $\Im$ is open and closed therefore $\{x\}$ is closed also $\Rightarrow \{x) = \{\bar{x}\}$

And $\{x\}$ is a compact subset of x.

Therefore every point of X has a neighbourhood whose closure is compact so it is locally compact.

Theorem 6: Prove that every closed subspace of a locally compact space is locally compact.

Proof: Let (Y, v) be a closed subspace of a locally compact space $(X, \Im)$, then Yis $\Im -$closed set. Let $y \in Y \subset X$.

To prove Y is locally compact.

We have$y \in Y \Rightarrow y \in X$ $\{Y \subset X\}$

It is given X is locally compact.

This implies there exists a neighbourhood U of y in X such that $\bar{U}$ is compact.

This implies $U \cap Y$ is open neighbourhood of y in Y

$$\Rightarrow U \cap Y \subset U$$

$$\Rightarrow \overline{U \cap Y} \subset \bar{U}$$

$\Rightarrow \overline{U \cap Y}$ is a closed subset of a compact set.

We know that a closed subset of a compact space is compact.

This implies for every point in y has a neighbourhood in Y where closure is compact.

Hence , Y is locally compact.

Theorem 7: Show that a compact topological space has BWP.

Proof: We will prove this theorem by contradiction.

Let A has no limit point in X .

This implies for every $x \in X$ there exists an open neighbourhood U_x of x which contains no point of A other then x.

This implies collection of such neighbourhood is an open cover for X .

i.e., $\{U_x : x \in X\}$ is an open cover for X .

It is given that X is compact space.

This implies it has a finite subcover.

$$\Rightarrow X = U\left\{U_x : i \in (1, 2, ..., N)\right\} \qquad (1)$$

Also it is given $\qquad A \subset X \qquad (2)$

From (1) and (2), we have

$$A \subset U\left\{U_{x_i} : i \in (1, 2, ..., N)\right\}$$

Since each U_x contains at most one point of A therefore $U\{U_{x_i} : i \in (1,2,....,N)\}$ will contains at

Most n points and A is given to be infinite set which is contradiction because an infinite set cannot be subset of a finite set.

Hence, A must have a limit point in X .

Therefore compact topological space has BWP.

Theorem 8: Prove that a metric space is sequentially compact if it satisfies the BWP.

Proof: Let (X,d) be a metric space. Also X is sequentially compact and $A \subset X$.

To show that it has a BWP.

Let $A \subset X$ be an infinite set.

To show that A has a limit point in X.

Since A is an infinite set this implies there exists any collection $\{x_n\}$ of distinct points in A also it is given that the space is sequentially compact.

This implies $\{x_n\}$ has a sequence $\{x_{nk}\}$ which converges to a point x in X .

We know that if a convergence sequence in a metric space has infinitly distinct points then its limit is a limit point in x .

Thus, set of the points of the sequence of this x converges this implies x is the limit point of the set of point of the subsequence and since the set is a subset of A i.e., also a limit point of A .

Conversely, let every infinite subset of X has a limit point then to show that X is sequentially compact. Let $\{X_n\}$ is a sequence in X . Then

(1) This sequence may have a point which is infinitely repeated its implies it has a constant subsequence which is convergent.

(2) If no point of $\{X_n\}$ is infinitely repeated i.e., $\{x_n\}$ has infinitly distinct points.

The set A of this sequence is inifinite it is given infinite set has a limit point x in X .

This implies there exist a sequence $\{x_n\}$ which converge to x. Hence, X is sequentially compact.

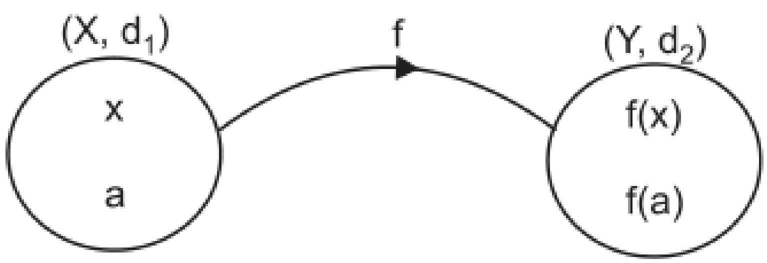
14.10 Uniformly Continuous

Uniform continuity is an important concept in analysis and topology that helps us to understand the behavior of functions in a controlled and uniform manner. Uniformly continuous functions behave well with respect to compact sets. Specifically, a uniformly continuous function maps compact sets to compact sets.

Let (X, d_1) and (Y, d_2) be two metric space. A mapping f defined on a metric space X and Y is uniformly continuous if $\in > 0$ then there exist $\delta > 0$ depending on $\in$ alone such that

$$d_2(f(x), f(a)) < \in$$

$$d_1(x, a) < \delta$$



Note: In case of continuity δ depends upon $\in$ and point a. But in case of uniformly continuous δ depending upone $\in$ alone. Using definition, for given

$$|x-a|<\delta \text{ if } \delta>0 \quad \text{implies} \quad |f(x)-f(a)|<\in$$

14.11 Lebesgue Covering Lemma

The Lebesgue Covering Lemma is a fundamental result in topology that provides a way to cover a compact set with a collection of open sets, while controlling the size of the covering sets.

Every open cover of a sequentially compact metric space has a lebesgue number.

Note: Let (X,d) be a metric space and $C=\{G_\alpha : \alpha \in \wedge\}$ be an open cover of X. A real number $l>0$ is said to be a lebesgul number for C if and only if every subset of X with diameter less than is contained in at least one G_α .

14.12 Heine-Borel Theorem

Heine-Borel Theorem is a key result in topology and analysis that provides a fundamental characterization of compact sets in Euclidean space, with wide-ranging applications in mathematics.

A subset of the real line is compact if and only if it is closed and bounded.

Proof: Let (R,U) be the usual topological space and let $A\subset R$ is compact.

Let us consider the family of open sets (open intervals) defined as

$$\{G_\alpha : \alpha \in A \text{and } G_\alpha =]\alpha-1, \alpha+1[\} \qquad (1)$$

Clearly this is an open cover of A . Since it is given A is compact this implies there exist a finite subcover of A

i.e., $$U\left\{G_{\alpha_1} : i \in (1,2,...,n)\right\}$$

$$A \subseteq U\left\{G_{\alpha_1}, G_{\alpha_2}, ..., G_{\alpha_n}\right)\Big\} \qquad (2)$$

Let $$M_0 = \max\left\{\alpha_1, \alpha_2, ..., \alpha_n\right\}$$

$$m_0 = \min\left\{\alpha_1, \alpha_2, ..., \alpha_n\right\}$$

$$\Rightarrow A \subseteq \left[m_{0-1}, m_{0+1}\right]$$

$$\Rightarrow A \text{ is bounded.}$$

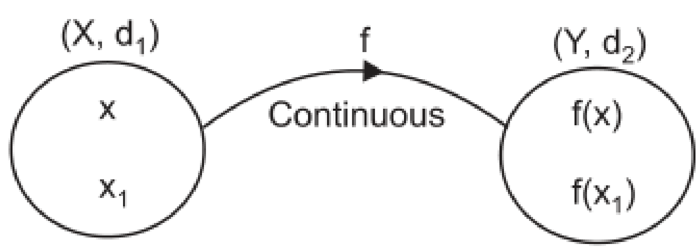
Let (R,U) be a T_2 -space and we know that a compact subset of a T_2 -space is closed.

This implies A is closed. Conversely, let A be bounded and closed.

To show that A is compact.We know that every bounded and closed interval on R is compact.Thus, A is compact.

Theorem 9: Prove that any continuous mapping of a compact metric space into a metric space is uniformly continuous.

Proof: Let f be a continuous mapping from a compact metric space X into a metric space Y with metric d_1 and d_2 respectively.



Since $\in > 0$ then for each point $x \in X$ then there exist an open ball $S_{\in/2}\left[f(x)\right]$ centred at the pont $f(x)$ This implies inverse image of all these open balls are open set in X .

This implies class of all such images is an open cover for X and it is given X is compact.

$\Rightarrow$ Open cover has a lebesgue number δ if x and $x_1 \in X$ such that

$d_1(x, x_1), \delta$ then the set $\{x, x_1\}$ is the set with diameter $< \delta$. Also both the points x and x_1 belongs to the inverse image of the open ball centred at $f(x)$ and $f(x_1)$

$$\Rightarrow d_2\{f(x), f(x_1)\}, \delta$$

Hence, the mapping f is uniformly continuous.

14.13 Product Topology

The product topology is a fundamental concept in topology that allows us to study the properties of product spaces and is used in various areas of mathematics. The product topology is used in algebraic topology to define the product of topological spaces, which is important for defining operations on homology and cohomology groups. The product topology allows us to study the properties of product spaces, which are used in various branches of mathematics. For example, in functional analysis, the product of Banach spaces is a common construction. The product topology is a way to construct a topology on the Cartesian product of two or more topological spaces.

Let X_1 and X_2 be any two sets. Then the Cartesian product of X_1 and X_2 written as $X_1 \times X_2$ is the set of all ordered paris (x_1, x_2) such that $x_1 \in X_1$ and $x_2 \in X_2$

i.e., $$X_1 \times X_2 = \{(x_1, x_2) : x_1 \in X_1, x_2 \in X_2\}$$

Let $(X_1, \Im_1)$ and $(X_2, \Im_2)$ be two topological spaces. The topology $\Im$ whose base is $B = \{G_1 \times G_2 : G_1 \in \Im_1 \text{ and } G_2 \in \Im_2\}$ is said to be product topology for $X_1 \times X_2 = X$. The corresponding topological space $(X, \Im)$ is known as product space of X_1 and X_2 ..

Examples

Example. 2: Let $\Im_1 = \{X, \phi, \{1\}\}$ be a topology for $X_1 = \{1,2,3\}$ and $\Im_2 = \{X, \phi, \{a\}, \{b\}, \{a,b\}, \{c,d\}, \{a,c,d\}, \{b,c,d\}\}$ be a topology for $X_2 = \{a,b,c,d\}$. Find a base for the product topology of $\aleph_1 \times \aleph_2$.

Solution: We know that

$$B_1 = \{\{1\}, \{1,2,3\}\} \text{ is a base for } \Im_1$$

And $$B_2 = \{\{a\}, \{b\}, \{c,d\}\} \text{ is a base for } \Im_2$$

Hence, a base for the product topology is given by

$$B = \{\{1) \times a; \{1\} \times \{b\}; \{1\} \times \{c,d\}; \{1,2,3\} \times \{a\}; \{1,2,3\} \times \{b\}; \{1,2,3\} \times \{c,d\}\}$$

$$= \{\{1,a); \{1,b) \ \}; \{(1,c), (1,b)\}; \{1,a); (2,a), (3,a)\}; \{(1,b), (2,b), (3,b)\}; \{(1,c), (1,d), (2,c), \{2,d), \{3,c), \{3,d)\}\}$$

14.14 Projection Mappings

Projection mappings are used extensively in topology and related areas of mathematics. They play a crucial role in defining and studying product spaces, and they provide a way to decompose a product space into its component spaces. They are also used in algebraic topology to define operations on homotopy and homology groups, and in functional analysis to define operations on function spaces.Projection mappings are a fundamental concept in mathematics, particularly in the context of Cartesian products and product topologies.

The mappings

$$\pi_x : X \times Y \to X \text{ such that}$$

$$\pi_x(x, y) = x; \forall (x, y) \text{ in } X \times Y$$

And

$$\pi_y : X \times Y \to Y \text{ such that}$$

$$\pi_y(x, y) = y; \forall (x, y) \text{ in } X \times Y$$

Are called projection mappings of the product $X \times Y$.

Embeddings

An embedding of a topological space Xinto another space Y, we mean a mapping $f : X \to Y$ which defines a homeomorphism of X onto $f(X)$.

Theorem 10: If $\{X \times Y, T)$ is the product space of $(X, \Im)$ and (Y, v) then the projection mapping π_x and π_y are continuous and open.

Proof: Let G be any $\Im$ -open subsets of X .Then by definition of π_x, we have

$$x_x^{-1}(G) = G \times Y$$

Which is a basic $T-$open subset of $X \times Y$. For $G \in \Im, Y \in v \Rightarrow G \times Y \in B$ where B is a base for T. Hence π_x is $T - \Im$ continuous. Similarly it can be proved that π_y is $T - v$ continuous. Hence, the projection mapping x_x, π_y are continuous. Now we shall to show that π_x and π_y are also open. Suppose W be a $T-$open subset of $X \times Y$. Then by definition of base B for T, we have

$$W = \left[U\left\{G\times H : G\in\Im, H\in v \text{ and } \mathrm{G}\times\mathrm{H}\in\mathrm{B}^{*}\right\}\right]$$

Now we have

$$x_x(W) = \pi_x\left[U\left\{G\times H : G\in\Im, H\in v \text{ and } \mathrm{G}\times\mathrm{H}\in\mathrm{B}^{*}\right\}\right]$$

Where $B^{*}\subset B$ since $G = U\left\{B_i, B_j \subset B\right\}$

$$= U\left\{\pi_x\left[G\times H\right] : G\in\Im, H\in v \text{ and } \mathrm{G}\times\mathrm{H}\in\mathrm{B}^{*}\right\}$$

$$= U\left\{G : G\in\Im \text{ and } \mathrm{G}\times\mathrm{H}\in\mathrm{B}^{*}\right\} \qquad \{\text{using definition of } \pi_x\}$$

$$\therefore \qquad \pi_x(w)\in\Im$$

Hence, π_x is an open mapping. Similarly π_y is also an open mapping.

Theorem 11: The product of two second countable space is a second countable space.

Proof: Let $(X_1,\Im_1)$ and $(X_2,\Im_2)$ be two second countable spaces to show that $(X_1\times \mathrm{X}_2, T)$ is also a second countable space.

Let $\mathcal{B}_1 = \{B_i = i\in N\}$ and $\mathcal{B}_2 = \{C_i : \mathrm{j}\in N\}$ be countable bases for $\Im_1$ and $\Im_2$ respectively. Consider the countable collection

$$C = \left\{B_i\times C_j : i\in N, j\in N\right\}$$

Let (x_1,x_2) be any point of $X_1\times X_2$ and let N be a neighbourhood of (x_1,x_2)

Since $\mathcal{B} = \{G_1\times G_2 : G_1\in\Im_1, G_2\in\Im_2\}$ is a base for the prodecut topology there exists a member $G_1\times G_2$ of $\mathcal{B}$ such that $(x_1x_2)\times G_1\times G_2\subset N$

Since $G_1\in\Im$ and $\mathcal{B}$ is a base for $\Im_1$ there exist some $B_i\in\mathcal{B}_1$ such that

$$x_1 \in B_i \subset G_1$$

Similarly, $$x_2 \in C_j \subset G_1$$

$\therefore$ $$(x_1, x_2) \in B_i \times C_j \subset G_1 \times G_2$$

Thus, $$(x_1, x_2) \in B_i \times C_j \subset N$$

Hence, $\{B_i \times C_j : i, j \in N\}$ is a base for the product topology T.

Therefore product space of two second countable space is also a second countable space.

Theorem 12: Product of two Housdorff spaces is a Hausdorff space.

Proof: Let $(X, \Im)$ and (y, v) be two T_2 -spaces. Suppose (x_1, y_1) and (x_2, y_2) are two distinct points of $X \times Y$. Then either $x_1 \neq x_2$ or $y_1 \neq y_2$

Suppose $x_1 \neq x_2$.

Since $(X, \Im)$ is a T_2 -space then there exist $\Im -$ open disjoint subsets G_1 and G_2 such that

$$x_1 \in G_1$$

And $$x_2 \in G_2$$

Then $G_1 \times Y$ and $G_2 \times Y$ are open subsets of $X \times Y$ such that

$$(x_1, y_1) \in G_1 \times Y$$

$$(x_2, y_2) \in G_2 \times Y$$

And $$(G_1 \times Y) \cap (G_2 \times Y) = (G_1 \cap G_2) \times Y$$

$$= \phi$$

Hence, product of two T_2 spaces is a T_2-space.

Theorem 13: Each projection $\pi_\alpha : X \to X_\alpha$ on a product space $X = \pi_i \{X_i\}$ is an open mapping.

Proof: Let G be an open subset of the product space $X = \pi_i \{X_i\}$. For every point p in G there exists a member B of the defining B base for the product topology T such that

$$p \in B \subset G$$

Thus, for any projection $\pi_\alpha : X \to X_\alpha$

$$\pi_\alpha (p) \in \pi_\alpha [B] \subset \pi_\alpha [G]$$

Since B belongs to the definiting base for X .

$$\therefore \qquad B = \pi \{X_\alpha : \alpha \neq \lambda_1, \lambda_2, \lambda_3, ..., \lambda_m\} \times \{G_{\lambda_1}, \times G_{\lambda_2} \times G_{\lambda_3}, ..., \times G_{\lambda_m}\}$$

Where G_{λ_k} is an open subset of X_{λ_k}

Thus, for any projection $\pi : x_\alpha \to X_\alpha$

$$\pi_\alpha (B) = \begin{Bmatrix} X_\alpha & \text{if } \alpha \neq \{\lambda_1, \lambda_2, ..., \lambda_m\} \\ G_\alpha & \text{if } \alpha = \{\lambda_1, \lambda_2, ..., \lambda_m\} \end{Bmatrix}$$

In either case $\pi_\alpha [B]$ is open set. Hence , $\pi_\alpha [B]$ is open set, i.e., each point $\pi_\alpha [p]$ in $\pi_\alpha [G]$ belong to an open set

$$\pi_\alpha [B] \subset \pi_\alpha [B]$$

Therefore, $\pi_\alpha\left[G\right]$ is an open set. Hence, each projection mapping π_α is an open mapping.

Theorem 14: Let X and Y be two topological spaces. Then the product space $X \times Y$ is connected if and only if X and Y are connected.

Prof: Let $X \times Y$ be connected space.

Since the projection mapping π_α and π_y are onto and continuous it follows that X and Y are also connected spaces.

Conversely, let X and Y are connected space. Then we have to prove that $X \times Y$ is also connected.

Let $\left(x_1, y_1\right)$ and $\left(x_2, y_2\right)$ be any two elements of $X \times Y$. Then $\left(x\right\} \times Y$ is homeomorphic to Y and $X \times \left\{Y_2\right\}$ is homeomorphic to X .

It follows that $\left\{x_1\right\} \times Y$ and $X \times \left\{y_1\right\}$ are connected spaces. Since these two spaces interseat at the point $\left(x_1, y_2\right)$. It follows the their union is a connected space.

Conversely, let X and Y are connected space. Then we have to prove that $X \times Y$ is also connected. Let $\left(x_1, y_1\right)$ and $\left(x_2, y_2\right)$ be any two elements of $X \times Y$. Then $\left(x\right\} \times y$ is homeomorphic to Y and $X \times \left\{y_2\right\}$ is homeomorphic to X.

It follows that $\left\{x_1\right\} \times Y$ and $X \times \left\{y_2\right\}$ are connected spaces.

Since these two spaces intersect at the point $\left(x_1, y_2\right)$ it follows the their union is a connected space. Since the union contains $\left\{x_1, y_1\right)$ and $\left(x_2, y_2\right)$ it follows that $X \times Y$ is connected.

Theorem 15: The product space $X = x\left\{X_\lambda : \lambda \in \wedge\right\}$ is Housdorff if and only if each space X_α is Housedorff.

Proof: Let each co-ordinate space X_λ be T_2 space and let $x=\{x_\lambda : \lambda \in \wedge\}$ and $y=\{y_\lambda : \lambda \in \wedge\}$ be two distinct points of the product space X. Then $x_\mu \neq y_\mu$ for some $\mu \in \wedge$, where $x_\mu \in X_\mu$ and $y_\mu \in X_\mu$ since X_μ is T_2 -space there exist open sets G_μ and H_μ in X_μ such that

$$x_\mu \in G_\mu, y_\mu \in H_\mu, \text{ and } x_\mu \in G_\mu, y_\mu \in H_\mu, \qquad (1)$$

Since $\pi_\mu(x)=x_\mu$ and $\pi_\mu(y)=y_\mu$

Using (1) , we have $\quad x \in \pi_\mu^{-1}[G_\mu], y \in \pi_\mu^{-1}[H_\mu]$

And $\quad \pi_\mu^{-1}[G_\mu \cap H_\mu] = \pi_\mu^{-1}[\phi] = \phi$

Or $\quad \pi_\mu^{-1}[G_\mu] = \pi_\mu^{-1}[H_\mu] = \phi$

But $\pi_\mu^{-1}[G_\mu]$ and $\pi_\mu^{-1}[H_\mu]$ are open in X being sub-baisc members of the product topology. Thus, we have to show that each pair x, y of distinct points of X there exist two disjoint open sets one containing x and the other containing y.

It follows that the product space X is T_2 -space conversely, let the product space X be T_2 -space. We shall show that the co-ordinate space X_μ is Housdorff for arbitrary $\mu \in \wedge$

Let a_μ and b_μ be any two distinct points of X_μ choose x and y in X such that x and y differ only in the μ th co-ordinate and such that $\quad x_\mu = a_\mu$ and $y_\mu = b_\mu$

Since space is T_2, there exist open sets g and H in X such that

$$x \in G \text{ and } y \in H \text{ and } G \cap H = \phi$$

There exist basic open sets $\quad U = x\{U_\lambda : \lambda \in \wedge\}$

And $$V = x\{V_\lambda : \lambda \in \wedge\}$$

Such that $x \in U \subset G$ and $y \in V \subset H$ and $U \subset V = \phi$

It is follows that U_μ and V_μ are open sets in X_μ such that $x_\mu = a_\mu \in U_\mu$ and $y_\mu = b_\mu \in V_\mu$ and

$$U_\mu \cap V_\mu = \phi$$

Hence, X_μ is Housdorff space.

14.6 Summary

Let $(X, \Im)$ be a topological space. Let A be a subset of X. A family $\mathcal{A}$ of subsets of X is said to be cover for $(X, \Im)$ if only if $U(v : v \in \mathcal{A}) = X$. Also if $\mathcal{B} \subset \mathcal{A}$ such that $\mathcal{B}$ is also a cover for X then $\mathcal{B}$ is a subcover of $\mathcal{A}$.

An open cover of A is a family $\{v : v \in \mathcal{A}\}$ of $\Im -$ open subset of X such that each point in X belongs to at least one number of the class $\{v : v \in \mathcal{A}\}$ i.e., $A \subseteq \{v : v \in \mathcal{A}\}$.

A cover of a topological space $(X, \Im)$ is said to be a finite cover if it has only a finite number of member.

Let $(X, \Im)$ be a topological space. Then $(X, \Im)$ is said to be compact if and only if every open cover has a finite subcover.

(i) $U\{G_i : i \in I\} = X$ (open cover)

(ii) $U\{G_i : i \in (1, 2, ..., N)\} = X$ (finite subcover)

Let $(X, \Im)$ be a topological space. A set A of X is said to be compact if every $\Im$ -open cover of A has a finite subcover.

A collection of subsets of X is said to have finite intersection property if and only if the intersection of member of each finite sub-collection is non-empty.

Let $(X,\Im)$ be a topological space. Then $(X,\Im)$ is said to be locally compact space if and only if every point of the set has at least one neighbourhood whose closure is compact.

Let $(X,\Im)$ be a topological space. Then $(X,\Im)$ is said to be lindelof space if every open cover of X has a countable sub cover.

Let $(X,\Im)$ be a topological space. Then $(X,\Im)$ is said to have the Bolzano-Weiertrass property if every infinite subset of X has a limit point. Any space with Bolzano-weiertrass property is called Frechet compact space.

Let (X,d) be a metric space. Then (X,d) is said to be sequentially compact if every sequence in X has a convergent sub-sequence. For example, the set of all real number in $(0,1)$ is not sequentially compact. For the sequence $\left(\frac{1}{2},\frac{1}{3},.....\right)$ in $(0,1)$ converges to a $0 \notin (0,1)$ on the other hand $[0,1]$ is sequentially compact.

Let (X,d_1) and (Y,d_2) be two metric spaces. A mapping f defined on a metric space X and Y is uniformly continuous if $\in > 0$ then there exist $\delta > 0$ depending on $\in$ alone such that

$$d_2\left(f(x),f(a)\right) < \in \text{ and } \qquad d_1(x,a) < \delta$$

Every open cover of a sequentially compact metric space has a Lebasque number.

A subset of the real line is compact if and only if it is closed and bounded.

Let $(X_1,\Im_1)$ and $(X_2,\Im_2)$ be two topological spaces. The topology $\Im$ whose base is $B=\{G_1\times G_2 : G_1\in\Im_1 \text{and } G_2\in\Im_2\}$ is said to be product topology for $X_1\times X_2 = X$. The corresponding topological space $(X, \Im)$ is known as product space of X_1 and X_2 ..

The mappings $\pi_x : X \times Y \to X$ such that

$$\pi_x(x, y) = x; \forall (x, y) \text{ in } X \times Y$$

And $\pi_y : X \times Y \to Y$ such that

$$\pi_y(x, y) = y; \forall (x, y) \text{ in } X \times Y$$

are called projection mappings of the product $X \times Y$.

An embedding of a topological space x into another space Y we mean a mapping $f : X \to Y$ which defines a homeomorphism of X onto $f(X)$.

14.7 Terminal Questions

Q.1. Write the Bolzano Weiertrass property.

Q.2. Explain the Heine-Borel theorem.

Q.3. What do you mean by Heine-Borel theorem?

Q.4. Define compact space and compact set.

Q.5. Let $X = \{a, b, c, d\}$ and $\Im = \{X, \phi, \{a\}, \{d\}, \{b, c\}, \{a, b, c\}, \{a, d\}, \{b, c, d\}\}$. Let $C = \{\{a, b\}, \{b, c\}, \{d\}\}$ is an open cover of X and $\{\{a, d\}, \{b, c\}\}$ is a finite subcover of C. To show that $(X, \Im)$ is compact space.