U. P. RajarshiTandon
Open University

Master of Science
PGMM -103N
Discrete Mathematics

Block: 1 Set Theory

Block: 2 Logic

Block: 3 Boolean Algebra

Block: 4 Graph Theory

Course Design Committee

Prof. Ashutosh Gupta, **Chairman**
Director, School of Sciences, UPRTOU, Prayagraj

Prof. A. K. Malik **Coordinator**
School of Sciences, UPRTOU, Prayagraj

Prof. Mukesh Kumar **Member**
Department of Mathematics, MNIT, Prayagraj

Dr. A. K. Pandey **Member**
Associate Professor, ECC, Prayagraj

Dr. Raghvendra Singh **Invited Member/Secretary**
Assistant Professor (Contractual), Mathematics
School of Sciences, UPRTOU, Prayagraj

Course Preparation Committee

Dr. Raghvendra Singh **Author (Unit 1 - 14)**
Assistant Professor (Contractual), Mathematics
School of Sciences, UPRTOU, Prayagraj

Prof. Satish Kumar **Editor (Unit 1 - 14)**
Department of Mathematics
D N College, Meerut

Prof. A. K. Malik **Program Coordinator**
School of Sciences, UPRTOU, Prayagraj

Dr. Raghvendra Singh **Course Coordinator/Invited Member/Secretary**
Assistant Professor (Contractual), Mathematics
School of Sciences,UPRTOU, Prayagraj

PGMM –103N: DISCRETE MATHEMATICS

ISBN-978-81-19530-44-1

Printed By: Chandrakala Universal Pvt. 42/7 Jawahar Lal Neharu Road, Prayagraj.

Syllabus

PGMM-103N/MAMM-103N: Discrete Mathematics

Block-1: Set Theory

Unit-1: Sets

Introduction, Representation of sets, types of sets, subset, universal set, Venn diagram, operations on sets, and algebra of sets.

Unit-2: Relations

Introduction, inverse relation, representation of relations, types of relations, equivalence relation, and partial order relation.

Unit-3: Functions

Introduction, inverse function, types of functions, real valued function, identity function, constant function, composition of functions.

Unit-4: Techniques of counting

Introduction, partition, principle of inclusion-exclusion, pigeonhole principle, permutations and combinations.

Block-2: Logic

Unit-5: Mathematical Logic

Introduction, proposition, basic logical operations, truth table, logical equivalence, algebra of propositions, Tautology, contradiction.

Unit-6: Normal Form

Introduction, normal form, disjunctive normal form, conjunctive normal form, logic in proof, universal and existential quantifiers.

Unit-7: Mathematical Induction

Introduction, methods of proof, principle of mathematical induction.

Unit-8: Recurrence Relations

Introduction, generating function, properties of generating functions, numeric function, recurrence relation, solution of recurrence relation.

Block-3: Boolean Algebra

Unit-9: Boolean Algebra

Introduction, binary operations, algebraic structure, Boolean algebra, Boolean expression, Boolean functions and logic gates.

Unit-10: Lattices

Introduction, Lattice, properties of lattice, principle of duality, semi and complete lattice, sublattice, isomorphic and bounded lattice.

Block-4: Graph Theory

Unit-11: Introduction to Graph

Definition of a graph, simple and multi-graph, degree of a vertex, types of graph: null graph, complete graph, regular graph.

Unit-12: Advanced Graph Theory

Path, cycle and circuit, Eulerian and Hamiltonian graph, matrix representation of graph, planner graph, graph coloring.

Unit-13: Tree

Introduction, tree, types of tree.

Unit-14: Rooted and Binary Tree

Introduction, rooted tree, spanning tree, minimal spanning tree, binary tree.

Master of Science
PGMM -103N
Discrete Mathematics

U. P. RajarshiTandon Open University

Block

1 Set Theory

Block-1

Set Theory

In this block we shall discuss about the set theory, relation, function & techniques of counting and its applications. Set theory has a great importance in the study of mathematics and computer sciences. A German mathematician Georg Cantor (1845-1918) introduces the idea of set theory. The concept of set theory has a great contribution in analysis. Finite sets are very important for the study of combinatory theory of counting. George Cantor (1874) discussed the term of countable set. Countable sets have great importance in real and discrete mathematics. In the set theory of real numbers, R can be geometrically demonstrated through the points on a straight line. Set theory and real number system are the fundamental of the Mathematics. The key concept of analysis must be based on an exactly defined on the concept of number.

Set theory forms the groundwork for mathematics, offering a structure for defining mathematical entities and systems. Its applications span diverse mathematical disciplines, including algebra, analysis, and topology. Relations are pivotal in mathematics, especially in discrete mathematics, logic, and computer science, where they model diverse relationships between objects. Functions are core to mathematics, describing connections between quantities, modeling real-world phenomena, and defining mathematical operations. They are a cornerstone in calculus, algebra, and various other mathematical fields.

In the first unit we deal with Introduction to sets, and representation of sets, types of sets, subset, universal set, Venn diagram, operations on sets, and algebra of sets. Second unit we shall discuss with the relations, inverse relation, and representation of relations, types of relations, equivalence relation, and partial order relation. In the third unit we deal with functions, inverse function, types of functions, real valued function, identity function, constant function, composition of functions. Partition, principle of inclusion-exclusion, pigeonhole principle, permutations and combinations are discussing in details in unit fourth.

Unit – 1: Sets

Structure

1.1 Introduction:

In modern mathematics, the words set and element are very common and appear in the most of the texts. They are even overused. There are instances when it is not appropriate to use them. For example, it is not good to use the word element as a replacement for other, more meaningful words. When you call something an element, then the set whose element is this one should be clear. The word element makes sense only in combination with the word set, unless we deal with a nonmathematical term (like chemical element), or a rare old-fashioned exception from the common mathematical terminology (sometimes the expression under the sign of integral is called an infinitesimal element; lines, planes, and other geometric images are also called elements). In dictionary the word set is defined as a collection, a group, a class or an assemblage etc.

1.2 Objectives:

After reading this unit we should be able to understand the:

- set theory and its properties
- represent sets by the Listing Method and Set-builder Method
- types of sets: Empty or Null Set, Singleton Set, Set of Sets, Multi Set
- Comparable and Non-comparable Set, Finite and Infinite Sets
- Equality of Sets, Cardinality of a set, Power Set, Index sets and Disjoint Sets
- Subset, Superset, Proper Subset, Improper Subset
- Universal set and Venn diagram,
- Operations on sets: Complement, Union, Intersection, Difference, Symmetric Difference
- Algebra of sets and Some Important Theorems on Set Theory

1.3 Sets:

In our daily life we usually use the word 'set' as set of natural numbers, set of real numbers, set of integers, tea set, set of books of an author, set of an examination papers, set of authors of this book, etc. In all of these, the meaning of the word 'set' is a collection of well-defined objects. A *set* is a well-defined collection of objects. Suppose A is a set and a is an element of A, then we write $a \in A$ (a belongs to A). If a is not an element of A, then we write $a \notin A$ (a does not belongs to A). Some examples of sets are following:

(i) All the states of India.
(ii) Rivers of India.
(iii) Mountains of India.
(iv) All the students of an engineering college.
(v) All the letter in the word 'Topology'.
(vi) A collection of numbers 1 to 9.
(vii) Members in a family.
(viii) People live in Rajasthan.
(ix) The set of all teachers in a college.
(x) Set of triangles in a plane.
(xi) All the straight lines passing through a given point.
(xii) Leaves of a tree.

Here in above sentences the objects belonging to a set may be anything i.e., numbers, states, rivers, letters, points, students, members, people, teachers, triangles etc. These objects are called elements or members of the set. Sets are denoted by capital letters and their element by lower case letter. Some sets of numbers are defined by name and the notations are the following:

(1) C is denoted a set of complex numbers (all real and imaginary numbers).

(2) R is denoted by a set of all real numbers (all rotational and irrational numbers).

(3) Q is denoted by a set of rational numbers (all integers and fractions with positive and negative both).

(4) I is denoted by a set of all integer numbers (all negative and positive integers with zero).

(5) N is denoted by a set of natural numbers.

(6) P is denoted by a set of all prime numbers.

(7) ϕ is denoted by an empty set or null set.

1.4 Representation of Sets:

Sets are generally defined by two methods (forms), as below:

Listing Method

Let A be the set A= {1, 3, 5, 7, 9, 11} ….. (i)

Here $1 \in A$, $3 \in A$, $5 \in A$, $7 \in A$, $9 \in A$, $11 \in A$ but $2 \notin A$. The form of presentation of set A in (i) is known as *listing method or tabular method* or *roster method*.

Set-builder Method

Also the equation (i) can be written as

$$A = \{x \mid x \text{ is an odd positive integer and } x < 13\} \quad \text{…. (ii)}$$

It means that A is the set of all odd positive integers which are less than 11. The form of presentation of set A in (ii) is known as *set-builder method* or *rule method.*

For example. The set consisting of all the letters in the word "DELHI" can be written as

{D, E, L, H, I}

or $\{x \mid x$ is a letter in the word "DELHI"$\}$.

For example. The set consisting of all even positive integers is denoted by {2, 4, 6, 8, 10, 12, ………} or $\{x \mid x$ is an even positive integer$\}$.

For example. The set consisting of fourth roots of unity is denoted by $\{1, -1, i, -i\}$ or $\{x \mid x^4=1\}$.

Note: The order is not preserved in case of a set, *i.e.*, each of the sets $\{a, b, c\}$, $\{b, c, a\}$, $\{a, c, b\}$, $\{c, a, b\}$ denote the same set.

1.5 Types of Sets:

Empty or Null Set

A set is said to be *empty set* or *null set* or *void set* if it contains no element. It is denoted by ϕ or {}.

For example. Let A $=\{x \mid x$ is a real number and $x^2 = -1\}$, B = $\{x \mid x < x\}$ and C = $\{x \mid x \in$ I and $1<x<2\}$.

Here we see that the sets A, B and C have no element, i.e., A, B and C are empty sets.

Singleton Set

A set is said to be *singleton set* or *unit set* if it contains only one element.

For example. Let A $=\{x \mid x$ is a positive integer and $x^2 = 4\}$ and B = {0}. Then we can write A={2} and B={0}. Here A and B both are singleton sets.

Set of Sets

If a set contains a number of sets as its elements, then it is known as *set of sets* or *family of sets* or *class of sets*.

For example. Let A = {{*a*}, {*a*, *b*}, {*a*, *b*, *c*}, {*a*, *b*, *c*, *d*}, {*a*, *b*, *c*, *d*, *e*}} and B = {{0}, {0, 1}, {0, 1, 2}}. Here A and B are set of sets.

Example.1. Explain the following sets: (i) {0} (ii) ϕ and (iii) {ϕ}.

Solution: (i) {0} is a set containing only one element 0. So {0} is called singleton set.

(ii) ϕ is a set containing no element. So ϕ is called empty set.

(iii) {ϕ} is a family of sets containing only one set ϕ.

Multi Set

A *multi set* is an unordered collection of objects in which an object can appear more than once.

For example. Let A = {*a*, *a*, *b*, *b*, *b*, *c*}. Here *a* appears two times, *b* appears three times and *c* appear one time. Therefore, A is called a multi set.

Comparable and Non-comparable Set

Let A and B be any two sets. Then A and B are said to be *comparable* if all the elements of A belongs to B or all the elements of B belongs to A (*i.e.*, $A \subseteq B$ or $B \subseteq A$). But if $A \not\subset B$ or $B \not\subset A$, then A and B are known as *non-comparable set*.

Note: Every set is comparable with itself, i.e., $A \subset A$, therefore A and A (itself) are comparable sets.

Example.2. Let A = {1, 2, 3, 4, 5, 6, 7, 9}, B = {1, 3, 5, 7} and C = {2, 5, 6, 7, 9}.

Solution: Here we see that all the elements of B belongs to A, *i.e.*, $B \subset A$, therefore A and B are comparable sets.

Also $C \subset A$, therefore A and C are comparable sets. But $C \not\subset B$ or $B \not\subset C$, therefore B and C are non-comparable sets.

Example.3. Let A = {*a*, *m*, *a*, *r*, *j*, *e*, *e*, *t*} and B = {*a*, *j*, *e*, *e*, *t*}.

Solution: Here we see that all the elements of B belongs to A, i.e., $B \subset A$, therefore A and B both are comparable sets.

Finite and Infinite Sets

A set is said to be *finite set* if it contains finite number of elements, otherwise it is infinite.
Let A be the set of all students of an engineering college, B is the set of vowels and N is the set of natural numbers.
Here A and B are finite sets and N is infinite set.

Equality of Sets

Let A and B be any two sets. If all the elements of A belongs to B and all the elements of B belongs to A, (i.e., $A \subseteq B$ and $B \subseteq A$) then A and B are said to be *equal sets* and written as A = B.

Example.4. Let A={N, I, R, A, N, J, A, N} and B={N, I, R, A, J}.
Solution: Here all the elements of A belongs to B and all the elements of B belongs to A , therefore A and B are equal set, i.e., A=B.

Cardinality of a set

Let A be any finite sets. The number of distinct elements contained in A is known as *cardinality* of the set A. It is denoted by $n(A)$ or $|A|$.

For example. Let A be the set A= {1, 2, 3, 4, 5}.

Here the number of distinct element in set A are 5, therefore $n(A) = 5$.

Note: For an empty set, $n(\phi)=0$.

Power Set

Let A be any set. The power set of A is the set of all subsets of A. It is denoted by P(A).

For example. Let A={a, b, c} be the set. Then the power set of A is defined as

$P(A) =\{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

Note: The number of elements in a P(A) is 2 raised to the cardinality of A i.e., the number of P(A) = $2^{n(A)}$. If A = {*a*, *b*, *c*, *d*}, then number of P(A) = 2^4 = 16.

Index sets

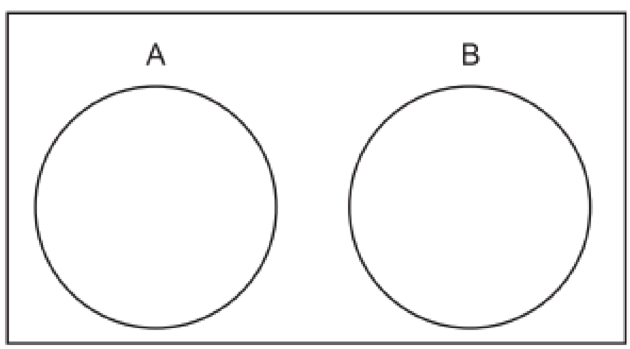
Index set is a set whose elements are used as names. It is usually denoted by Λ. An index set may be finite or infinite.

For example. Let A={*a*, *b*, *c*,…..}, B={α, β, γ, …..} and C={*i*, *j*, *k*, …..} be any three sets. Here we see that the all elements of A, B and C are used as names. Therefore A, B and C are index sets.

Disjoint Sets

Let A and B be any two sets. Then A and B are said to be *disjoint sets* if they have no common elements.

For example. Let A = {1, 3, 5, 7} and B = {2, 4, 6, 8} be any two sets. Here we see that the sets A and B have no common elements, i.e., A∩B=ϕ. Therefore A and B are disjoint sets.



1.6 Subsets:

Let A and B be any two sets. If all the elements of A belong to B, then A is said to be *subset* of B. It is denoted as A⊂B, read as "A is a subset of B" or "A is contained in B".

For example. Let A be the set A= {a, b, c}. Then ϕ, {*a*}, {b}, {c}, {*a*, b}, {*a*, c}, {b, c} and A are all subsets of A.

For example. If A= {*a*, b, c}, then {b, d} is not a subset of A because d∉A.

Superset

Let A and B be any two sets. A⊂B is also expressed by writing as B⊃A and is read as "B contains A" or B is a super set of A.

For example. Let A = {1, 2, 3} and B = {1, 2, 3, 4}.

Here we see that all elements of set A belong to set B, i.e., A⊂B, i.e., B contains A**.** Therefore B is a super set of A.

Proper Subset

Let A and B be any two sets. Then A is said to be proper subset of B if A⊂B, A≠ϕ and A≠B.

Improper Subset

Let A be any set then ϕ and itself A are improper subsets of A.

Note: Singleton sets has only improper subsets.

For example. Let A be the set A= {*a*, b, c}. Then the all proper subsets of A are

ϕ, {*a*}, {b}, {c}, {*a*, b}, {*a*, c}, {b, c} and A.

Here {*a*}, {b}, {c}, {*a*, b}, {*a*, c} and {b, c} are all subsets of A.

ϕ and A are improper subsets of A.

1.7 Universal Set

A set which contains all the sets considered in a particular discussion as its subset is called *universal set*. For the sets of numbers, the set of complex numbers C will be the universal set. It is denoted by U.

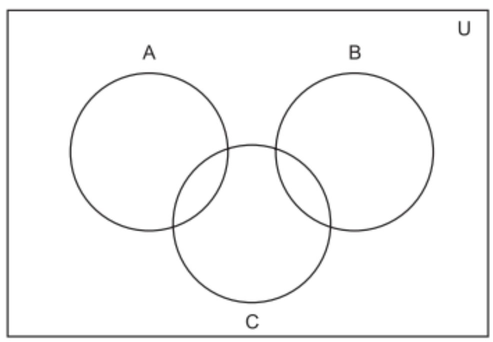
or

Consider a big set U which contains a number of smaller sets as its subsets. Suppose we are considering these smaller sets in a certain discussion, then U is called a *universal set*.

For example Let A = {1, 2, 3} and U = {1, 2, 3, 4, 5, 6, 7, 8}. Here all the elements of A belongs to U. Then U may be considering a universal set for the set A.

1.8 Venn Diagram

A *Venn diagram* is a pictorial representation of sets which it is represented by a rectangle and the sets by circles.



1.9 Operations on Sets:

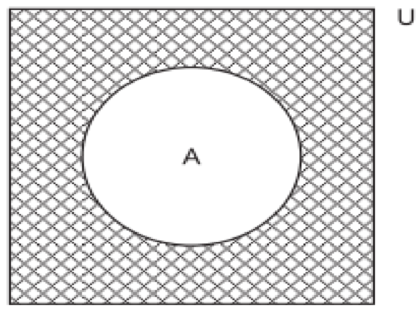
Here we introduce and study some basic operations in this section. Using these operation, we can construct new sets by combining the elements of the given sets.

Complement of a Set

Let U be the universal set. The *complement* of a set A with respect to U is the set of elements which belong to U but do not belong to A. It is denoted by U-A or $\overline{A}$ *or* A' or A^c and is defined as $\overline{A}$ ={x: $x \in$U and $x \notin$A}.

For example. Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9} and A = {1, 3, 5, 7, 9} be any two sets. Then complement of A or U-A or $\overline{A}$ = {2, 4, 6, 8}.

For example. Let U= {*x*: *x* is a letter in English alphabet} and A = {*x*: *x* is a vowel} be any two sets. Then complement of A or $\overline{A}$ = {*x*: *x* is a consonant}.

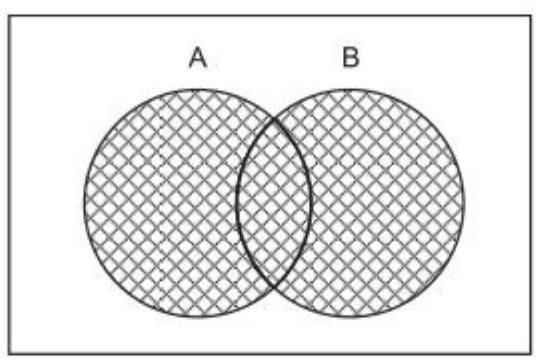


Union of Sets

Let A and B be any two sets. The *union* of A and B is the set of all elements which belong to A or to B. It is denoted by A∪B = {*x*: *x*∈A or *x*∈B}.

For example. Let A = {1, 2, 3, 4, 5} and B = {2, 4, 6, 8, 10} be any two sets.

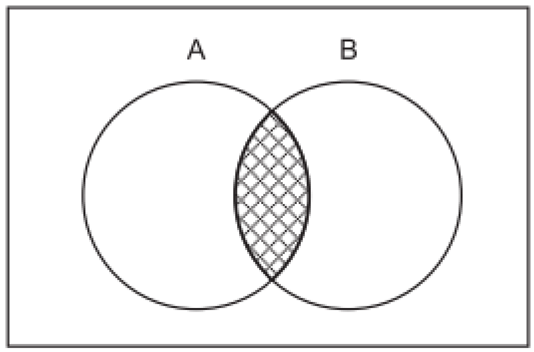
Then union of A and B or (A∪B) = {1, 2, 3, 4, 5, 6, 8, 10}.



Intersection of Sets

Let A and B be any two sets. The *intersection* of A and B is the set of elements which belong to both A and B and is denoted by $A \cap B = \{x: x \in A \text{ and } x \in B\}$.

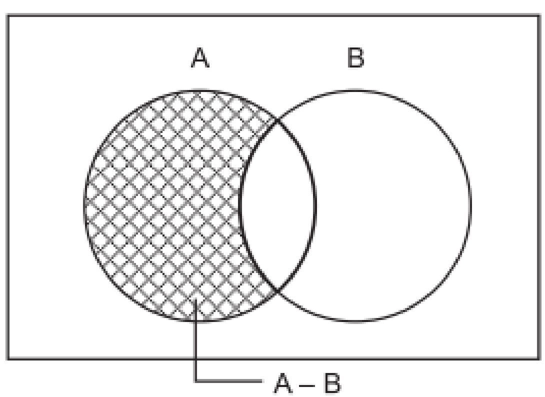
For example. Let A = {1, 2, 3, 4, 5} and B = {2, 4, 6, 8, 10} be any two sets. Then intersection of A and B or $A \cap B = \{2, 4\}$.



Difference of Sets

Let A and B be any two sets. The *difference* of A and B is the set of elements which belong to A but do not belong to B. It is denoted by A-B *or* A~B *or* $A/B = \{x: x \in A \text{ and } x \notin B\}$.

For example. Let A = {1, 2, 3, 4, 5, 6, 8} and B = {3, 4, 5, 6, 7, 10} be any two sets. Then the difference of A and B is A-B = {1, 2, 8} and the difference of B and A is B-A = {7, 10}.



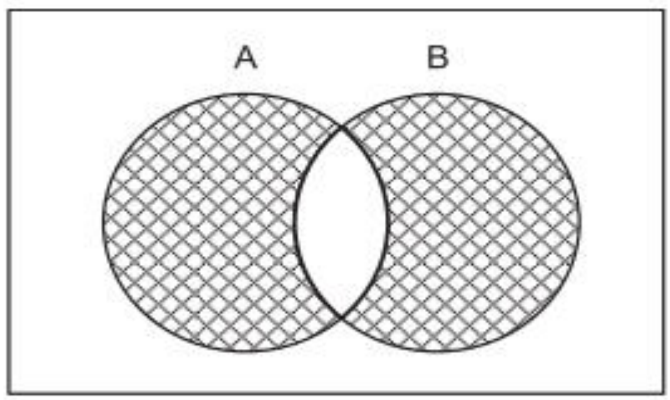
Symmetric Difference of Sets

Let A and B be any two sets. The *symmetric difference* of A and B is the set of elements which belong to A or B but do not belong to both A and B. It is denoted by A⊕B and defined as

$$A \oplus B = \{x : (x \in A \text{ and } x \notin B) \text{ or } (x \notin A \text{ and } x \in B)\}$$

or $$A \oplus B = (A - B) \cup (B - A).$$

For example. Let A = {1, 2, 3, 4, 5} and B = {1, 3, 5, 7} be any two sets. Then the symmetric difference of A and B or A⊕B = {2, 4, 7}.



1.10 Algebra of Sets:

In this section we discussed some important properties of set theory, which are following:

(*a*) For empty set

(i) $A \cup \phi = A$ (ii) $A \cap \phi = \phi$

(*b*) For universal set

(i) $A \cup U = U$ (ii) $A \cap U = A$

(*c*) For idempotent

(i) $A \cup A = A$ (ii) $A \cap A = A$

(*d*) For Complements

(i) $\phi' = U$ (ii) $U' = \phi$ (iii) $(A')' = A$

(iv) $A \cup A' = U$ (v) $A \cap A' = \phi$ (vi) $(A \cup B)' = A' \cap B'$

(vii) $(A \cap B)' = A' \cup B'$

(*e*) For Associative

(i) $A \cup (B \cup C) = (A \cup B) \cup C$ (ii) $A \cap (B \cap C) = (A \cap B) \cap C$

(*f*) For Commutative

(i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$

(*g*) For Distributive

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Some Important Theorems on Set Theory

In this section we discussed some important theorems on set theory, which are following:

Theorem:1 Let *A*, *B* and *C* be any three sets, then

(i) $A \cup (B \cup C) = (A \cup B) \cup C$ **(ii)** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(iii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ **(iv)** $A - (B \cup C) = (A - B) \cap (A - C)$

(v) $A - (B \cap C) = (A - B) \cup (A - C)$ **(vi)** $(A - C) \cap (B - C) = (A \cap B) - C$

(vii) $\boldsymbol{A\times(B\cup C)=(A\times B)\cup(A\times C)}$

Proof: **(i)** Let x be any element of $A\cup(B\cup C)$. Then we have

$x\in A\cup(B\cup C)$

$\Leftrightarrow$ $x\in A$ or $x\in(B\cup C)$

$\Leftrightarrow$ $x\in A$ or ($x\in B$ or $x\in C$)

$\Leftrightarrow$ $x\in A$ or $x\in B$ or $x\in C$

$\Leftrightarrow$ ($x\in A$ or $x\in B$) or $x\in C$

$\Leftrightarrow$ $x\in(A\cup B)$ or $x\in C$

$\Leftrightarrow$ $x\in[(A\cup B)\cup C]$

Hence, $A\cup(B\cup C)=(A\cup B)\cup C$.

(ii) Let x be any element of $A\cup(B\cap C)$. Then we have

$x\in[A\cup(B\cap C)]$

$\Leftrightarrow$ $x\in A$ or $x\in B\cap C$

$\Leftrightarrow$ $x\in A$ or ($x\in B$ and $x\in C$)

$\Leftrightarrow$ ($x\in A$ or $x\in B$) and ($x\in A$ or $x\in C$)

$\Leftrightarrow$ ($x\in A\cup B$) and ($x\in A\cup C$)

$\Leftrightarrow$ $x\in[(A\cup B)\cap(A\cup C)]$

Hence, $A\cup(B\cap C=(A\cup B)\cap(A\cup C)$.

(iii) Let x be any element of $A\cap(B\cup C)$. Then we have

$x\in A\cap(B\cup C)$

$\Leftrightarrow \quad x \in A$ and $x \in B \cup C$

$\Leftrightarrow \quad x \in A$ and ($x \in B$ or $x \in C$)

$\Leftrightarrow \quad$ ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$)

$\Leftrightarrow \quad x \in (A \cap B)$ or $x \in (A \cap C)$

$\Leftrightarrow \quad x \in [(A \cap B) \cup (A \cap C)]$

Hence, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(iv) Let x be any element of $A - (B \cup C)$. Then we have

$\quad x \in A - (B \cup C)$

$\Leftrightarrow \quad x \in A, x \notin (B \cup C)$

$\Leftrightarrow \quad x \in A$, ($x \in B$ or $\in C$)

$\Leftrightarrow \quad (x \in A, x \notin B)$ and $(x \in A, x \notin C)$

$\Leftrightarrow \quad x \in (A - B)$ and $x \in (A - C)$

$\Leftrightarrow \quad x \in [(A - B) \cap (A - C)]$

Thus, $A - (B \cup C) \subseteq (A - B) \cap (A - C)$

and $(A - B) \cap (A - C) \subseteq A - (B \cup C)$

Hence, $A - (B \cup C) = (A - B) \cap (A - C)$.

(v) Let x be any element of $A - (B \cap C)$. Then we have

$\quad x \in A - (B \cap C)$

$\Leftrightarrow \quad x \in A, x \notin (B \cap C)$

$\Leftrightarrow \quad x \in A$, x belongs either to B' or to C'

$\Leftrightarrow \quad x \in A,\ x \in B'$ or $x \in A,\ x \in C'$

$\Leftrightarrow \quad x \in A - B$ or $x \in A - C$

$\Leftrightarrow \quad x \in (A - B) \cup x(A - C)$

Thus, we have

$A - (B \cap C) \subseteq (A - B) \cup (A - C)$

and $(A - B) \cup (A - C) \subseteq A - (B \cap C)$

Hence, $A - (B \cap C) = (A - B) \cup (A - C)$.

(vi) Let x be any element of $(A - C) \cap (B - C)$. Then we have

$x \in [(A - C) \cap (B - C)]$

$\Leftrightarrow \quad x \in A - C$ and $x \in B - C$

$\Leftrightarrow \quad x \in A,\ x \notin C$ and $x \in B,\ x \notin C$

$\Leftrightarrow \quad x \in A$, and $x \in B,\ x \notin C$

$\Leftrightarrow \quad x \in (A \cap B),\ x \notin C$

$\Leftrightarrow \quad x \in (A \cap B) - C$

Hence, $(A - C) \cap (B - C) = (A \cap B) - C$.

(vii) First we have to prove

$A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$

Let (a, b) be any element of $A \times (B \cup C)$. Then we have

$(a, b) \in A \times (B \cup C)$

$\Rightarrow \quad a \in A$ and $b \in B \cup C$

$\Rightarrow \quad a \in A$ and $(b \in B$ or $b \in C)$

$\Rightarrow \quad (a \in A$ and $b \in B)$ or $(a \in A$ and $b \in C)$

$\Rightarrow \quad (a, b) \in A \times B$ or $(a, b) \in A \times C$

$\Rightarrow \quad (a, b) \in (A \times B) \cup (A \times C)$

Thus, $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$

Now we have to prove

$(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$

Let (x, y) be any element of $(A \times B) \cup (A \times C)$. Then we have

$\quad (x, y) \in (A \times B) \cup (A \times C)$

$\Rightarrow \quad (x, y) \in A \times B$ or $(x, y) \in A \times C$

$\Rightarrow \quad (x \in A$ and $y \in B)$ or $(x \in A$ and $y \in C)$

$\Rightarrow \quad x \in A$ and $(y \in B$ or $y \in C)$

$\Rightarrow \quad (x, y) \in A \times (B \cup C)$

Thus, $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$

Hence, $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Note: Cartesian product of two set is not commutative.

Theorem:2 Let A and B be any two sets, then

(i) $A - B = A \cap B'$ **(ii)** $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

(iii) $B - A \subseteq A'$ **(iv)** $B - A' = B \cap A$

(v) $\boldsymbol{A - B \subset A}$ **(vi)** $\boldsymbol{(A - B) \cap B = \phi}$**.**

(vii) $\boldsymbol{(A \cup B)' = A' \cap B'}$ **(viii)** $\boldsymbol{(A \cap B)' = A' \cup B'}$

Proof: (i) We have

$$A - B = \{x : x \in A, x \notin B\}$$

$$= \{x : x \in A, x \in B'\}$$

$$= \{x : x \in (A \cap B')\}$$

Hence, $A - B = A \cap B'$

(ii) Let x be any element of $(A - B) \cup (B - A)$. Then we have

$$x \in (A - B) \cup (B - A)$$

$\Leftrightarrow$ $x \in (A - B)$ or $x \in (B - A)$

$\Leftrightarrow$ $(x \in A \text{ } or \text{ } x \notin B)$ or $(x \in B, x \notin A)$

$\Leftrightarrow$ $(x \in A$ or $x \in B)$ but x does not belong to both A and B

$\Leftrightarrow$ $x \in (A \cup B)$ but $x \notin (A \cap B)$

$\Leftrightarrow$ $x \in (A \cup B) - (A \cap B)$

Thus $(A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B)$

and $(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A)$

Hence, $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

(iii) Let x be any element of $B - A$. Then we have

$x \in B - A$

$\Rightarrow$ $x \in B, x \notin A$

$\Rightarrow \quad x \in B, x \in A'$

i.e., each element of $B - A$ belongs to A'.

Hence, $B - A \subseteq A'$.

(iv) Let x be any element of $B - A'$. Then we have

$B - A' = \{x : x \in B, x \notin A'\}$

$= \{x : x \in B, x \in A\}$

$= \{x : x \in B \cap A\}$

$= B \cap A$

Hence, $B - A' = B \cap A$.

(v) Using definition of difference, we know that the all the elements of $A - B$ are the elements of A. So, we have $A - B \subset A$.

Hence, $A - B \subset A$.

(vi) Let x be any element of $(A - B) \cap B$. Then we have

$(A - B) \cap B = \{x : x \in A - B \text{ and } x \in B\}$

$= \{x : x \in A, x \notin B \text{ and } x \in B\}$

Here, there is no element in A which belongs to B and also does not belong to B.

Hence, $(A - B) \cap B = \phi$.

(vii) Let x be any element of $(A \cup B)'$. Then we have

$x \in (A \cup B)'$

$\Leftrightarrow \quad x \notin A \cup B$

$\Leftrightarrow \quad x \notin A$ and $x \notin B$

$\Leftrightarrow \quad x \in A'$ and $x \in B'$

$\Leftrightarrow \quad x \in (A' \cap B')$

Thus, $(A \cup B)' \subseteq (A' \cap B')$

and $(A' \cap B') \subseteq (A \cup B)'$

Hence, $(A \cup B)' = A' \cap B'$.

(viii) Let x be any element of $(A \cap B)'$. Then we have

$x \in (A \cap B)'$

$\Leftrightarrow \quad x \notin (A \cap B)$

$\Leftrightarrow \quad x \notin A$ or $x \notin B$

$\Leftrightarrow \quad x \in A'$ or $x \in B'$

$\Leftrightarrow \quad x \in (A' \cup B')$

Thus, $(A \cap B)' \subseteq (A' \cup B')$ and $(A' \cup B') \subseteq (A \cap B)'$

Hence, $(A \cap B)' = A' \cup B'$.

Theorem:3 Let A and B be any two sets such that $A \subseteq B$. Then

(i) $A \cap B = A$ (ii) $A \cup B = B$ (iii) $B' \subseteq A'$

Proof: (i) It is given that $A \subseteq B$. Then we have,

$A \cap B \subseteq A$

Now if, $x \in A \Rightarrow x \in B$

$\Rightarrow \quad x \in A,\ x \in B$

$\Rightarrow \quad x \in A \cap B$

$\therefore \quad A \subseteq A \cap B$

Hence, $A \cap B = A$

(ii) We have, $B \subseteq A \cup B$

Now we have,

$x \in A \cup B$

$\Rightarrow \quad x \in A$ or $x \in B$

$\Rightarrow \quad x \in B \qquad \{\because\ A \subset B\}$

$\therefore \quad A \cup B \subseteq B$

Hence, $A \cup B = B$.

(iii) Let x be any element of B'. Then we have

$x \in B'$

$\Rightarrow \quad x \notin B$

$\Rightarrow \quad x \notin A \qquad \{\because\ \mathrm{A} \subseteq B\}$

$\Rightarrow \quad x \in A'$

Hence, $B' \subset A'$.

Theorem:4 Let A and B be any two sets, then

(i) $(A \cup B) \cap B' = A$ *iff* $A \cap B = \phi$ (ii) $A - B = A$ *iff* $A \cap B = \phi$

(iii) $\boldsymbol{A \oplus A = \phi}$ **(iv)** $\boldsymbol{A \oplus \phi = A}$

(v) $\boldsymbol{A \oplus B = \phi}$ ***iff*** $\boldsymbol{A = B}$**.**

Proof: (i) Using distributive law, we have

$(A \cup B) \cap B' = (A \cap B') \cup (B \cap B')$

$= (A \cap B') \cup \phi$ $\{\because B \cap B' = \phi\}$

$= A \cap B'$

Now we have to prove, $A \cap B = \phi$ *iff* $A \cap B' = A$

Let $A \cap B' = A$

$\Rightarrow$ $A \subseteq B'$

$\Rightarrow$ $A \cap B = \phi$

Again we have

$A \cap B = \phi$

$\Rightarrow$ $A \subseteq B'$

$\Rightarrow$ $A \cap B' = A$

Hence, $(A \cup B) \cap B' = A$ *iff* $A \cap B = \phi$.

(ii) We have

$A - B = \{x : x \in A, x \notin B\}$

$= \{x : x \in A, x \in B'\}$

$= \{x : x \in A \cap B'\}$

Thus, $A - B = A \cap B'$

Now if, $A - B = A$ then we have

$$A = A \cap B'$$

$\Rightarrow \quad A \subseteq B'$

$\Rightarrow$ all elements of A belong to B'

$\Rightarrow$ none element of A belong to B

$\Rightarrow \quad A \cap B = \phi$.

Conversely, if $A \cap B = \phi$.

Then we have

$$A - B' = \phi$$

$\Rightarrow \quad A \subseteq B'$

Now we have,

$$A \cap B' = A$$

$\Rightarrow \quad A - B = A$.

(iii) We have

$$A \oplus A = (A - A) \cup (A - A)$$

$$= \phi \cup \phi$$

$$= \mathrm{A}$$

(iv) We have

$$A \oplus \phi = (A - \phi) \cup (\phi - A)$$

$$= A \cup \phi$$

$= A$

(v) We have

$A \oplus B = (A - B) \cup (B - A)$

Let $A \oplus B = \phi$, then we have

$\Leftrightarrow \quad (A - B) \cup (B - A) = \phi$

$\Leftrightarrow \quad A - B = \phi$ and $B - A = \phi$

$\Leftrightarrow \quad A = B$

Hence, $A \oplus B = \phi$ *iff* $A = B$.

1.10 Summary:

A set is a well-defined collection of objects. The objects in a set are known as members or elements or points. A multi set is an unordered collection of objects in which an object can appear more than once.

A set is said to be empty set or null set or void set if it contains no element. It is denoted by ϕ or {}. Let A and B be any two sets. If all the element of A belongs to B, then A is said to be subset of B.

If a set contains a number of sets as its elements then it is known as set of sets or family of sets or class of sets.

Two sets A and B are said to be disjoint sets if they have no common elements. A set is said to be finite s*et* if it contains finite number of elements, otherwise it is infinite. Let A be any set. The power set of A is the set of all subsets of A. Index set is a set whose elements are used as names.

The difference of A and B is the set of elements which belong to A but do not belong to B. The symmetric difference of A and B is the set of elements which belong to A or B but do not belong to A and B.

1.11 Terminal Questions:

Q.1 List of elements of the following sets:

(a) $\{x : x \in I , x^2 < 11\}$

(b) $\{x : x \in N$, x is even and $x < 17\}$

(c) $\{x : x \in N$, x is prime and $x < 21\}$

(d) $\{x : x$ is a solution of $x^2 + 3x + 2 = 0\}$

(e) $\{x : x \in I , x < 3\}$

(f) $\{x : x \in N, x + 5 = 3\}$

(g) $\{x : x$ is a month with exactly 30 days$\}$

Q.2 Let $U = \{1, 2, 3, \ldots, 9, 10\}$ be the universal set and $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 7. 9\}$, $C = \{2, 5, 6, 8\}$. Find

(a) A', B', C'

(b) $A \cup B$, $B \cup C$, and $A \cup C$

(c) $A \cap B$, $B \cap C$, $A \cap C$

(d) $A - B$, $B - A$, $B - C$, $C - B$, $A - C$ and $C - A$.

(e) $A \oplus B$, $B \oplus C$, and $A \oplus C$

Q.3 Which of the sets are equal?

(a) $\{x : x$ is a letter in the word 'wolf '$\}$

(b) $\{x : x$ is a letter in the word 'follow'$\}$

(c) The lettters *f, l, o, w*.

(d) The letters which appear in the word 'flow'.

Q.4 Is a set A comparable with itself?

Q.5. Find the power set of $\{1, 2\}$

Q.6. Let $A = \{a, b, c\}$ and $B = \{c, d, e, f\}$. Find the $A - B$, $B - A$ and $A \oplus B$.

Q.7. Prove that $A \cap (B - C) = A \cap B) - (A \cap C)$

Q.8. If $A = \{a, b, c\}$. find all the subsets of A.

Q.9. Let $A = \{1, 2,\}$ and $B = \{3, 4\}$. Find $A \times B$ and $\mathrm{B} \times \mathrm{A}$.

Q.10. Prove that $A \subset B$ and $C \subset D \Rightarrow (A \times C) \subset (B \times D)$

Q.11. Prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Q.12. Prove that $A \cup B = (A \oplus B) \oplus (A \cap B)$

Answer

1. {(a) $\{-3, -2, -1, 0, 1, 2, 3\}$

(b) $\{2, 4, 6, 8, 10, 12, 14, 16\}$

(c) $\{2, 3, 5, 7, 11, 13, 17, 19\}$

(d) $\{-1, -2\}$

(e) {... –, –3, –1, 0, 1, 2}

(f) ϕ or { }

(g) {April, June, September, November}.

2. (a) A' = {5, 6, 7, 8, 9, 10}, B' = 1, 2, 5, 6, 8, 10}, C' = {1, 3, 4, 7, 9, 10}

(b) $A \cup B$={1, 2, 3, 4, 7, 9}, $B \cup C$={2, 3, 4, 5, 6, 7, 8, 9} and $A \cup C$={1, 2, 3, 4, 5, 6, 8}

(c) $A \cap B$ = {3, 4}, $B \cap$ C = φ and $A \cap C$ = {2}.

(d) $A - B$ = {1, 2}, $B - A$ = {7, 9}, $B - C$ = {3, 4, 7, 9}, $C - B$ ={2, 5, 6, 8},

$A - C$ = {1, 3, 4} and $C - A$ = {5, 6, 8}.

(e) $A \oplus B$ = {1, 2, 7, 9}, $B \oplus C$ = {2, 3, 4, 5, 6, 7, 8, 9} and $A \oplus C$ = {1, 3, 4, 5, 6, 8}.

3. All the given sets are equal.

4. Yes

5. ϕ, {1}, {2}, {1, 2}.

6. {*a*, *b*}, (*d*, *e*, *f*} and {*a*, *b*, *d*, *e*, *f*}

8. *A*, ϕ, {*a*}, {*b*}, {*c*}, {*a*, *b*}, {*b*, *c*}, {*a*, *c*}, {*a*, *b*, *c*}

9. $A \times B$ = {(1, 3}, (1, 4), (2, 3), (2, 4)}

$B \times A$ = {(3, 1), {3, 2}, (4, 1), (4, 2)}.

UNIT- 2 : Relations

Structure

2.1 Introduction

Relations are essential in mathematics, especially in fields like discrete mathematics, logic, and computer science. They serve as a foundational concept for modeling diverse relationships between objects. The term "relation" brings to mind familiar examples of relationships between individuals, such as father to son, mother to daughter, and etc. In mathematics, if A is the set of real numbers, there are numerous commonly used relations between two real numbers, such as "less than," "greater than," and "equality." These examples illustrate relationships between objects. A relation that describes the relationship between two objects is called a binary relation. If a relation describes the relationship among three objects, it is called a ternary relation. In set theory, relations are a fundamental concept that describe how elements of sets are related to each other. A relation between two sets, say A and B is a subset of their Cartesian product $A \times B$.

Relations have extensive applications in various fields, including social sciences, economics, engineering, and technology. In computer science, the concept of relations is a fundamental tool for understanding and learning. Hence, the concept of relations is pervasive and fundamental, providing a framework for understanding and describing relationships between objects in various contexts.

2.2 Objectives

After studying this unit, the learner will be able to understand the:

- Ordered pairs and the Cartesian Product of Sets
- Ordered Sets and Relations
- inverse relations, identity relation, universal relation
- types of relations, and equivalence relation and equivalence classes
- order relation and partial order relation

2.3 Ordered Pairs

An *ordered pair* is represented by (a, b) in which a is first element and b is second element. Let (a, b) and (x, y) are two ordered pairs. Then we have

$(a, b) = (x, y)$ if $a = x$ and $b = y$.

2.4 Cartesian Product of Sets

Let A and B be any two sets. The *Cartesian products* of A and B is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$

i.e., $A \times B = \{(a, b) : a \in A, b \in B\}$

and $B \times A = \{(b, a) : b \in B, a \in A\}$.

Examples

Example.1. Let $A = \{a, b\}$ and $B = \{1, 2, 3\}$ be any two sets. Find the Cartesian product of : (i) A × B (ii) A × A.

Solution: The Cartesian product of A and B is

$A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$

and the Cartesian product of A and A is

$A \times A = \{(a, a), (a, b), (b, a), (b, b)\}$.

In general, if $A_1, A_2, A_3, \ldots\ldots\ldots\ldots, A_n$ are n sets then the product set of all these sets is

$A_1 \times A_2 \times A_3 \times \ldots\ldots\ldots\ldots \times A_n = \{a_1, a_2, a_3 \ldots\ldots\ldots\ldots, a_n) : a_1 \in A_1, a_2 \in A_2, a_3 \in A_3, \ldots\ldots\ldots, a_n \in A_n\}$.

Note: 1. If one of them in two sets is infinite and the other is non-empty set then the Cartesian product of two set is also infinite set.

Theorem:1. If $A \subset B$, show that $A \times A \subset (A \times B) \cap (B \times A)$.

Proof: Let (a, b) be any two element of $A \times A$. Then we have

$a \in A$ and $b \in A$

Since $A \subset B$

$\Rightarrow \quad a \in B, \quad b \in B$

Now we have $a \in A, \; b \in B$

$\Rightarrow \quad (a, b) \in A \times B$

and $\quad a \in B, \; b \in A$

$\Rightarrow \quad (a, b) \in B \times A$

Therefore (a, b) belongs to $A \times B$ and $B \times A$ both.

i.e., $\quad (a, b) \in (A \times B) \cap (B \times A)$

Hence, $A \times A \subset (A \times B) \cap (B \times A)$.

2.5 Ordered Sets

A set X is said to be an *ordered set* if any order relation exist between every pair of distinct elements of set, i.e., for any two elements a and b such that (i) $a<b$ or $b<a$ if $a \neq b$ and (ii) $a<b$ or $b<c \Rightarrow a<c$

For example. The set of natural numbers and the set of integers are ordered sets, according to increasing or decreasing their magnitude.

Note.1. Subset of an ordered set are ordered set.

2. An empty set is an ordered set.

3. A singleton set is an ordered set.

2.6 Relation or Binary Relation

In our day to life, a word used relation means something like as marriage and friendship, etc. "Is the mother of", "is the father of", "is the sister of", is the brother of", "is the friend of", are all relations over the set of men. Similarly, "is equal to", is less than", "is greater than", "is the divisor of" are relations on the set of numbers. In this book we study binary relations. A binary relation is the relation between two objects.

For example, "is the son of" is a relation between two men a and b. Therefore the binary relation involves with certain ordered pair (a, b) in which the first element a is related to the second element b.

Let A and B be any two sets. A *relation* R from a set A to set B is a subset of $A \times B$ and defined as

xRy if and only if $(x, y) \in R$, $x \in A$ and $y \in B$

or $\quad xRy \Leftrightarrow (x, y) \in R$ and

$x \not{R}\, y \Leftrightarrow (x, y) \notin R$,

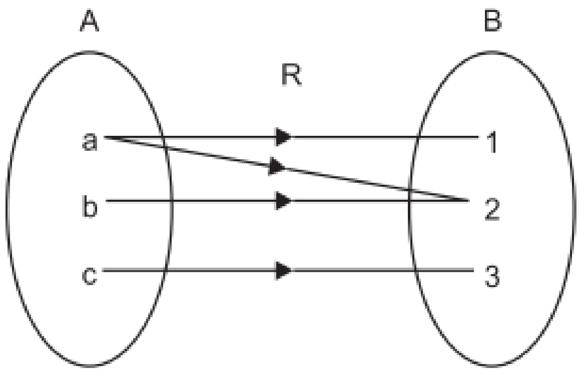
$x\,R\,y$ reads "x is R-related to y".

Note: (i) If R is a relation from A to A then R is known as relation on A.

(ii) A binary relation on a set A is a subset of A × A.

For example. Let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$ be any two sets.

Then $R = \{(a, 1), (a, 2), (b, 2), (c, 3)\}$ is a relation from A to B.



2.7 Inverse Relation

Let R be a relation from a set A to a set B. Then R^{-1} from B to A is known as the *inverse relation* of R if and only if

$$R^{-1} = \{(y, x) : (x, y) \in R\}.$$

For example. Let $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$ be any two sets.

Then $R = \{(1, 2), (1, 4), (2, 4), (3, 6)\}$ is a relation from A to B

and $R^{-1} = \{(2, 1), (4, 1), (4, 2), (6, 3)\}$ is an inverse relation from B to A.

Note: (i) Every relation has an inverse relation.

(ii) Let A= {1, 2, 3, 4} and R be the relation > (is greater than). Then

2.8 Identity Relation

Let $A = \{a, b, c\}$ be any set. Then a relation R on a set A is known as an *identity relation* if

$R = \{(a, a) : a \in A\}$.

For example. Let $A = \{a, b, c, d\}$ be any set. Then the relation $R = \{(a, a), (b, b), (c, c), (d, d)\}$ is an identity relation on A.

2.9 Universal Relation

Let $A = \{a, b, c\}$ be any set. Then a relation R on a set A is known as *universal relation* if

$$R = A \times A$$

or $R = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$

is a universal relation on A.

For example. Let $A = \{a, b\}$ be any set. Then the relation $R = \{(a, a), (a, b), (b, a), (b, b)\}$ is a universal relation on A.

Note: (i) If R is a relation from A to A then R is known as relation on A.

(ii) A binary relation on a set A is a subset of $A \times A$.

(iii) Every relation has an inverse relation.

(iv) Let $A=\{1, 2, 3, 4\}$ and R be the relation > (is greater than). Then we have

$$R=\{(2, 1), (3, 2), (3, 1), (4, 3), (4, 2), (4, 1)\}.$$

2.10 Types of Relation

Some of the important types of relations are as follows:

(i) Reflexive Relation

A relation R on a set A is known as *reflexive* relation if and only if $aRa, \forall a \in A$.

(ii) Symmetric Relation

A relation R on a set A is known as *symmetric* relation if and only if $aRb \Rightarrow bRa, \forall(a, b)\in R$.

(iii) Anti-symmetric Relation

A relation R on a set A is known as *anti-symmetric* relation if and only if $aRb, bRa \Rightarrow a = b, \forall(a, b)\in R$.

(iv) Transitive Relation

A relation R on a set A is known as *transitive* relation if and only if $aRb, bRc \Rightarrow aRc,$ $(a, b, c\in A)$.

Note: (i) In R, the relation "is equal to" is reflexive, symmetric and transitive.

(ii) In R, the relation "less than" is anti-symmetric and transitive.

(iii) The relation "is the friend of" on the set of all human beings is reflexive.

(iv) The relation "less than", "greater than", "is the father of", "is the wife of" on the set of people are not reflexive.

(v) The relation "a divides b" on set of natural numbers is anti-symmetric for a divides b and b divides a if and only if $a = b$.

(vi) The relation "is the brother of" on any set of men is transitive for a is brother of b, b is brother of c then a is brother of c.

(vii) The relation "is the father of" is not transitive.

Examples

Example.2. Write a relation which is reflexive but neither symmetric nor transitive.

Solution: Let $A = \{a, b, c\}$ be any set and the relation R on A defined as

$R = \{(a, a), (a, c), (b, a), (b, b), (c, b), (c, c)\}$.

Then we have

(i) Reflexive: We have $(a, a) \in R$, $\forall a \in A$.

Therefore R is reflexive on A, *i.e.*, $(a, a), (b, b), (c, c) \in R$.

(ii) Symmetric: We have $(a, c) \in R$ but $(c, a) \notin R$.

Therefore R is not symmetric on A, i.e., $(a, c), (b, a), (c, b) \in R$ but $(c, a), (a, b), (b, c) \notin R$.

(iii) Transitive: We have $(a, c), (c, b) \in R$ but $(a, b) \notin R$.

Therefore R is not transitive on A. Hence R is reflexive but neither symmetric nor transitive.

2.11 Equivalence Relation

A relation *R* on a set *A* is known as an *equivalence relation* if and only if it is reflexive, symmetric and transitive. Equivalence relation is denoted by ~.

For example. The relation "is the brother of". On any set of men, "is equal to" on the set of numbers are all equivalence relation.

For example. Let $A = \{a, b, c\}$ and the relation *R* on *a* set *A* is defined as

$R = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$ is an equivalence relation.

Note: A universal relation R on any set A always satisfied the properties of equivalence relation.

Examples

Example.3. Let I be an integer set and R is a relation on I defined as

$R = \{(a, b): a<b \text{ and } a, b\in I\}$ is not an equivalence relation.

Solution: Let $R = \{(a, b): a<b \text{ and } a, b\in I\}$.

Then (i) Reflexive: We have $(a, a)\notin R$, i.e., a is not less than a, $\forall a\in I$.

Therefore R is not reflexive on I.

(ii) Symmetric: Suppose $(a, b)\in R$ i.e., $a<b \Rightarrow (b, a)\notin R$, i.e., b is not less than a.

Therefore R is not symmetric on I, i.e., $(a, b)\in R \Rightarrow (b, a)\notin R$ (because if a is less than b then b is not less than a)

(iii) Transitive: We have $(a, b), (b, c)\in R \Rightarrow (a, c)\in R$ i.e., $a<b$ and $b<c\Rightarrow a<c$.

Therefore R is transitive on I. Hence R is transitive but neither reflexive nor symmetric.

Example.4. If R is an equivalence relation on a set A then show that R^{-1} is also an equivalence relation on A.

Solution: Let $A = \{a, b, c\}$ be any set and the relation R on a set A. Suppose R is an equivalence relation, i.e., R is reflexive, symmetric and transitive. To show that R^{-1} is an equivalence relation.

Then (i) R is reflexive: We have $(a, a)\in R$, $\forall a\in A$

$\Rightarrow (a, a)\in R^{-1}$, $\forall a\in A$

Therefore R^{-1} is reflexive on A.

(ii) R is symmetric: We have $(a, b)\in R \Rightarrow (b, a)\in R$.

Now we have $(a, b)\in R^{-1} \Rightarrow (b, a)\in R$

$\Rightarrow (a, b)\in R$

$\Rightarrow (b, a)\in R^{-1}$.

Therefore $(a, b)\in R^{-1} \Rightarrow (b, a)\in R^{-1}$,

i.e., R^{-1} is symmetric on A.

(iii) R is transitive: We have $(a, b), (b, c)\in R \Rightarrow (a, c)\in R$.

Now we have $(a, b), (b, c)\in R^{-1} \Rightarrow (b, a), (c, b)\in R$

$\Rightarrow (c, b), (b, a)\in R$

$\Rightarrow (c, a)\in R$

$\Rightarrow (a, c)\in R^{-1}$

Therefore $(a, b), (b, c)\in R^{-1} \Rightarrow (a, c)\in R^{-1}$, i.e., R^{-1} is transitive on A.

Hence R^{-1} is reflexive on A i.e., R^{-1} is reflexive, symmetric and transitive.

2.12 Equivalence Classes

Let R be an equivalence relation on a set A. Let a be any arbitrary element of A. The set of all element $x \in A$ such that xRa constitute a subset of A (say $[a]$).

Thus subset $[a]$ is known as *equivalence class* of a with respect to R, denoted as

$[a]=\{x : x\in A \text{ and } xRa\}$.

2.13 Order Relation

A relation which is transitive but not an equivalence relation is known as an *order relation*.

If R is an order relation on a set X, then

$$x\mathrm{R}y \text{ and } y\mathrm{R}z \Rightarrow x\mathrm{R}z,\ \forall\ x, y, z \in \mathrm{X}.$$

2.14 Partial Order Relation

A relation R on a set X is said to be a *partial order relation* if it is at the same time

(i) Reflexive

(ii) Anti-symmetric and

(ii) Transitive.

It is denoted by the symbol $\leq$. A set X together with a partial order relation defined on it, *i.e.*, $(X, \leq)$ is known as a *partial ordered set*.

For example. The relation "x divides y" on the set of natural numbers is a partial order relation. The relation "sub-set of" on the set of all sub-sets of a set is a partial order relation.

2.15 Summary

The Cartesian products of A and B is the set of all ordered pairs (a, b) such that $a\in A$ and $b\in B$ *i.e.*, $A\times B = \{(a, b) : a\in A, b\in B\}$ and $B\times A = \{(b, a) : b\in B, a\in A\}$. Let R be a relation from a set A to a set B. Then R^{-1} from B to A is known as the inverse relation of R if and only if $R^{-1} = \{(y, x) : (x, y)\in R\}$.

Let $A = \{a, b, c\}$ be any set. Then a relation R on a set A is known as an identity relation if $R =\{(a, a) : a \in A\}$.

A relation R on a set A is known as reflexive relation if and only if aRa, $\forall a \in A$. A relation R on a set A is known as symmetric relation if and only if $aRb \Rightarrow bRa\ \forall (a, b) \in R$. A relation R on a set A is known as anti-symmetric relation if and only if $aRb, bRa \Rightarrow a = b\ \forall (a, b) \in R$. A relation R on a set A is known as transitive relation if and only if $aRb, bRc \Rightarrow aRc, (a, b, c \in A)$. A relation R on a set A is known as an equivalence relation if and only if it is reflexive, symmetric and transitive.

A relation which is transitive but not an equivalence relation is known as an order relation. If R is an order relation on a set X, then $x\mathrm{R}y$ and $y\mathrm{R}z \Rightarrow x\mathrm{R}z$, $\forall\ x, y, z \in \mathrm{X}$. A relation R on a set X is said to be a partial order relation if it is at the same time (i) Reflexive (ii) Anti-symmetric and (ii) Transitive.

A set X together with a partial order relation defined on it, *i.e.*, $(X, \leq)$ is known as a partial ordered set. If A and B are two sets such that $A \leq B$ and $B \leq A$, then $A \sim B$.

2.16 Terminal Questions

Q.1. What do you means by Cartesian product of sets?

Q.2. Write a short note on types of relations.

Q.3. Explain the Equivalence relation with example.

Q.4. If $A = \{a, b, c, d\}$ and $R = \{(a, a), \{b, b), (c, c), (d, d)\}$. Prove that R is reflexive.

Q.5. Give an example of a relation which is symmetric and transitive but not reflexive.

Q.6. Give an example of a relation that is reflexive but neither symmetric nor transitive.

Q.7. Give an example of a relation which is transitive but not reflexive or symmetric.

Answer

5. $A = \{a, b, c\}$ and $R = \{(a, a), \{b, b), (a, b), \{b, a)\}$.

6. $A = \{a, b, c\}$ and $R = \{(a, a), (b, b), (c, c), (a, b), (b, c)\}$.

7. $A = \{a, b, c, d\}$ and $R = \{(a, b), (b, c), (a, c)\}$.

Unit -3: Functions

Structure

3.1 Introduction:

Functions are a versatile and powerful tool used to model and analyze complex systems and relationships. In mathematics, a function represents a relation between a set of inputs (the domain) and a set of possible outputs (the codomain), where each input is uniquely related to exactly one output. Functions find numerous applications across various fields, including mathematics, physics, engineering, economics, computer science, biology, statistics, and finance. In engineering, functions are used to model and analyze systems such as electrical circuits, mechanical systems, and control systems, as well as in signal processing and image processing.

In mathematics, functions are fundamental for modeling relationships between quantities, describing geometric shapes, and solving equations, particularly in calculus, algebra, and analysis. In computer science, functions play a fundamental role in defining the algorithms, data structures, flow chart, and software applications, and are used in areas such as cryptography and computer graphics.

3.2 Objectives:

After reading this unit you should be able to understand the:

- Defining of a function and types of functions
- Inverse of a Mapping, Inclusion Mapping and Identity Mapping
- Real Valued Mapping, Characteristic Function and Constant Function
- Zero Function, Injective and Bijective Mapping
- Equality of Mapping
- Composition of Functions or Product of Functions

3.3 Functions or Mapping

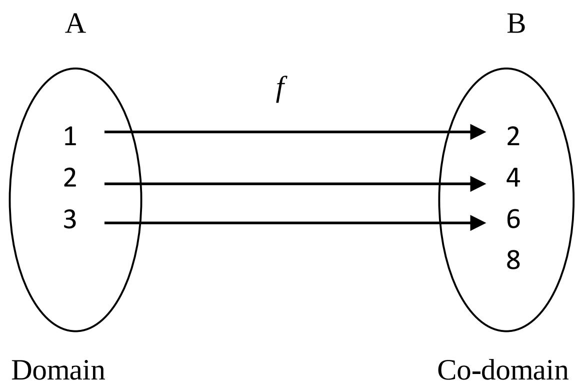
Let A and B be any two non-empty sets. If there exists a rule or a correspondence f which associate each element of A has a unique image in B then f is a function or mapping from A to B. This mapping is denoted by

$$f : A \to B$$

or $$A \xrightarrow{f} B.$$

Here the set A is known as domain and the set B is known as co-domain of the function f.

For example. Let $A = \{1, 2, 3\}$, $B = \{2, 4, 6, 8\}$ and $f : A \to B$ is defined as



Here range is $\{2, 4, 6\}$. We know that the range is a subset of co-domain.

Examples

Example.1. If $A = \{1, 2, 3\}$ and $B = \{a, b, c, d\}$ then does

(i) $\{(1, a), (2, c), (3, d)\}$

(ii) $\{(1, a), (2, b), (2, c), (3, d)\}$

(iii) {(1, *a*), (2, *b*)}

(iv) {(1, *a*), (2, *b*), (3, *a*)} represent a function from $f : A \rightarrow B$.

Solution: (i) Here we see that $f(1)=a$, $f(2)=c$ and $f(3)=d$. Therefore *f* is a function from A to B because every element of A has a unique image in B.

(ii) Here we see that $f(1)=a$, $f(2)=b$, $f(2)=c$ and $f(3)=d$. Therefore *f* is not a function from A to B because every element of A has not a unique image in B, i.e., one element (2) of A has two images (*b, c*) in B.

(iii) Here we see that $f(1)=a$ and $f(2)=b$. Therefore *f* is not a function from A to B because every element of A has not a unique image in B, i.e., one element (3) of A has not any image in B.

(iv) Here we see that $f(1)=a$, $f(2)=b$ and $f(3)=a$. Therefore *f* is a function from A to B because every element of A has a unique image in B.

Example.2. If $A = \{1, 2, 3, 4\}$ and $f(1)=2$, $f(2)=3$, $f(3)=4$ and $f(4)=2$, then does *f* represent a function.

Solution: We have $A = \{1, 2, 3, 4\}$ and $B = \{2, 3, 4\}$.

Here we see that every element of A has a unique image in B. Therefore *f* is a function from A to B.

3.4 Types of Functions

Here we discuss some types of functions which as follows:

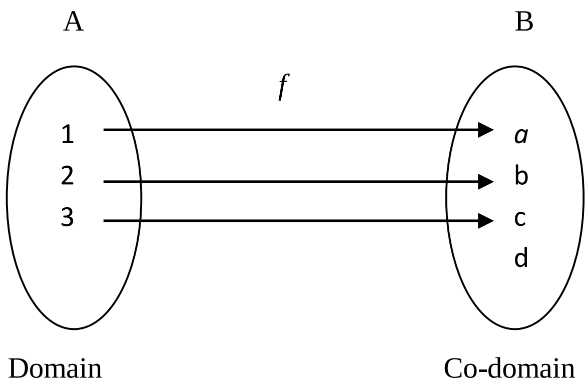
(i) One-One Function

A function $f: A \rightarrow B$ is called *one-one* if $x_1, x_2 \in A$, we have

$x_1 = x_2 \quad \Rightarrow \quad f(x_1) = f(x_2)$

or $\quad x_1 \neq x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2)$.

For example. Let $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$ and $f : A \rightarrow B$ is defined as



Here f is known as one-one function and range of f is $\{a, b, c\}$.

Examples

Example.3. If $A = \{1, 2, 3, 4, 5\}$, $B = \{a, b, c\}$ and f is defined as $f(1)=a$, $f(2)=b$, $f(3)=a$, $f(4)=a$ and $f(5)=c$, then state whether f is a function from A to B or not, if yes write its type.

Solution: Here we see that

$$f(1)=a,\ f(2)=b,\ f(3)=a,\ f(4)=a \text{ and } f(5)=c,$$

Therefore f is a function from A to B because every element of A has a unique image in B.

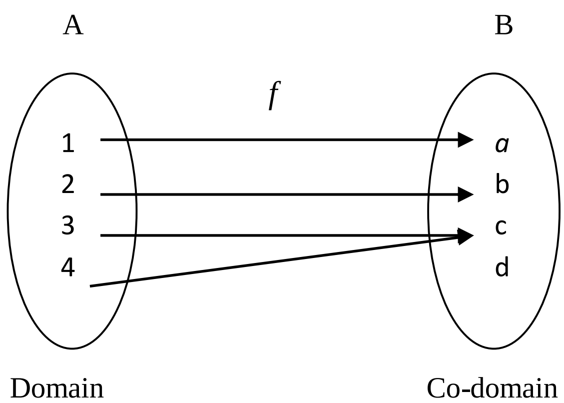
Hence f is a function from A to B.

Also we see that three elements (1, 3, 4) of A has same image (a) in B. Hence f is not one-one function from A to B, i.e., one element (2) of A has two images (b, c) in B.

(ii) Many-One Function

A function $f : A \rightarrow B$ is said to be *many-one* if at least one element of B has two or more than two pre-image in A.

For example. Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$ and $f : A \rightarrow B$ is defined as

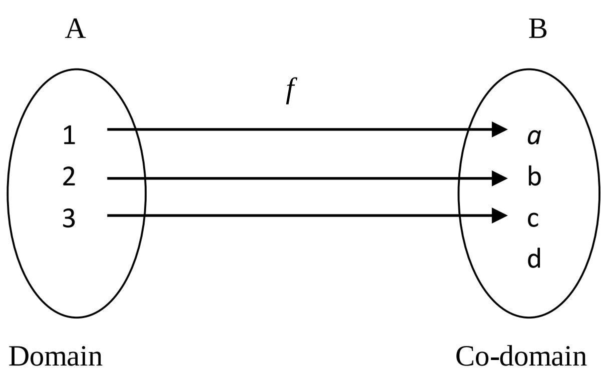


Here f is known as many-one function and range of f is $\{a, b, c\}$.

(iii) Into Function

A function $f : A \rightarrow$ B is said to be *into* if there is at least one element of B, has no pre-image in A.

For example. Let $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$ and $f : A \rightarrow B$ is defined as

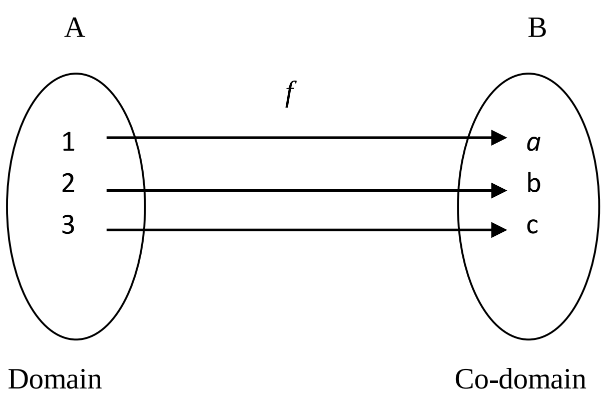


Here one element d of the set B has no pre-image in the set A. Then f is known as into function and range of f is $\{a, b, c\}$.

(iv) Onto Function

A function $f : A \rightarrow B$ is said to be *onto* if there is no element of B, which is not an image of some element of A.

For example. Let A {1, 2, 3}, $B = \{a, b, c\}$ and $f : A \rightarrow B$ is defined as



Here f is known as onto function and range of f is $\{a, b, c\}$.

3.5 Inverse of a Mapping

Let $f : X \rightarrow Y$ be a *one-one onto mapping* and

$$f(x)=y, \forall x \in X, \forall y \in Y.$$

Now we define a mapping $f^{-1}: y \rightarrow X$ such that

$$f^{-1}(y)=x, \forall x \in X, \forall y \in Y,$$

where f^{-1} is called the inverse of f. Here f is inversible mapping because inverse of f is exists.

Examples

Example.4. Let f be a function $f: R \to R$ is defined as $f(x)=x^2$, $\forall\ x \in R$, where R is the set of real numbers. Find the value of $f^{-1}(9)$.

Solution: It is given that f be a function $f: R \to R$ is defined as $f(x)=x^2$, $\forall\ x \in R$.

We have $f^{-1}(9) = \{x \in R: f(x) = 9\}$

$= \{x \in R: x^2 = 9\}$

$= \{x \in R: x = 3, -3\}$

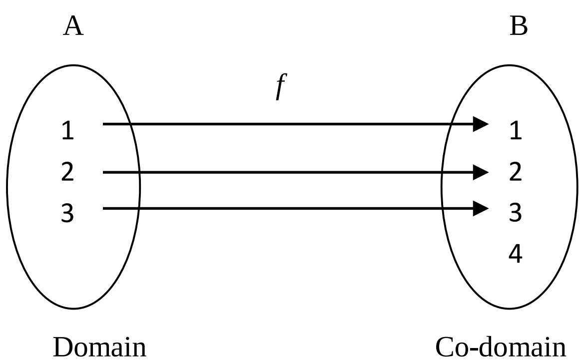
$= \{3, -3\}$.

3.6 Inclusion Mapping

Let X be any subset of Y. Then the mapping $f: X \to Y$ is said to be *inclusion mapping* if

$$f(x) = x, \forall x \in X.$$

For example. Let $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4\}$ and $f: A \to B$ is defined as

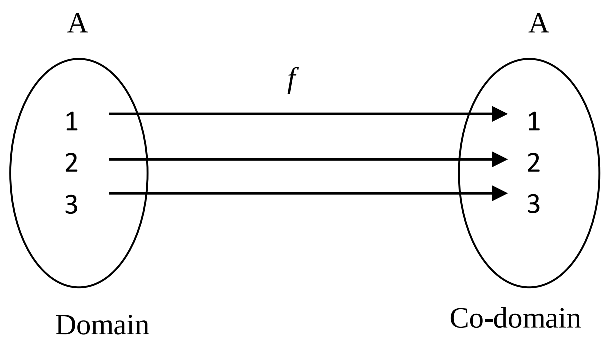


Here f is known as inclusion mapping.

3.7 Identity Mapping

Let $f : X \to X$ be a mapping. Then f is said to be *identity mapping* if $f(x) = x, \forall x \in X$.

For example. Let $A = \{1, 2, 3\}$ and $f : A \to A$ is defined as



Here f is known as identity mapping.

3.8 Real Valued Mapping

A mapping $f : X \to R$, where R is the set of real numbers, is known as *real valued mapping*.

3.9 Characteristic Function and Constant Function

Let U be the universal set and A be a subset of U. Then the real valued function f: $U \to \{0, 1\}$

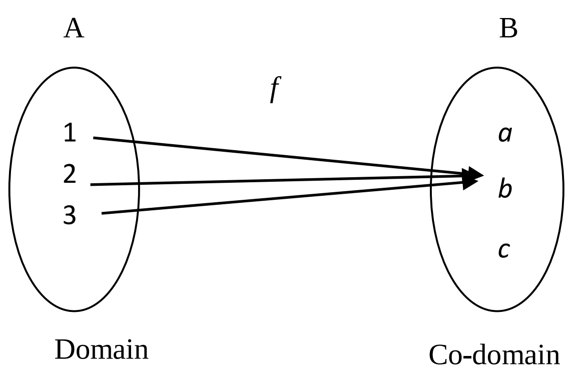
such that $$f_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

is known as characteristic function of A.

Let $f: X \to Y$ be a function. Then f is said to be *constant function* if $f(x) = a, \forall x \in X$

i.e., a function $f : X \to Y$ is known as *constant function* if each element of X is mapped onto a single element of Y.

For example. Let $A = \{1, 2, 3\}$, $B=\{a, b, c\}$ and $f : A \to B$ is defined as



Here f is known as constant function, *i.e.*, $f(1) = b$, $f(2) = b$, $f(3) = b$.

3.10 Zero Function

The function $f : X \to Y$ is known as *zero function* if the image of each element of X under f is zero *i.e.*, $f(x) = 0$.

3.11 Injective and Bijective Mapping

A mapping f is said to be *injective* (or *injection*) which is either one-one into or one-one onto.

A mapping f is said to be *bijective* (or *bijection*) which is both one-one and onto.

3.12 Equality of Mapping

Let $f : X \rightarrow$ and $g : X \rightarrow Y$ be two mapping. Then the mapping f and g are said to be *equal mapping* if and only if

$$f(x) = g(x),\ \forall x \in X.$$

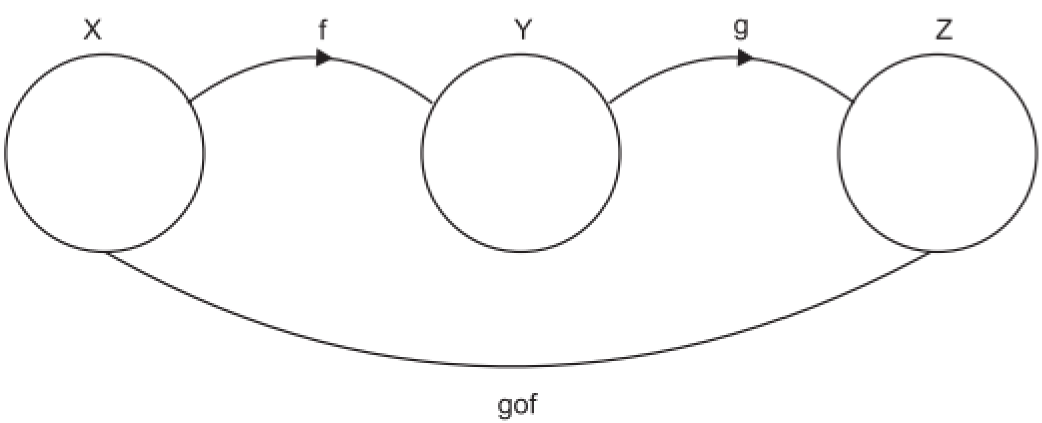
In case of equal mappings, the domains of mappings must be the same.

3.13 Composition of Functions or Product of Functions

Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be any two mapping. Then a function $gof : X \rightarrow Z$ is defined as

$$gof = g[f(x)],\ \forall x \in X$$

is known as *composition of functions*.



Examples

Example.5. Let $f(x)=x^2$, $g(x)=x+3$, $\forall x \in R$. Find gof and fog.

Solution: Here $gof = g[f(x)]$

$= g(x^2)$

$= x^2 + 3$

and $fog = f[g(x)]$

$= f(x + 3)$

$= (x + 3)^2$

$= x^2 + 6x + 9.$

Example.6. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be any two mapping such that $f(x)=\log(1+x)$, $g(x)=e^x$, then find the value of $gof(x)$ and $fog(x)$.

Solution: Here we have $gof : X \rightarrow Z$ is a mapping such that

$gof(x) = g[f(x)]$

$= g[\log(1+x)]$

$= e^{\log(1+x)}$

$= (1+x).$

Now we have $fog(x) = f[g(x)]$

$= f(e^x)$

$= \log(1+ e^x).$

Example.7. Let $f : R \rightarrow R$ and $g : R \rightarrow R$ be any two mapping such that $f(x)=x^2$, $g(x)= x^3$, $\forall x \in R$. Find the values of $gof(x)$ and $fog(x)$.

Solution: Here we have $gof : R \rightarrow R$ is a mapping such that

$gof(x) = g[f(x)]$

$= g(x^2)$

$= (x^2)^3$

$= x^6$.

and $fog = f[g(x)]$

$= f(x^3)$

$= (x^3)^2$

$= x^6$.

Theorem:1. If $A_1 \subset B \subset A$ and if $A \sim A_1$ then $A \sim B$.

Proof: Suppose if A is equivalent to A_1, then there exists a one-one onto function f from $A \to A_1$. Also it is given $B \subset A$, so f to B is also one-one. This means that the B is equivalent to a subset B_1 of A_1. Therefore the function f:$B \to B_1$ is one-one and onto , and so $B \sim B_1$. Continuing in this way we get the equivalent sets A, A_1, A_2, …….. and B, B_1, B_2, ……… such that

$A \supset B \supset A_1 \supset B_1 \supset A_2 \supset B_2 \supset A_3$ ………

Suppose $T = A \cap B \cap A_1 \cap B_1 \cap A_2 \cap B_2 \cap A_3$ ………

Thus we have

$A = (A-B) \cup (B-A_1) \cup (A_1-B_1) \cup (B_1-A_2) \cup$ …………………$\cup T$

$B = (B-A_1) \cup (A_1-B_1) \cup (B_1-A_2) \cup$ ………………………$\cup T$

Now define a mapping g:$A \to B$ such that

$g(A-B) = A_1-B_1$

$g\,(A_1\text{-}B_1) = A_2\text{-}B_2$

$g\,(A_2\text{-}B_2) = A_3\text{-}B_3$

….   …..

….   …..

$g\,(B\text{-}A_1) = B\text{-}A_1$

$g\,(B_1\text{-}A_2) = B_1\text{-}A_2$

….   …..

….   …..

$g\,(T) = T$.

Thus the mapping g is one-one and onto. Hence A ~ B.

Theorem:2. Let $f: X \to Y$ be a one-one and onto mapping, then $f^{-1}: Y \to X$ is also a one-one and onto mapping.

Proof: Suppose $f : X \to Y$ is a one-one and onto mapping. To show that $f^{-1}: Y \to X$ is a one-one and onto mapping.

Let $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ such that

$$f(x_1) = y_1 \text{ and } f(x_2) = y_2$$

If f^{-1} denotes the inverse of f, we have

$f^{-1}(y_1) = x_1$ and $f^{-1}(y_2) = x_2$

Now we have

$$f^{-1}(y_1) = f^{-1}(y_2)$$

$\Rightarrow \qquad x_1 = x_2$

$\Rightarrow \qquad f(x_1) = f(x_2)$ (because f is one-one)

$\Rightarrow \qquad y_1 = y_2$.

Therefore f^{-1} is a one-one mapping.

Again f^{-1} is also an onto mapping for each element $x \in X$ is the inverse image of the element $y \in Y$, where $y = f(x)$. Hence, the mapping $f^{-1}: Y \to X$ is always a bijective mapping.

Theorem:3. Let $f : X \to Y$ and, $A, B \subset X$ then

(i) $f[A \cup B] = f(A) \cup f(B)$ (ii) $f[A \cap B] \subset f(A) \cap f(B)$ But $f[A \cap B] \neq f(A) \cap f(B)$.

Proof: (i) Suppose $y \in f[A \cup B]$

$\Rightarrow \qquad y = f(x)$ for some $x \in A \cup B$

i.e., $\qquad y = f(x)$ for some $x \in A$ or $x \in B$

Now $\quad x \in A \quad \Rightarrow \quad y \in f(A)$

And $\quad x \in B \quad \Rightarrow \quad y \in f(B)$

Hence, $y \in f[A \cup B]$

$\Rightarrow \qquad y \in f(A) \cup f(B)$

$\therefore \qquad f[(A \cup B] \subseteq f(A) \cup f(B)$ ….(1.1)

Now, if $y \in f(A) \cup f(B)$

$\Rightarrow$ either $y \in f(A)$ or $y \in f(B)$

If $y \in f(A)$

$\Rightarrow$ there is an $x \in A$ such that $y = f(x)$

$\therefore \quad y \in f(A \cup B)$

If $y \in f(B)$

$\Rightarrow$ there is an $x \in B$ such that $y \in f(x)$

$\therefore \quad y \in f(A \cup B)$.

Hence. $y \in f(A) \cup f(B)$

$\Rightarrow \quad y \in f[A \cup B]$

$\therefore \quad f(A) \cup f(B) \subseteq f[A \cup B]$ ….(1.2)

Using Eqns.(1.1) and (1.2), we get

$f[A \cup B] = f(A) \cup f(B)$.

(ii) Suppose $y \in f[A \cap B]$

$\Rightarrow \quad y = f(x)$ for some $x \in A \cap B$

Since $x \in A \quad \Rightarrow \quad f(x) \in f(A)$ and $x \in B$

$\Rightarrow \quad f(x) \in f(B)$

Hence, $x \in A \cap B$

$\Rightarrow \quad x \in A$ and $x \in B$

$\Rightarrow \quad f(x) \in f(A)$ and $f(x) \in f(B)$

$\Rightarrow \quad f(x) \in f(A) \cap f(B)$

$\therefore \quad f(A \cap B] \subset f(A) \cap f(B)$

Now to show that

$f[A \cap B] \neq f(A) \cap f(B)$

Consider a mapping $f: R \to R$ such that

$f(x) = x^2$

Let $A = [-1, 0]$ and $B = [0, 1]$

Then $A \cap B = \{0\}$ so that $f[A \cap B] = f(0) = \{0\}$

But $f(A) = [0, 1]$; $f(B) = [0, 1]$

$\therefore \quad f(A) \cap f(B) = [0, 1]$

Hence, $f(A \cap B] \neq f(A) \cap f(B)$.

Theorem:4. To show that composite of mappings is associative

Proof: Consider the functions $f: X \to Y$, $g: Y \to Z$ and $h: Z \to W$.

To show that $ho(gof) = (hog)of$

Suppose $x \in X$, then we have

$[ho(gof)](x) = h[(gof)(x)]$

$= h[g(f(x)]$ (if $f(x) = y$)

$= h[g(y)]$ (if $g(y) = z$)

$= h(z)$ ….(1.3)

Now we have

$[(hog)of](x) = (hog)[f(x)]$

$= (hog)(y)$ (if $f(x) = y$)

$= h[g(y)]$ (if $g(y) = z$)

$= h(z)$ …..(1.4)

Using Eqns.(1.3) and (1.4), we have $ho(gof) = (hog)of$

Hence the composite of mappings is associative.

Theorem:5 If $f : X \to Y$ and $g : Y \to Z$ be one-one and onto mappings then *gof* is inversible mapping and $(gof)^{-1} = f^{-1}og^{-1}$.

Proof: Consider $f : X \to Y$ and $g : Y \to Z$ both are the one-one onto mapping. First we show that *got* is inversible, *i.e.*, *gof* is one-one onto mapping because a one-one onto mapping is always inversible.

Let $x_1, x_2 \in X$, we have $(gof)(x_1) = (gof)(x_2)$

$\Rightarrow \quad g(f(x_1) = g(f(x_2))$

$\Rightarrow \quad f(x_1) = f(x_2)$, for g is one-one

$\Rightarrow \quad x_1 = x_2$, for f is one-one.

Hence, $(gof)(x_1) = (gof)(x_2)$

$\Rightarrow \quad x_1 = x_2$

which show that *gof* is one-one.

Now let $z \in Z$. Since g is one-one onto mapping and therefore exist one and only one element $y \in Y$

such that $g(y) = z$.

Again f is one-one onto, there exist a unique element $x \in X$ such that $f(x) = y$,

$(gof)(x) = g(f(x))$

$= g(y)$

$= z$

i.e., *gof* is an onto mapping.

Now we show that

$(gof)^{-1} = f^{-1}og^{-1}$.

Given that

$f : X \rightarrow Y, \quad g : Y \rightarrow Z$

$gof : X \rightarrow Z, \quad (gof)^{-1} : Z \rightarrow X$

$\because \quad f^{-1} : Y \rightarrow X, \quad g^{-1} : Z \rightarrow Y$

$\therefore \quad f^{-1}og^{-1} : Z \rightarrow X$.

We have

$(gof)(x) = z \quad \Rightarrow \quad x = (gof)^{-1} z$ ($\because$ *got* is one one-one onto)

$f(x) = y \quad \Rightarrow \quad x = f^{-1}(y)$ ($\because$ *f* is one-one onto)

$g(y) = z \quad \Rightarrow \quad y = g^{-1}(z)$ ($\because$ *g* is one-one onto)

$$(f^{-1}og^{-1})(z) = f^{-1}(g^{-1})(z)$$

$$= f^{-1}(y)$$

$$= x$$

$$= (gof)^{-1}(z)$$

Hence, $(f^{-1}og-1)(z) = (gof)^{-1}(z) \ \forall z \in Z$

Here $f^{-1}og^{-1}$ and $(gof)^{-1}$ both the mapping have the same domain z.

Using definition of equal mapping, we have

$$(gof)^{-1} = f^{-1}og^{-1}.$$

3.14 Summary:

Let A and B be any two non-empty sets. If there exists a rule or a correspondence f which associate each element of A has a unique image in B then f is a function or mapping from A to B. A function $f: A \rightarrow B$ is called one-one if $x_1, x_2 \in A$, we have $x_1 = x_2 \Rightarrow f(x_1) = f(x_2)$ or $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$. A function $f: A \rightarrow B$ is said to be many-one if at least one element of B has two or more than two pre-image in A. A function $f: A \rightarrow B$ is said to be many-one if at least one element of B has two or more than two pre-image in A. A function $f: A \rightarrow$ B is said to be into if there is at least one element of B, has no pre-image in A.

A function $f: A \rightarrow B$ is said to be onto if there is no element of B, which is not an image of some element of A. Let $f: X \rightarrow Y$ be a one-one onto mapping and $f(x) = y$, $\forall x \in X$, $\forall y \in Y$. Now we define a mapping $f^{-1}: y \rightarrow X$ such that $f^{-1}(y) = x$, $\forall x \in X$, $\forall y \in Y$, where f^{-1} is called the inverse of f. Let $f: X \rightarrow X$ be a mapping. Then f is said to be identity mapping if $f(x) = x$, $\forall x \in X$. A mapping $f: X \rightarrow R$, where R is the set of real numbers, is known as real valued mapping. Let $f: X \rightarrow Y$ be a function. Then f is said to be constant function if $f(x) = a$, $\forall x \in X$ *i.e.*, a function $f: X \rightarrow Y$ is known as constant function if each element of X is mapped onto a single element of Y.

The function $f: X \rightarrow Y$ is known as zero function if the image of each element of X under f is zero *i.e.*, $f(x) = 0$. A mapping f is said to be injective (or injection) which is either one-one into or one-one onto. A mapping f is said to be bijective (or bijection) which is both one-one and onto. Let $f: X \rightarrow$ and $g: X \rightarrow Y$ be two mapping. Then the mapping f and g are said to be equal mapping if and only if $f(x) = g(x)$ $\forall x \in X$. Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be any two functions. Then a function $gof: X \rightarrow Z$ is defined as $gof = g[f(x)]$, $\forall x \in X$ is known as composition of functions.

3.15 Terminal Questions:

Q.1. What do you means by Inverse of a mapping?

Q.2. Write a short note on types of functions.

Q.3. Explain the Composition of Functions with example.

Q.4. Define identity, constant, zero, real valued and characteristic functions with examples.

Q.5. Define injective and bijective mappings with examples.

Q.6. If the function $f : R \to R$ be defined by $f(x) = x^2$, find $f^{-1}(g)$ and $f^{-1}(-g)$.

Q.7. If the function $f : R \to R$ be defined by $f(x) = x^2 - 1$ then find $f^{-1}(-2)$ and f^{-1} {8, 15}.

Q.8. If $X = \{x_1, x_2, x_3\}$ and $y = \{y_1, y_2, y_3\}$. To show that $f = \{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$ is an into mapping.

Q.9. If $f : X \to Y$, $g : Y \to Z$ such that $f(x) = \log(1 + x)$, $g(x) = e^x$, then find $(gof)(x)$.

Q.10. If $f : R \to R$ and $g : R \to R$ be functions such that $f(x) = x^2$ and $g(x) = x^3$, find $(gof)(x)$ and $(gof)(3)$.

Q.11. If $f : R \to R$ be defined by $f(x) = x^3 - 3$, where R is the set of all real numbers, then find

(a) 1/3 (b) $f(2)$ and (c) $f(-g)$

Q.12. If $f : R \to R$, $g : R \to R$ be mappings such that $f(x) = x^2 + 2$, $g(x) = 2x + 1$ then find $(fog)(x)$ and $(gof)(x)$.

Answer

---

6. {3, –3}, ϕ

7. ϕ, {3, –3, 4, – 4}.

9. $1 + x$.

10. x^6, 3^6.

11. (a) -80/ 27 (b) 5 (c) –732.

12. $4x^2 + 12x + 10$ and $4x^2 + 4x + 3$, $2x^2 + 5$.

Unit-4: Techniques of counting

Structure

4.1 Introduction

4.2 Objectives

4.3 Partition

4.4 Principle of Inclusion-exclusion

4.5 Pigeonhole Principle

4.6 Permutations

4.7 Combinations

4.8 Summary

4.9 Terminal Questions

4.1 Introduction

Counting techniques are methods used to determine the number of possible outcomes in a given situation. The most common techniques are counting principle, permutations, combinations, binomial coefficients, pigeonhole principle, and inclusion and exclusion principle etc. These techniques are foundational in combinatorics and are used in various fields, including mathematics, computer science, and statistics. The Counting Principle, also known as the Fundamental Counting Principle, is a basic principle in combinatorics that allows you to determine the total number of outcomes in a multi-step scenario by multiplying the number of choices at each step. The Counting Principle can be extended to more complex scenarios involving multiple events and choices. For example, if you have multiple independent choices in a sequence, you can find the total number of outcomes by multiplying the number of choices at each step.

The Counting Principle is fundamental in combinatorics and is often used in probability theory and other areas of mathematics to calculate the total number of possible outcomes in various scenarios. In this unit we shall discuss about partition, principle of inclusion-exclusion, pigeonhole principle, permutations and combinations.

4.2 Objectives

After reading this unit the learner should be able to understand the:

- Partition
- principle of inclusion- exclusion
- Pigeonhole principle
- Permutation and combination

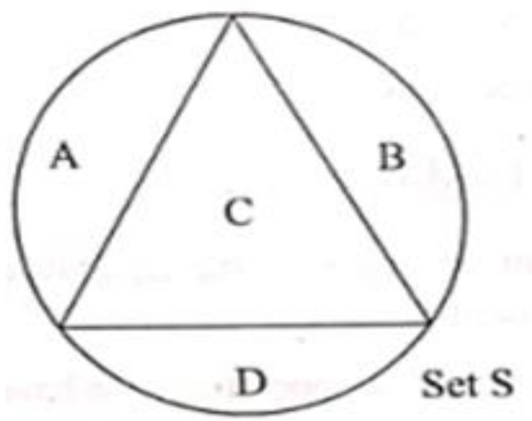
4.3 Partitions

A set $\{A, B, C, ...\}$ of the non-empty subsets of a set S, is called the partition of S if

(i) $A \cup B \cup C \cup ..... = S,$

(ii) The intersection of every pair of distinct subsets is the empty set, where the subsets A, B, C,… are called its members (elements) or blocks.

There is no restriction pertaining to the number of elements in every partition.



For example: Let S = {…., - 5, -4, -2, -1, 0, 1, 2, 3, 4, ….}.

Then the collection $\{A_1, A_2, A_3\}$,

where $A_1 = \{…, -6, -3, 0, 3, 6, 9, ….\}$,

$A_2 = \{…., -5, -2, 1, 4, 7, 10, ….\}$

and $A_3 = \{…, -4, -1, 2, 5, 8, 11, ……..\}$ is a partition of S because $A_1 \cup A_2 \cup A_3 = S$ and $A_i \cap AA_j = \phi$ when $A_i \neq A_j$.

Examples

Example.1. Consider the subsets: $A = \{3, 6, 9, 12, ..., 24\}$ **,** $B = \{1, 4, 7, 10, ..., 25\}$ **and** $C = \{2, 5, 8, 11, ..., 23\}$ **of** $S = \{1, 2, 3, ..., 25]$**. To show that** $\{A, B, C\}$ **is a partition of S.**

Solution: It is given that

$A = \{3, 6, 9, 12, ..., 24\}$,

$B = \{1, 4, 7, 10, ..., 25\}$,

$C = \{2, 5, 8, 11, ..., 23\}$,

and $S = \{1, 2, 3, ..., 25]$**.**

Obviously, $A \cup B \cup C = S$

and $A \cap B = A \cap C = B \cap C = \phi,$

Hence $\{A, B, C\}$ is a partition of S.

Example.2: Consider the following collection of subsets $\{A, B, C\}$ **of a set**

$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

(a) **[{1,5}, {2, 6, 3}, {4, 8, 9, 3},**

(b) **[{1}, {2, 4, 8}, {5, 7, 9}, and**

(c) **[{1, 5}, {2, 4, 6, 8},{3, 7, 9}]**

Determine which one is a partition of the set.

Solution:

(a) is not a partition as $B \cap C = \{3\} \neq \phi$

(b) is not a partition as $A \cup B \cup C = \{1, 2, 4, 5, 7, 8, 9\} \neq S$

(c) is a partition a $A \cap B = \phi = B \cap C = C \cap A$ and $A \cup B \cup C = S$.

4.4 Principle of Inclusion-Exclusion

The Principle of Inclusion-Exclusion is a counting technique used to calculate the number of elements in the union of finite sets by subtracting the sizes of their intersections. It states that the size of the union of two or more sets is the sum of the sizes of the individual sets, minus the sum of the sizes of the intersections of all possible pairs of sets, plus the sum of the sizes of the intersections of all possible triplets of sets, and so on, depending on the number of sets involved. This principle can be generalized to any finite number of sets. It is a fundamental tool in combinatorics and is used to solve problems involving counting when there are overlapping sets.

Theorem 1. Let A and B be any two finite disjoint sets then we have

$$n(A \cup B) = n(A) + n(B)$$

Proof: Consider that A have m_1 elements, so we have

$$n(A) = m_1 \qquad \ldots(1)$$

Again consider suppose that B have m_2 elements, so we have

$$n(B) = m_2 \qquad \ldots(2)$$

Since A and B are two disjoint sets (having no element in common) therefore A $\cup$ B will have all the elements of A and all the elements of B.

Therefore number of elements in $A \cup B$ is

$m_1 + m_2$. …(3)

From equations (1), (2) and (3), we get

$$n(A \cup B) = n(A) + n(B).$$

Theorem.2. Let A and B be any two finite sets then we have

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

Proof: We know that

$$(A - B) \cup (A \cap B) \cup (B - A) = A \cup B \quad \text{…..(1)}$$

and $A - B, A \cap B$ and $B - A$ are pair wise disjoint.

Therefore, we have

$$n(A \cup B) = n(A - B) + n(A \cap B) + n(B - A) \quad \text{……(2)}$$

Further, we have

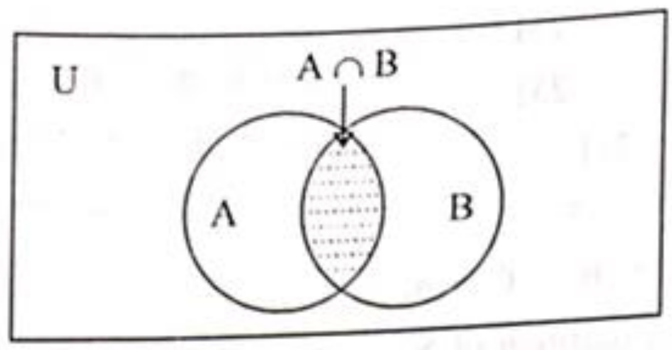
$$A = (A - B) \cup (A \cap B) \quad \text{and} \quad (A - B) \cap (A \cap B) = \phi$$

Therefore we have

$$n(A) = n(A - B) + n(A \cap B) \quad \text{…….(3)}$$

Similarly, we have

$n(B) = n(A \cap B) + n(B - A)$ ……(4)



Adding equations (3) and (4), we have

$$n(A) + n(B) = \{n(A - B) + n(A \cap B) + n(B - A)\} + n(A \cap B)$$

$$= n(A \cup B) + n(A \cap B)$$ [using equation (2)]

Hence, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

Theorem 3: Let A, B and C be the finite sets then we have

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C).$$

Proof: Consider $A \cup B = X$.

From the above theorem 2 we have

$n(X) = n(A \cup B) = n(A) + n(B) - n(A \cap B)$ .....(1)

Now we have

$$n(A \cup B \cup C) = n(X \cup C)$$

$$= n(X) + n(C) - n(X \cap C)$$

$$= n(X) + n(C) - n[(A \cup B) \cap C]$$

$$= n(X) + n(C) - n[(A \cap C) \cup (B \cap C)]$$

$$= n(X) + n(C) - \{n(A \cap C) + n(B \cap C) - n(A \cap C \cap B)\} \quad .....(2)$$

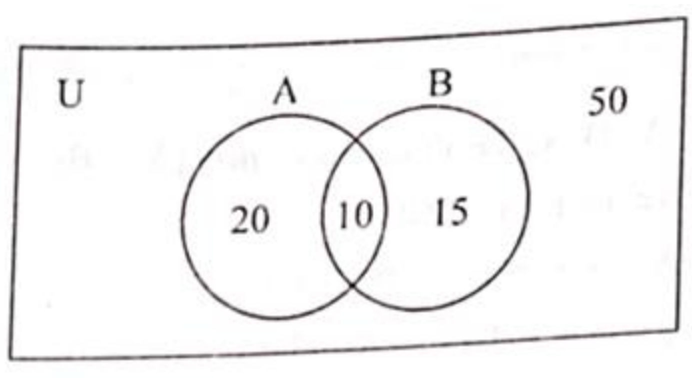
$$= n(A) + n(B) - n(A \cap B) + n(C) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$

$$= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C).$$

Examples

Example.3. In a class of 50 students, 30 are studying English and 25 French language and 10 are studying both languages. How many students are studying either language?

Solution: Let U be the class of 50 students, A is a set of those students studying English and B be the set of those studying French. Therefore we have to find $n(A \cup B)$.



It is given that

$$n(A) = 30, n(B) = 15, n(A \cap B) = 10$$

We know that

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$\therefore \qquad n(A \cup B) = 30 + 25 - 10 = 45$$

Therefore, the number of students who studying either of languages are 45.

Example.4: If 55% of teachers like tea where 75% like coffee. What can be said about the percentage of teachers who like both tea and coffee?

Solution: Let us consider n(s) = total number of teachers = 100.

A = {x: x likes tea},

B = {x: x likes coffee}

Then we have $n(A) = 55$, $n(B) = 75$,

We know that

$A \cap B$ = {x: x likes tea and coffee both}

We have

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$= 55 + 75 - 100 = 30$

Therefore $n(A \cap B) = 30$. Hence the 30% of teachers like tea and coffee both.

Hence, 39% teachers like both oranges and apples.

Example.5. Let A, B, C be three sets, then $(A - B) \cap (A - C) = A - (B \cup C)$.

Solution: We have to prove that

(i) $(A-B)\cap(A-C)\subseteq A-(B\cup C)$

(ii) $A-(B\cup C)\subseteq(A-B)\cap(A-C)$

Let $x\in(A-B)\cap(A-C)$ then we have

$x\in(A-B)\cap(A-C)\Leftrightarrow x\in(A-B)$ and $x\in(A-C)$

$\Leftrightarrow x\in A, x\notin B$ and $x\in A, x\notin C$

$\Leftrightarrow x\in A, x\notin(B\cup C)$

$\Leftrightarrow x\in A-(B\cup C)$

Which implies that

$(A-B)\cap(A-C)=A-(B\cup C).$

Hence $(A-B)\cap(A-C)=A-(B\cup C).$

Example.6: In a group of 1000 people, there are 750, who can speak Hindi and 400, who can speak English. How many can speak Hindi only? How many can speak English only? How many can speak?

Solution: Let A = {x: x speaks Hindi}

and B = {x: x speaks English)

Then we have

$A – B$ = {x: x speaks Hindi and cannot speak English}

$B – A$ = {x: x speaks English and cannot speak Hindi}

$A \cap B = \{x : x$ speaks Hindi and English both$\}$

It is given that

$n(A) = 750$, $n(B) = 400$ $n(A \cup B) = 1000$

We have

$$n(A \cap B) = n(A) + n(B) - n(A \cup B)$$

$$= 750 + 400 - 1000$$

$$= 1150 - 1000$$

$$= 150$$

Therefore, 150 people are speaking the both languages Hindi and English.

Again, we have

$n(A) = n(A - B) + n(A \cap B)$,

$$n(A - B) = n(A) - n(A \cap B)$$

$$= 750 - 150$$

$$= 600$$

Hence, 600 people are speaking Hindi only,

Finally, we have

$$n(\mathrm{B} - A) = n(B) - n(A \cap B)$$

$$=400 - 150$$

$$= 250.$$

Hence 250 people are speaking English only.

Example.7: A computer company must hire 50 programmers to handle systems programming jobs and 80 for the application programming. Of the hired persons, 20 will have to do the job of both types. Find how many programmers must be hired?

Solution: Suppose the computer company hire to handle system programming jobs is

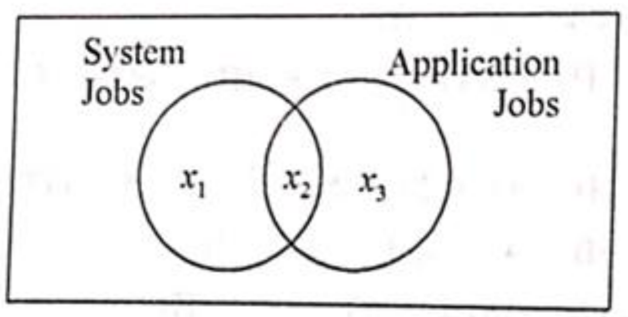
$$x_1 + x_2 = 50 \qquad \ldots(1)$$

The computer company hire to handle the application programming jobs is

$$x_2 + x_3 = 80 \qquad \ldots(2)$$

The computer company hired persons doing both the jobs are

$$x_2 = 20 \qquad \ldots.(3)$$



From equations (1) and (3), we have

$$x_1 = 50 - 20 = 30$$

From equations (2) and (3), we have

$$x_3 = 80 - 20 = 60$$

Thus, we have

$$x_1 + x_2 + x_3 = 30 + 20 + 60 = 110.$$

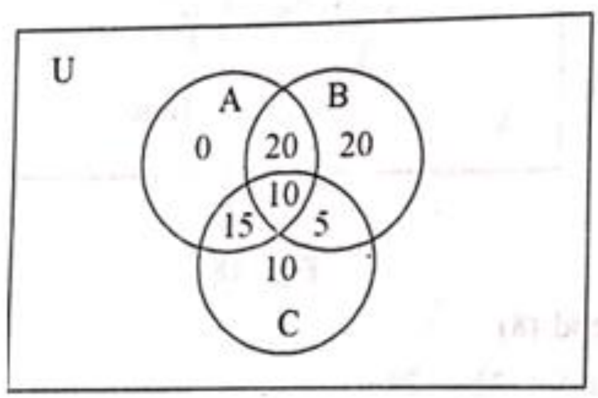
Hence the computer company 110 programmer should be hired.

Example.8: In a town 45% read magazine A, 55% read magazine B, 40% read magazine C, 30% read magazines A and B, 15% read magazines B and C, 25% read C and A, 10% read all the three magazines. Find what percentage do not read any magazine? What percentage reads exactly two of the magazines?

Solution: Let A, B and C denote the set of all those who read magazines A, B and C respectively. Then $n(A)=45, n(B)=55, n(C)=40, n(A\cap B)=30, n(B\cap C)=15, n(A\cap C)=25,$

$$n(A\cap B\cap C)=10.$$

Consider $n(U)=100$ (say)



The diagram given here make the understanding clear the following. The number of persons who read only A and B but not C = 30 -10 = 20.

The number of persons who read B and c but not A= 15 – 10 = 5

The number of persons who read only C and A but not B = 25 – 10 = 15

The number of persons who read only A = 45 – (20 + 10 + 15) = 0

The number of persons who read only B = 55 – (20 + 10 + 5) = 20

The number of persons who read only C = 40 – (15 + 10 + 5) = 10

Thus the number of persons who read at least one magazine

= 0 + 20 + 10 + 20 + 15 + 5 + 10 = 80

Thus the persons those who do not read any magazine are 20 in number i.e., 20% do not read any magazine. Hence the number of persons who read exactly two magazines = 20 + 5 + 15 = 40 i.e., 40% read two of the magazines.

4.5 Pigeonhole Principle

Suppose that a flock of pigeons flies into a set of pigeonholes to roost. The pigeonhole principle states that if there are more pigeons than pigeonholes, then there must be at least one pigeonhole with at least two pigeons in it. Of course, this principle applies to other objects besides pigeons and pigeonholes.

Theorem.4: It R + 1 or more objects are placed into R boxes, then there is at least one box containing two or more of the objects.

Proof: Let none of the R boxes contains more than one object. Then the total number of objects would be at most R. This is a contradiction, since there are at least R+ 1 objects. Thus there is at least one box containing two or more of the objects.

Theorem.5. Let $n-$**pigeons are assigned R pigeonholes, then one of the pigeonhole must contain at least** $\left\lfloor \frac{n-1}{R} \right\rfloor + 1$ **pigeons, where** $\left\lfloor \frac{n-1}{R} \right\rfloor$ **is the floor of** $\frac{n-1}{R}$ **i.e., the greatest integer less than or equal to** $\frac{n-1}{R}$.

Proof: We have $n-$pigeons and $R-$pigeonholes such that $n > R.$ Assuming that each of R pigeonhole contains not more than $\left[\frac{n-1}{R}\right]$pigeons, then total number of pigeons in the R pigeonholes must be less than or equal to $R\times\left[\frac{n-1}{R}\right] \le R\times\frac{n-1}{R} = n-1$

But there are n- pigeons, so this contradicts our assumption that a pigeonhole contains not more than $\left[\frac{n-1}{R}\right]$ pigeons. So one of the pigeonholes must contain at least $\left[\frac{n-1}{R}\right]+1$ pigeons.

Examples

Example.9: Let there are 5 separate departments in a departmental store and the total number of employee are 36. Show that one of the department must have at least 8 employee.

Solution; Let 36 employees are pigeons and 5 departments as pigeonholes, then according to the pigeonhole principle's, one of the department will have at least

$$\left[\frac{36-1}{5}\right]+1 \text{ employees}$$

$$\Rightarrow \quad \left[\frac{35}{5}\right]+1$$
$$= 7+1$$
$$= 8.$$

Example.10: Prove that if any six numbers from the set [1, 2, 3, 4, 5, 6, 7, 8, 9] are chosen, then two of them will add up to 10.

Solution: From the set we from different sets containing two numbers that add up to 10 as follows:

$S_1 = \{1,9\}$, $S_2 = \{2,8\}$, $S_3 = \{3,7\}$, and $S_4 = \{4,6\}$ and we left with a singleton set $S_5 = \{5\}$. Therefore, there are four such sets consisting of 8 numbers and one number 5 is left unused then 5 numbers are selected from S_1 to S_5, and one number must be selected from S_1 to S_4, therefore, two number will be chosen from any one of S_1 to S_4, thus the sum of there will be 10.

Example.11: Prove that if we allot 26 rooms to the students in a P.G. hostel from the room numbered between 1 and 50 inclusive, at least two are consecutively numbered.

Solution: Let us consider $R_1, R_2, ..., R_i, ..., R_{26}$ .....(1)

Be the chosen room numbers from the room numbers $[1, 2, 3, ..., 50]$ where R_i indicate the i^{th} room number chosen from the room numbers $[1, 2, 3, ..., 50]$.

The room numbers

$$R_1 + 1, R_2 + 1, ..., R_{26} + 1 \qquad ....(2)$$

Together with room numbers of (1) which are 52 in numbers lie in the range [1, 2, 3,…,50, 51]. Hence 52 rooms are selected from the room numbers 1 to 51, therefore, by the pigeonhole principle at least one of the room number from (1) coincides with room number from (2) [since (1) are distinct and so also (2) are distinct]. Thus, for some i and j, we have

$$R_j = R_j + 1, i, j = 1, 2, ..., 26$$

And room number R_i follows R_j i.e., R_i and R_j are consecutively numbered.

4.6 Permutation

Permutation is a counting problem which comes under the branch of mathematics called combinatorics. A permutation is an arrangement of a finite set of objects in a particular order.

For example, there are six different permutations of the set {a, b, c}. They are abc, acb, bac, bca, cab and cba.

Any arrangement on n distinct object taken r at a time ($r \leqslant n$) is called r-permutation. The number of permutations of n distinct objects taken r at a time is given by

$$^nP_r = n\,(n-1)\,(n-2)\,\ldots\ldots\ldots\,(n-r+1) = \frac{n!}{(n-r)!}$$

If out n objects in a set, p objects are exactly alike of one kind, q objects exactly alike of second kind and r objects exactly alike of third kind and the remaining objects are all different then the number of permutation of n objects taken all at a time is

$$\frac{n!}{p!q!r!}.$$

Example.12: How many ways are there to arrange the nine letters in the word 'ALLAHABAD'?

Solution: Since the word ALLAHABAD contains 4A's and 2L's.

Therefore there are $= \frac{9!}{4!2!} = 7560$ ways to arrange the nine letters in the word 'ALLAHABAD'.

4.7 Combinatorics:

Combinations are used in various areas of mathematics and combinatorics, such as in probability theory, where they are used to calculate the number of favorable outcomes in an experiment.

Combinations are a way to select items from a larger set without considering the order in which the items are selected. In other words, combinations are a selection of items where the order does not matter.

The number of combinations of n items taken r at a time is denoted by "n choose r" and is calculated using the formula: ${}^{n}C_{r} = \frac{n!}{r!(\text{n}-\text{r})!}$

For example, if you have a set of {A, B, C, D}, the different ways to choose 2 items (combinations) without considering the order would be {AB, AC, AD, BC, BD, CD}.

4.8 Summary:

A set $\{A, B, C, \ldots.\}$ of the non-empty subsets of a set S, is called the partition of S if

(i) $A \cup B \cup C \cup ..... = S,$

(ii) The intersection of every pair of distinct subsets is the empty set, where the subsets A, B, C,… are called its members (elements) or blocks.

Let A and B be any two finite disjoint sets then we have

$$n(A \cup B) = n(A) + n(B)$$

Let A and B be any two finite sets then we have

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

Let A, B and C be the finite sets then we have

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C).$$

It R + 1 or more objects are placed into R boxes, then there is at least one box containing two or more of the objects.

Any arrangement on n distinct object taken r at a time (r ⩽ n) is called r-permutation. The number of permutations of n distinct objects taken r at a time is given by

$$^{n}P_r = n\,(n-1)\,(n-2)\,\ldots\ldots\ldots\,(n-r+1) = \frac{n!}{(n-r)!}$$

The number of combinations of n items taken r at a time is denoted by "n choose r" and is calculated using the formula: $^{n}C_r = \dfrac{n!}{r!(n-r)!}$.

4.9 Terminal Questions:

Q.1. Define partition of a set.

Q.2. State the principle of inclusion-exclusion.

Q.3. What do you mean by pigeonhole principle?

Q.4. A survey is taken on method of commuter travel. Each respondent is asked to check BUS, TRAIN or AUTOMOBILE as a major method of travelling to work. More than one answer is permitted. The results reported were as follows;

(i) 30 people checked BUS;

(ii) 35 people checked TRAIN;

(iii) 100 people checked AUTOMOBILE;

(iv) 15 people checked BUS and TRAIN;

(v) 15 people checked BUS AND AUTOMOBILE;

(vi) 20 people checked TRAIN and AUTOMOBILE;

(vii) 5 people checked all three methods.

How many respondents completed their surveys?

Q.5. A survey shows 74% of Indians like apples and 68% like oranges. What percentage like both apples and oranges/

Q.6. For any sets A and B, prove

(i) $\left(A-B\right)\cap B=\phi$

(ii) $A-B=A\cap B'=B'-A'$

(iii) $\left(A\cap B\right)\cup\left(A\cap B'\right)=A$

(iv) $A\cup\left(A\cap B\right)=A$

(v) $A\cap\left(A\cup B\right)'=\phi$

Q.7. In a school, assuming sport participation is compulsory. In a class of 80 student, 60 play football and 40 play basketball. Find:

(i) How many play both the games.

(ii) Play football only.

Q.8. Write a short note on permutation and combinations.

Answer

4. 120.

5. 42%

7. (i) 20 (ii) 40.

U. P. Rajarshi Tandon Open University

Master of Science
PGMM -103N
Discrete Mathematics

Block

2 Logic

Block-2

Logic

We study about mathematical logic and its different aspects which is most basic unit of this block as it introduces the concept of statements, statement variables and the five elementary operations and associated logical connectives. We introduce the well-formed statement formulae, tautologies and equivalence of formulae. The law of duality is explained and established. It has got tremendous application in almost every field, social, economy, engineering, technology etc. In computer science concept of logic is a major tool to learn to understand it more clearly. The main problem in logic is the investigation of the process of reasoning. In Mathematics, a certain set of statements (propositions) is assumed and from this set, other statements are derived by logical reasoning. In this section, we shall investigate those processes which can be accepted as valid in the derivation of a statement from the given set of statements. The given set of statements is called premises or hypothesis and the statement derived from the given statement is called conclusion. This is most basic unit of this block as it introduces the concept of The principle of Mathematical induction is of great help in proving results involving a natural member for every n or for every $n \geq$ some positive integer m. If $P(n)$ is a statement involving a positive integer n.If $P(l)$ is true $\Rightarrow$truth of $P(l+1)\ \forall l \geq$ m.

Then $P(n)$ is true for every $n \geq m$. The particular case of this result for $m = 1$ is usually referred to as the principle of mathematical induction and in fact the general version stated above can be obtained from version stated above can be obtained from this particular case. The above principle is popularly stated as if a statement holds for $n = 1$ and whenever it is true for $n = t$, it holds for $n = t + 1$, then it holds for all natural numbers n. There was a reason for looking the further generalization, apart from mathematical interest. The reason was the many applications. Apart from the ones we mentioned at the beginning, the binomial theorem has several applications in probability theory, calculus and approximating numbers like $(1.02)^{15}$. We shall discuss a few of them in this unit. A recurrence relation of the sequence $\{a_n\}$ is an equation that expresses an in terms of one or more of the previous terms of the sequence namely $a_0, a_1, a_2, \ldots a_{n-1}$ for all integers n with $n \geq n_0$ is non negative integers.

A recurrence relation are also called difference equations solution of recurrence relation = A sequence is called a solution of recurrence relation of its term satisfy of the recurrence relation.

This is most basic unit of this block as Recursively defined functions should be well defined. It means for every positive integer, the value of the function at this integer is determined in an unambiguous way.

Assume a is a nonzero real number and n is a nonnegative integer. Give a recursive definition of a_n, n some recursive functions, The values of the function at the first k positive integers are specified A rule is given to determine the value of the function at larger integer from its values at some of the preceding k integers.

UNIT-5 : Mathematical Logic

Structure

5.1 Introduction

5.2 Objectives

5.3 Statements (Proposition)

5.4 Logical connectives

5.5 Truth functional rules

5.6 Elementary Logical Operations

5.7 Converse, Inverse and Contrapositive of $p \rightarrow q$

5.8 Tautology & Contradiction

5.9 Tautological equivalence

5.10 Algebra of Proposition

5.11 Summary

5.12 Terminal Questions

5.1 Introduction

This is most basic unit of this block as it introduces the concept of statements, statements, statement variables and the five elementary operations and associated logical connectives. We introduce the well-formed statement formulae, tautologies and equivalence of formulae. The law of duality is explained and established. It has got tremendous application in almost every field, social, economy, engineering, technology etc. In computer science concept of logic is a major tool to learn to understand it more clearly. Mathematics has a language of its own like most other sciences, which is very precise and communicates just what is required-neither more nor less.

Language basically consists of words and their combinations called 'expression' or 'sentences'. However in Mathematics any expression or statement will not be called a 'sentence'. Mathematics has a language of its own like most other sciences, which is very precise and communicates just what is required-neither more nor less. Language basically consists of words and their combinations called 'expression' or 'sentences'. However in Mathematics any expression or statement will not be called a 'sentence'.

5.2 Objectives

After reading this unit we should be able to understand the:

- concept of statement and statement variables
- elementary operations like Conjunction, Disjunction, Negation, Implication, Double
- Implication
- Tautology & Contradiction
- statement formulae, tautologies to equivalence of formulae

5.3 Statements

A statement (or proposition) is a sentence which is either true or false but not both.

Examples

Example.1. Which of the following are statements?

(a) Indira Gandhi was one of the Prime Ministers of India.

(b) 8 is greater than 10.

(c) $2 + 4 = 6$

(d) Blood is green.

(e) It is raining

(f) The sun will come out tomorrow.

Solution:

(a) is a statement because it is true.

(b) is a statement because it is false.

(c) is a statement because it is true.

(d) is a statement because it is false.

(e) is a statement because the sentence " it is raining" is either true of false but not both a given time.

(f) is a statement since it is either true or false but not both. Although, we would have to wait until tomorrow whether it is true or false.

If a sentence is a question (interrogative type) or a command or not free of ambiguity then the sentence cannot be answered as true or false and therefore such sentences are not statements.

Example.2: The following are not statements.

(a) Is the number 6 a prime?

(b) $2 – x = 6$

(c) What are you studying?

(d) Open the door.

(e) This statement is false.

Explanation:

(a) is not a statement because it is a question

(b) is not a statement because it is true or false depending on the value of x.

(c) is not a statement because it is a question.

(d) is a command and therefore it is not a statement.

(e) is not a statement because it is not possible to assign a definite true or false value to it. If we assume that sentence (e) is true then it says that statement (e) is false.

On the other hand, if we assume that sentence (e) is false then it implies that statement (e) is true. Hence it is not a statement.

5.4 Logical connectives:

There are some key words and phrases which are used to form new sentences from given sentences, as for example 'and' 'or', 'not', 'if…. then …., if and only if' etc. They are called sentential or logical connectives. **A** Sentence with some logical connective is called a ‘Compound sentence’ and a sentence without logical connective is called an ‘atomic sentence.

For example: A triangle is a plane figure. Water is cold, are atomic sentences. But the followings are the compound sentences.

(a) A triangle is a plane figure and is bounded by three straight lines.

(b) A real number is rational or irrational.

(c) 2013 is not a leap year.

(d) If a triangle is equilateral then it’s all angles are equal.

(e) If a triangle is isosceles then two of its angles are equal.

A part of a compound sentence that itself is a sentence is called a component of the sentence – thus the components of the sentence are also sentence.

5.5 Truth functional rules or truth tables:

The rules by which the truth or falsity of a compound sentence is determined from the truth or falsity of its components are called *truth functional rules*. The table giving the truth or falsity of the compound sentence depending upon the truth or falsity of its components is called its *truth table*.

We shall say that *T* or *F* according as the sentence is true or false respectively.

5.6 Elementary Logical Operations:

The formation of compound sentence from given sentences by using the logical connectives are called elementary logical operations which are five in number in accordance with the five logical connectives used.

They are:

(1) Conjunction

(2) Disjunction

(3) Negation

(4) Implication

(5) Double implication.

Note: When we form compound sentence by using any of the five logical connectives, it is not necessary that the components of compound sentence should be related in the nation however absurd is permitted. As for example consider the compound sentence ‘Ram is a player and the earth revolves about the Sun.

Here the components of the compound sentence are not related in the usual sense of conversation.

1. Conjunction

A sentence obtained by conjoining two sentences P, Q by using the connective ‘and’ is called the *conjunction* of the two sentences and will be denoted by $P \wedge Q$ (read as P and Q).

Example: Let P= U.S.A. sent Apollo 11 to the Moon, Q = Russia sent Luna 15 to the Moon. Then $P \wedge Q$= U.S.A. sent Apollo 11 and Russia sent Luna 15 to the Moon.

Truth functional rule for conjunction:

$P \wedge Q$ is true if and only if both the sentences P, Q are true. How this truth functional rule is obtained is a matter of sophisticated logical reasoning and is beyond the purview of the present discussion.

Truth-Table for Conjunction:

The following table gives the truth-values of $P \wedge Q$ for all possible truth values of P and Q:

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

2. Disjunction:

A sentence obtained by joining two sentences P, Q, by the connective ‘or’ is called the *disjunction* of the two sentences and will be denoted by $P \vee Q$ (read as P or Q).

For example: P = Ram is intelligent, Q = Ram is hard working, $P \vee Q$ = Ram is intelligent or hard working.

Truth functional rule for disjunction:

$P \vee Q$ is true if at least one of P, Q is true, that is,$P \vee Q$ is false only when both P and Q are false.

Truth Table for disjunction*:*

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

3. Negation

A sentence which has a truth value opposite to that of a sentence P is called the negation of P and is denoted by – P or ~ P. Negation of an atomic sentence is obtained by using the connective 'not' at proper place.

For example: If P= The water is cold, then – P =The water is not cold.

Negation of $P \wedge Q$ is $(- P) \vee (- Q)$,

that is, $-(P \wedge Q) \equiv (- P) \vee (- Q)$.

Thus the negation of 'Ram is poor and honest' is 'Ram is not poor or not honest. This can be verified by the following Truth Table:

P	Q	$P \wedge Q$	$-P$	$-Q$	$(-P) \vee (-Q)$
T	T	T	F	F	F
T	F	F	F	T	T
F	T	F	T	F	T
F	F	F	T	T	T

The above table shows that the truth-values of $P \wedge Q$ (as given in the third column) are exactly opposite to those of $(-P) \vee (-Q)$ as given in the last column.

The negation of $P \vee Q$ is $(-P) \wedge (-Q)$,

that is, $-(P \vee Q) \equiv (-P) \wedge (-Q)$

The negation of 'Mohan or Sohan has failed' is 'neither Mohan nor Sohan has failed' that is, 'Mohan has not failed and Sohan has not failed'.

4. Implication or a conditional sentence

A conditional sentence obtained by using the connective 'If ….then…' is called an implication. For example: P = you read, Q = you will pass, By using the connective 'if ……… then' we get 'If you read then you will pass' which can be denoted by 'If P then Q'. It is also written as $P \Rightarrow Q$ (read as P implies Q).

In the implication $P \Rightarrow Q$, P is called the *hypothesis* or antecedent and Q is called the *conclusion or consequent.*

The Truth functional rule for implication:

$P \Rightarrow Q$ is false if P is true and Q is false;' otherwise it is true. The Negation of $P \Rightarrow Q$ is $P \wedge (-Q)$ that is, $\sim(P \Rightarrow Q) \equiv P \wedge (-Q)$. This is proved by the following Truth Table:

P	Q	$P \Rightarrow Q$	$-Q$	$P \wedge (-Q)$
T	T	T	F	F
T	F	F	T	T
F	T	T	F	F
F	F	T	T	F

Truth values of $P \Rightarrow Q$ as given in third column are exactly opposite to those of

$P \wedge (-Q)$ as given in the last column. Thus the Negation of the sentence 'If you read then you will pass' is 'You read and you will not pass.

Note that 'If you do not read then you will not pass' is not the negation of the given sentence.

5. Double Implication

A bi-conditional sentence obtained by using the connective 'If and only if' (briefly written as *iff*) between two sentences P, Q is called a double implication and is written a 'P *iff* Q'. It is also written as $P \Leftrightarrow Q$ (read as P implies and implied by Q)**.** Thus we find that $P \Leftrightarrow Q$ is precisely the conjunction of $P \Rightarrow Q$, $Q \Rightarrow P$, that is $P \Leftrightarrow Q \equiv (P \Rightarrow Q) \wedge (Q \Rightarrow P)$.

The double implication $P \Leftrightarrow Q$ is true only when both P and Q are true or both are false. This is proved by the following table:

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Leftrightarrow Q$ i.e. $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
T	*T*	*T*	*T*	*T*
T	*F*	*F*	*T*	*F*
F	*T*	*T*	*F*	*F*
F	*F*	*T*	*T*	*T*

Note: If P= the Sun revolves about the earth, Q= The year consists of 400 days. Then 'P iff Q' or $P \Leftrightarrow Q$ = the Sun revolves about the earth iff the year consists of 400 days – which statement is true though P and Q are both false. The Negation of $P \Leftrightarrow Q$ is $(P \wedge - Q) \vee (Q \wedge - P)$. Thus the Negation of the sentence 'One is good teacher iff one is a good scholar' is 'One is a good teacher and a bad scholar or one is a good scholar and a bad teacher'.

Note: The symbols $\vee$, $\wedge$, ~, $\rightarrow$ and $\leftrightarrow$ defined above are called **connectives.**

Examples

Example.3: Construct the truth table for $\sim p \vee q$.

Solution: We must consider all possible combinations of truth values of p and q. All possible combinations of the truth values of the statements p and q are listed in the first two columns of the table. The truth values of $\sim p$ are entered in the third column and the truth values of $\sim p \vee q$ are entered in the fourth column.

P	q	$\sim p$	$\sim p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Truth table for $\sim p \vee q$

Example.4: Construct the truth table for $p \wedge \sim p$.

Solution: Since the statement $p \wedge \sim p$ has only one distinct atomic statement. We have to consider 2 possible combinations of truth values. The truth table for $p \wedge \sim p$ is given below:

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

Truth table for $p \wedge \sim p$

Example.5: Construct the truth-table for $\sim(p\wedge\sim q)$.

Solution: In the first two columns, we list all the variable and the combinations of their truth values. In the third column, we write truth values for $\sim q$. The truth values of $p\wedge\sim q$ are listed in the next column. Finally we obtain the truth values of the proposition $\sim(p\wedge\sim q)$. Thus we have the following truth table:

p	q	$\sim q$	$p \wedge \sim q$	$\sim(p \wedge \sim q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

Example.6: Construct the truth-table for $(p \vee q) \wedge (p\vee r)$.

Solution: Here, we have three atomic statements. Therefore we shall require eight rows to list all possible combinations of the truth values of statements p, q and r. Rest of the procedure will be the same as above.

We shall proceed in steps and in the final column we will have the truth values of the given statements:

p	q	r	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	F	T	F
F	F	F	F	F	F

Truth table for $(p \vee q) \wedge (p \vee r)$

Example.7: Prove that the truth values of the following pairs of sentences are the same.

(a) $P \wedge (Q \vee R)$ and $(P \wedge Q) \vee (P \wedge R)$

(b) $P \vee (Q \wedge R)$ and $(P \vee Q) \wedge (P \vee R)$

(c) $P \wedge (Q \wedge R)$ and $(P \wedge Q) \wedge R$

(d) $P \vee (Q \vee R)$ and $(P \vee Q) \vee R$

Solution: The truth table is

P	Q	R	$Q\vee R$	$P\wedge(Q\vee R)$	$P\wedge Q$	$P\wedge R$	$(P\wedge Q)\vee(P\wedge R)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

from columns fifth and eight we find that the truth values of $P\wedge (Q\vee R)$ ad $(P\wedge Q) \vee (P\wedge R)$ are the same in all cases. Solutions of other parts have been left as exercise.

Check your progress:

1. Which of the following are statements?

(a) Is 3 a positive number?

(b) $x^2 - 5x + 6 = 0$

(c) There will be snow in December.

(d) Give me ten rupees.

(e) Ramesh is poor but honest

(f) No triangles are squares.

2. Let p be the proposition "Mathematics is easy" and let q be the proposition "five is greater than four." Write in English the proposition, which corresponds to each of the following:

(a) $p \wedge q$ (b) $p \vee q$

(c) $\sim(p \wedge q)$ (d) $\sim p \wedge \sim q$

(e) $(p \wedge \sim q) \vee (\sim p \wedge q)$

3. Write the negation of each of the following statements:

(a) $2+7 \leq 1\ 3$

(b) 3 is an odd integer and 8 is an even integer.

(c) No nice people are dangerous.

4. Let p be the statement "Ravi is rich" and let q be the statement "Ravi is happy." Write the following statements in symbolic form:

(a) Ravi is poor but happy.

(b) Ravi is rich or unhappy.

(c) Ravi is neither rich nor unhappy.

(d) Ravi is poor or he is both rich and unhappy.

5. Construct the truth-table for the following functions:

(a) $(p'+q')'$ (b) $(p'q')'$

(c) $p(p+q)$ (d) $pqr+p'q'r'$

(e) $(p'+qr)'(pq+q'r)$

6. Given the truth values of *p* and *q* as true and those of *r* and *s* as false; find the truth values of the following:

(a) $p\vee (q\wedge r)$

(b) $(p \wedge (q \wedge r))\vee \sim((p \vee q)\wedge(r \vee s))$.

Answers

---

1. (c), (e) and (f) are statements.

2. (a) Mathematics is easy and five is greater than four.

(b) Mathematics is easy or five is greater than four.

(c) Either Mathematics is not easy or five is not greater than four.

(d) Mathematics is not easy and five is not greater than four.

(e) Either Mathematics is easy and five is not greater than four or Mathematics is not easy and five is greater than four.

3. (a) It is false that $2 + 7 \leq 1\ 3$

(b) Either 3 is not an odd integer or 8 is not an even integer.

(c) Some nice people are dangerous.

4. (a) $\sim p\wedge q$ (b) $p\vee \sim q$

(c) $\sim p \wedge q$ (d) $\sim p \vee (p \wedge \sim q)$

5. (a) True (b) True

5.7 Converse, Inverse and Contrapositive of $p \rightarrow q$

Let $p \rightarrow q$ be any conditional statement. Then,

(a) the converse of $p \rightarrow q$ is statement $q \rightarrow p$.

(b) the inverse of $p \rightarrow q$ is the statement $\sim p \rightarrow \sim q$.

(c) the contrapositive of $p \rightarrow q$ is the statement $\sim q \rightarrow \sim p$.

Example.8: Write the converse, inverse and contrapositive of the conditional statement "if 2 + 2 = 4 then I am not the Prime Minister of India."

Solution: Let p: 2+2 =4 and q: I am not the Prime Minister of India.

Then the given statement can be written as $p \rightarrow q$. Therefore, the converse is $q \rightarrow p$. That is, if I am not the Prime Minister of India then 2+2 = 4. The inverse of $p \rightarrow q$ is the statement $\sim p \rightarrow \sim q$. That is, if $2+2 \neq 4$ then I am Prime Minister of India.

The contra-positive of $p \rightarrow q$ is the statement $\sim q \rightarrow \sim p$.

That is, contra-positive of the given statement is "if I am Prime Minister of India then $2+2 \neq 4$."

Propositional Functions and Propositional Variables

By a propositional variable, we mean a symbol which represents an arbitrary statement (proposition). Thus propositional variable is a variable that can be replaced by a statement. We shall use the symbols p, q, r, ….or p_1, p_2, p_3,…… to denote propositional variables.

Propositional function is a function or statement which is formed by using propositional variables and connectives. For example, compound statement such as $p \vee q$, $p \wedge q$, $p \rightarrow q$ and $p \wedge (q \rightarrow r)$ are propositional functions. More formally, a propositional function is an expression, which is a combination of propositional variables and connectives. Propositional function are denoted as f (p, q, r,….), where p, q, r,….are the variable used in forming the function f. A propositional function f in n variables p_1, p_2,p_3,…..p_n will be denoted as $f(p_1, p_2, ,p_3,\ldots.p_n)$.

(1). $f(p, q) = \sim (p \wedge q)$ is a propositional function in propositional variables p and q

(2). $f(p, q, r) = p \rightarrow (p \rightarrow r)$ is propositional function in propositional variables p,q and r.

(3). $f(p_1, p_2, p_3) = (p_1 \rightarrow p_2) \wedge (p_2 \rightarrow p_3)$ is a propositional function in propositional variables p_1, p_2, and p_3.

Thus we see that propositional functions are compound statements formed by using finite number of simple (atomic) statements and connectives. We shall often use the word statement for propositional functions.

5.8 Tautology & Contradiction

A compound sentence is called a tautology if it is always true irrespective of the truth values of its component parts. *i.e.* A statement (or propositional function) which is true for all possible truth values of its propositional variables is called a tautology.

A statement which is always false is called a contradiction. A simple method to determine whether a given statement is a tautology is to construct its truth table.

If the statement is tautology then the column corresponding to the statement in the truth table contains only T.

Similarly a statement is contradiction if the column corresponding to the statement contains only *F*.

For example $P \vee -P$ is a tautology, since one of P and $-P$ must be true and so $P \vee -P$ is always true. Similarly $(-P \Rightarrow Q) \wedge -Q \Rightarrow P$ is a tautology as proved by the following table.

P	Q	$-P$	$-Q$	$-P \Rightarrow Q$	$(-P \Rightarrow Q) \wedge -Q$	$(-P \Rightarrow Q) \wedge -Q \Rightarrow P$
T	*T*	*F*	*F*	*T*	*F*	*T*
T	*F*	*F*	*T*	*T*	*T*	*T*
F	*T*	*T*	*F*	*T*	*F*	*T*
F	*F*	*T*	*F*	*F*	*T*	*T*

If $P \Rightarrow Q$ is a tautology then we also say $P \Rightarrow Q$ tautologically.

Thus in the preceding example we can say that $(-P \Rightarrow Q) \wedge -Q \Rightarrow P$ tautologically.

Note: $P \Rightarrow Q$ cannot be a tautology if both *P* and *Q* are atomic sentence.

5.9 Tautological equivalence

Two sentence *P* and *Q* are said to be *tautologically equivalent* if $P \Rightarrow Q$ tautologically. And also $Q \Rightarrow P$ tautological equivalence if $P \Rightarrow Q$ tautologically, and also $Q \Rightarrow P$ tautologically. *P* and *Q* are tautologically equivalent may be written as $P \equiv Q$. It is clear that two compound sentence *P* and *Q* are tautologically equivalent if they have the same truth values in all the cases. i.e. Two statement *p* and *q* are said to be logically equivalent or equal if they have identical truth values.

One method to determine whether any two statements are equal is to construct a column for each statement in a truth table and compare these to see if they are identical.

For example $P \Rightarrow Q$ is tautologically equivalent to $-Q \Rightarrow -P$ as proved by the following table:

P	Q	$P \Rightarrow Q$	$-Q$	$-P$	$-Q \Rightarrow -P$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

We find that the truth values of $P \Rightarrow Q$ and $-Q \Rightarrow -P$ are the same in all the cases.

Hence $[P \Rightarrow Q] \Rightarrow [-Q \Rightarrow -P]$ and $[-Q \Rightarrow -P \Rightarrow [P \Rightarrow Q]$ are both tautologies.

The sentence –Q⇒ –P is called the contra-positive of the sentence $P \Rightarrow$Q. Hence very often to prove P⇒Q we prove –Q⇒–P.

Note: If $P \Rightarrow Q$ is a tautology, hen if P is true then Q must be true, since the implication is always true except when P is true and Q false.

Example.9: Show that each of the following is a tautology

(a) $[p \wedge (p \to q)] \to q$

(b) $[(p \to q) \wedge (q \to r)] \to (p \to r)$

(a) We shall construct truth-table for the function $p \wedge (p \to q) \to q$

Solution:

p	q	$p \to q$	$p \wedge (p \to q)$	$[p \wedge (p \to q)] \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Truth table for $p \wedge (p \to q)\,] \to q$

Since the column for $[p \wedge p \to q\,)] \to q$ contains only T, it is a tautology

(c) Here we construct the truth-table for $[(p \to q) \wedge (q \to r)] \to p \to r)]$

p	q	r	$p \to q$	$q \to r$	$p \to r$	$(p \to r) \wedge (q \to r)$	$[p \to r) \wedge [q \to r)\,] \to p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T

F	*T*	*F*	*T*	*F*	*T*	*F*	*T*
F	*F*	*T*	*T*	*T*	*T*	*T*	*T*
F	*F*	*F*	*T*	*T*	*T*	*T*	*T*

Truth table for $[(p \to r) \wedge (q \to r)] \to (p \to r)$

Since the last column corresponding to $[(p \to q) \wedge (q \to r)] \to (p \to r)$ contains only *T*, it is a tautology.

Example.10: Show that the statement $p \wedge \sim p$ is a contradiction.

Solution: Consider the truth table for $p \wedge \sim p$.

P	~	$p \wedge \sim p$
T	*F*	*F*
F	*T*	*F*

Truth table for $p \wedge \sim p$

It follows from the table that $p \wedge \sim p$ is a contradiction.

Example.11: Prove that $p \to q = \sim p \vee q$.

Solution: We shall construct truth table for statement $p \to q$ and $\sim p \vee q$.

p	q	$p \to q$	$\sim q$	$\sim p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Truth table for $p \to q$ and $\sim p \vee q$

We observe that the truth values in the columns for $p \to q$ and $\sim p \vee q$ are identical. Hence $p \to q$ $= \sim p \vee q$.

Example.12: Show that the statement $(p \wedge \sim p) \vee q$ and q are equal.

Solution: Consider the truth table for given statement.

p	q	$\sim p$	$p \wedge \sim p$	$(p \wedge \sim p) \vee q$
T	T	F	F	T
T	F	F	F	F
F	T	T	F	T
F	F	T	F	F

Truth table for $(p \wedge \sim p) \vee q$ and q

From the truth table we see that columns for $(p \wedge \sim p) \vee q$ and q are identical. Hence they are equal.

Note. **(1)** Some authors have used the symbol '$\Leftrightarrow$' to denote equivalent or equal statements and symbol $\leftrightarrow$ is used for bi-conditional statement.

Example.13: Show that $p \to (q \to r) = (p \wedge q) \to r$

Solution: Consider the following truth table.

p	q	r	$q \to r$	$p \wedge q$	$p \to (q \to r)$	$(p \wedge q) \to r$
T	T	T	T	T	T	T
T	T	F	F	T	F	F
T	F	T	T	F	T	T
T	F	F	T	F	T	T
F	T	T	T	F	T	T
F	T	F	F	F	T	T
F	F	T	T	F	T	T
F	F	F	T	F	T	T

Truth table for $p \to (q \to r) \& (p \wedge q) \to r$

We see that columns for $p \to (q \to r)$ and $(p \wedge q) \to r$ are identical hence given statement are equal.

Check your progress:

(1). By constructing truth tables, show that the following are tautologies:

(a) $(P \wedge Q) \Rightarrow P$

(b) $(P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$

(c) $(P \Leftrightarrow Q) \wedge (Q \wedge R) \Rightarrow (P \Leftrightarrow R)$

(d) $(P \vee Q) \wedge - Q \Rightarrow P$

(e) $[P \Rightarrow Q] \Leftrightarrow [-P \vee Q]$

(2). Show that the following are tautological equivalences:

(a) $(P \Leftrightarrow Q) \equiv (P \Rightarrow Q) \wedge (- P \Rightarrow -Q)$

(b) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

(c) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$

(d) $-(P \wedge Q) \equiv (-P) \vee (-Q)$

(e) $-(P \vee Q) \equiv (-P) \wedge (-Q)$

(f) $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge (P \wedge R)$

(g) $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$

5.10 Algebra of Proposition:

The following theorem contains various laws satisfied by propositions. We shall use these laws for simplification of propositions.

Theorem 1: The following laws are satisfied by statements:

1. Commutative laws:

(a) $p \vee q = q \vee p$. (b) $p \wedge q = q \wedge p$.

2. Associative laws:

(a) $p \vee (q \vee r) = (p \vee q) \vee r$ (b) $p \wedge (q \wedge r) = (p \wedge q) \wedge r$

3. Distributive laws:

(a) $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$ (b) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

4. Idempotent laws:

(a) $p \vee q = p$ (b) $p \wedge p = p$

5. Laws of absorption:

(a) $p \vee (p \wedge q) = p$ (b) $p \wedge (p \vee q) = p$

6. Involution laws:

(a) $\sim(\sim p) = p$

7. Complement laws:

(a) $p \vee \sim p = T$ (b) $p \wedge \sim p = F$

8. De Morgan's laws:

(a) $\sim (p \vee q) = \sim p \wedge \sim q$ (b) $\sim (p \wedge q) = \sim p \vee \sim q$

9. Operation with *T*

(a) $p \vee T = T$ (b) $p \wedge T = p$

10. Operation with *F*

(a) $F \vee p = p$ (b) $F \wedge p = F$

Here *T* and *F* denote statements, which are tautology and contradiction respectively.

Proof: We shall prove 3(a) and 8(a). The remaining laws can be proved exactly in the same way by constructing truth tables.

To prove 3(a), consider the following truth-table

p	q	r	$p \vee q$	$p \vee r$	$q \wedge r$	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T
T	F	T	T	T	F	T	T
T	F	F	T	T	F	T	T
F	T	T	T	T	T	T	T

F	T	F	T	F	F	F	F
F	F	T	F	T	F	F	F
F	F	F	F	F	F	F	F

Since columns for $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are identical they are equal.

To prove 8(a), consider the following truth table:

P	q	$\sim p$	$\sim q$	$p \vee q$	$\sim (p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

It follows from the table that $\sim (p \vee q) = \sim p \wedge \sim q$.

Theorem 2: Show that (a) $p \rightarrow q = \sim p \vee q$, (b) $p \leftrightarrow q = (p \rightarrow q) \wedge = (q \rightarrow p)$

Proof: Using the definitions of $\rightarrow$ and $\leftrightarrow$ we have,

p	q	$\sim p$	$p \to q$	$q \to p$	$p \leftrightarrow q$	$\sim p \vee q$	$(p \to q) \wedge = (q \to p)$
T	T	F	T	T	T	T	T
T	F	F	F	T	F	F	F
F	T	T	T	F	F	T	F
F	F	T	T	T	T	T	T

Since the truth values in columns (4) and (7) are identical, we have (a), Similarly, since the truth values in columns (6) and (8) are identical, we have (b).

Theorem 3: Show that (a) $\sim (p \to q) = p \wedge \sim q$, (b) $\sim (p \leftrightarrow q) = p \leftrightarrow \sim q$

Proof: Using the definitions of $\to$ and $\leftrightarrow$, we construct the truth table

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim (p \to q)$	$p \wedge \sim q$	$p \leftrightarrow q$	$\sim (p \leftrightarrow q)$	$p \leftrightarrow \sim q$
T	T	F	F	T	F	F	T	F	F
T	F	F	T	F	T	T	F	T	T
F	T	T	F	T	F	F	F	T	T
F	F	T	T	T	F	F	T	F	F

$\sim(p \leftrightarrow q) = p \leftrightarrow \sim q$ have identical truth values, so $\sim(p \leftrightarrow q) = p \leftrightarrow \sim q$

Example.14: Prove that $\sim(p \wedge q) \rightarrow (\sim p \vee (\sim p \vee q)) = \sim p \vee q$, without constructing truth table.

Solution: We shall use theorems 1 and 2.

L.H.S = $\sim(p \wedge q) \rightarrow (\sim p \vee (\sim p \vee q))$

$= (p \wedge q) \vee (\sim p \vee (\sim p \vee q))$ Since $p \rightarrow q = \sim p \vee q$

$= (p \wedge q) \vee ((\sim p \vee \sim p) \vee q)$ by associative law

$= (p \wedge q) \vee (\sim p \vee q)$ by idempotent law

$= ((p \wedge q) \vee \sim p) \vee q$ by associative law

$= (\sim p \vee (p \wedge q)) \vee q$ by commutative law

$= ((\sim p \vee p) \wedge (\sim p \vee q)) \vee q$ by distributive law

$= (T \wedge (\sim p \vee q)) \vee q$ by complement law

$= (\sim p \vee q) \vee q$ by 9(b) of Theorem 1

$= \sim p \vee (q \vee q)$ by associative law

$= \sim p \vee q$ by idempotent law

=R.H.S.

5.11 Summary:

A statement (or proposition) is a sentence which is either true or false but not both.

There are some key words and phrases which are used to form new sentences from given sentences, as for example ‘and’ ‘or’, ‘not’, ‘if…. then …., if and only if’ etc. They are called sentential or logical connectives. **A** Sentence with some logical connective is called a ‘Compound sentence’ and a sentence without logical connective is called an ‘atomic sentence.

A sentence obtained by conjoining two sentences P, Q by using the connective ‘and’ is called the *conjunction* of the two sentences and will be denoted by $P \wedge Q$ (read as P and Q).

A sentence obtained by joining two sentences P, Q, by the connective ‘or’ is called the *disjunction* of the two sentences and will be denoted by $P \vee Q$ (read as P or Q).

A sentence which has a truth value opposite to that of a sentence P is called the negation of P and is denoted by $-P$ or ~ P. Negation of an atomic sentence is obtained by using the connective ‘not’ at proper place.

A conditional sentence obtained by using the connective ‘If ….then…’ is called an implication.

A bi-conditional sentence obtained by using the connective ‘If and only if’ (briefly written as *iff*) between two sentences P, Q is called a double implication and is written a ‘P *iff* Q’

Let $p \rightarrow q$ be any conditional statement. Then,

(i) the converse of $p \rightarrow q$ is statement $q \rightarrow p$.

(ii) the inverse of $p \rightarrow q$ is the statement $\sim p \rightarrow \sim q$.

(iii) the contrapositive of $p \rightarrow q$ is the statement $\sim q \rightarrow \sim p$.

A propositional function is an expression, which is a combination of propositional variables and connectives.

A compound sentence is called a tautology if it is always true irrespective of the truth values of its component parts.

5.12 Terminal Questions:

1. Prove that each of the following is a tautology:

(a) $p \to p$ (b) $p \wedge q \to p$

(c) $p \to (p \vee q)$ (d) $(p \wedge (p \to q)) \to q$

(e) $(p \to q) \to [(p \vee (q \wedge r)) \leftrightarrow q \wedge (p \vee r)]$

2. Write in words the converse, inverse, contra positive and negation of the implication "if she works then she will earn money."

3. Construct truth tables to determine whether each of the following is tautology or a contradiction:

(a) $p \wedge \sim p$ (b) $p \to (q \to p)$

(c) $p \to q \wedge p$ (d) $q \vee (\sim q \wedge p)$

4. Prove the following:

(a) $p \vee q = q \vee p$ (b) $p \wedge (q \wedge r) = (p \vee q) \wedge r$

(c) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r$ (d) $p \vee p = p$

(e) $\sim(p \wedge q) = \sim p \vee \sim q$ (f) $\sim(p \leftrightarrow q) = \sim p \leftrightarrow q$

5. Write in English the negation of each of the following:

(a) The weather is bad and I will not go to work.

(b) I grow fat only if I eat too much.

6. Show the following equivalences:

(a) $p \to (q \to q) \Leftrightarrow \sim p \to (p \to q)$

(b) $\sim(p \leftrightarrow q) \Leftrightarrow (p \wedge \sim q) \vee (\sim p \wedge q)$

7. We define $p \Rightarrow q$ if and only if $p \to q$ is tautology. Prove the following:

(a) $p \to q \Rightarrow p \to (p \wedge q)$

(b) $(p \to q) \to q \Rightarrow p \vee q$

8. Prove that for any propositions *p* and *q*.

(a) $p \vee T = T$ (b) $p \wedge F = F$

(c) $\{(p \vee \sim q) \wedge (\sim p \vee \sim q)\} \vee q = T$.

Answers

---

2. The converse of the statement is "if she earns money then the she works." The inverse is "if she does not work then she will not earn money." The contra-positive is "if she does not earn money then she does not work" The negation of the statement is "she works and she will not earn money."

3. (a) Contradiction (b) Tautology

(c) Neither tautology nor contradiction

(d) Neither tautology nor contradiction

5.(a) The weather is bad but I will go to work.

(b) I grow fat and (although) I don't eat too much.

UNIT-6: Normal form

Structure

6.1 Introduction

The main problem in logic is the investigation of the process of reasoning. In Mathematics, a certain set of statements (propositions) is assumed and from this set, other statements are derived by logical reasoning. In this section, we shall investigate those processes which can be accepted as valid in the derivation of a statement from the given set of statements.

The given set of statements is called premises or hypothesis and the statement derived from the given statement is called conclusion.

6.2 Objectives

After reading this unit the learner should be able to understand about:

- concept of Argument
- Rule of Detachment
- Invalidity of an Argument
- Indirect Method of proof
- Proof by Counter-Example

6.3 Normal form:

The main problem in logic is the investigation of the process of reasoning. In Mathematics, a certain set of statements (propositions) is assumed and from this set, other statements are derived by logical reasoning. The given set of statements is called premises or hypothesis and the statement derived from the given statement is called conclusion.

6.4 Disjunctive Normal Form:

A Boolean function in n variables $x_1, x_2, \ldots, x_n$, is said to be in disjunctive normal form (in short, D N form) if it is a sum of minterms. Also 1 and 0 are said to be in disjunctive normal form.

In other words, a Boolean function in n variables $x_1, x_2, \ldots, x_n$ is said to be in disjunctive normal form if the function is a sum of terms of the type $f_1(x_1)\, f_2(x_2)\, f_3(x_3)\, f_4(x_4)\, f_5(x_5) \ldots f_n(x_n)$ where $f_j(x_j) = x_j$ or x'_j for all j=1, 2,….n and no two terms are same. Also 1 and 0 are said to be in disjunctive normal from.

The disjunctive normal form is also called the (sum of products canonical form).

Observe that in a disjunctive normal form (sum of products canonical form) any particular minterms may or may not be present. Since there are 2^n minterms in n variables, we can have only 2^{2n}different DN forms. These DN forms include the DN form of 0 in which no minterms is present in the sum and also the DN form of 1 where all the minterms are present in the sum. In any case, every Boolean function given in DN form in n variables is equal to one of the 2^{2n} Boolean functions.

Example.1: Write the function $f=(xy' + zx)' + x'$ in DN form.

Solution: We have

$$f = (xy' + xz)' + x'$$

$$= (xy')'(xz)' + x'$$

$$= (x'+y)(x'+z') + x'$$

$$= x' + yz' + x' = x' + yz'$$

$$= x'(yz + yz' + y'z + y'z') + yz'x + yz'x'$$

$$= x'yz + x'yz' + x'y'z + x'y'z + x'y'z' + xyz' + x'yz'$$

$= x'yz+xyz'+x'yz'+x'y'z+x'y'z'$, because the minterm $x'y'z'$ is appearing twice.

Example.2: Write the Boolean function $f = x_1+x_2$ in sum of products canonical form in three variables x_1, x_2 and x_3.

Solution: We have

$$f = x_1+x_2$$

$$= x_1(x_2+x'_2)(x_3+x'_3) + x_2(x_1+x'_1)(x_3+x'_3)$$

$$= x_1(x_2x_3+x_2x'_3+x'_2x_3+x'_2x'_3) + x_2(x_1x_3+x_1x'_3+x'_1x_3+x'_1x'_3)$$

$$= x_1x_2x_3+x_1x_2x'_3+x_1x'_2x_3+x_1x'_2x'_3+x_1x_2x_3+x_1x_2x'_3+x'_1x_2x_3+x'_1x_2x'_3$$

$$= x_1x_2x_3+x_1x_2x'_3+x_1x'_2x_3+x'_1x_2x_3+x'_1x_2x'_3+x_1x'_2x_3.$$

Complete Disjunctive Normal Form:

As seen earlier, if there are n variables, then the total number of minterms will be 2^n. Therefore any DN form can have at most 2^n minterms.

A disjunctive normal form in n variables, which contains all the 2^n minterms is called the complete disjunctive normal form. For example, $f = xy + x'y + xy'+x'y'$ is the complete disjunctive normal form in two variables. It can be seen by simplification of the complete disjunctive normal form or by the following theorem that the complete disjunctive normal form is identically equal to 1.

Theorem 2: If each of n variables is assigned the value 0 or 1 in an arbitrary, but fixed manner then exactly one minterm of the complete disjunctive normal form in the n variables will have the value 1 and all other minterms will have the value 0.

Proof: Consider the complete disjunctive normal form in n variables $x_1, x_2, \ldots, x_n$. Then it has all the 2^n minterms of the form $f_1(x_1)\, f_2(x_2) \ldots f_n(x_n)$ where $f_i(x_i)=x_i$ or x'_I for each $i = 1, 2, \ldots, n$. Now assign the values 0 or 1 to the variables x_1, x_2,….x_n. Select a minterm from the complete normal form as follows: use x_i if x_i is assigned the value 1 and use x'_i if x_i is assigned the value 0 for each x_i, i=1, 2, …., n. The term so selected is then a product of n ones and hence is equal to 1. All other terms in the complete normal form will contain at least one factor 0 and hence will be 0.

Corollary 1: To functions with same minterms are obviously equal. Conversely, if two functions are equal, then they must have same value for every choice of value for each variable. In particular, they assume the same value for each set of values 0 and 1, which may be assigned to the variables. By theorem 2 above, the combinations of values of 0 and 1 which, when assigned to the variables, make the function assume the value 1 uniquely determine the terms which are present in the DN from for the function. Hence, both DN forms contain the same min-terms.

Corollary 2: To establish any identity in Boolean algebra, it is sufficient to check the value of each function (on both sides of the identity) for all combination of 0 and 1, which may be assigned to the variables.

We have seen in the preceding theorems that a Boolean function is completely determined by the value it take for each possible assignment of 0 and 1 to the respective variables. This suggests that Boolean functions could be easily specified by giving a table to represent such properties. If such a table has been given, then the function, in disjunctive normal form, may be written down by inspection. We simply look at the conditions where the function takes the value 1 then the sum of corresponding minterms (where function takes the value1) gives the function, although the function so obtained may not be in simplest form. The following example will explain this method.

Example.3: Find and simplify the function specified by the table as given.

Row	x	y	z	f(x,y,z)
1	1	1	1	0
2	1	1	0	1

3	1	0	1	0
4	1	0	0	1
5	0	1	1	0
6	0	1	0	0
7	0	0	1	0

Solution: We observe that function f(x, y, z) takes the value 1 for the normal form of f will contain two minterms each corresponding to the conditions given in rows 2 and 4. In row 2, the values of x, y, z are given respectively as 1, 1, 0 and so the corresponding minterm will be xyz'. Similarly the other minterm will be $xy'z'$ which is taken with respect to row 4.

Thus the function $f(x,y,z) = xyz' + xy'z'$.

We now simplify f(x, y, z) b using laws of Boolean algebra $\therefore f(x, y, z) = xz' (y + y') = xz'$. It may be verified that this function satisfies all the other rows also.

6.5 Conjunctive Normal Form:

Conjunctive normal form is a dual of disjunctive normal form. Thus all the results that we proved for DN forms can be extended to this form by duality.

A Boolean function is said to be in conjunctive normal form (in short, CN form) in n variables x_1, x_2, …., x_n, for $n > 0$ if the $f_1(x_1)f_2(x_2)+\ldots+f_n(x_n)$. Where $f_i(x_i) = x_i$ or x_i' for each i = 1, 2,….., n, and no two terms are same. Moreover 0 and 1 are also said to be in conjunctive normal form. The terms of the type

$f_1(x_1)+f_2(x_2)+\ldots+f_n(x_n)$, where $f_i(x_i)= x_i$ or x_i' for each i = 1, 2, ….., n are called max-terms or maximal polynomials.

Theorem 1: Every function in Boolean algebra, which contains no constants is equal to a function in E/n form.

Proof: Let f be the Boolean function which contains no constant. If f contains an expression of the form $(x+y)'$ or $(xy)'$ for some variables x and y then De Morgan's rule may be applied to get $x'y'$ and $x'+y'$, respectively. This process may be continued until each′ which appears applies only to a single variable.

Next, by applying the distribution law, f can be reduced to products. Now suppose some term does not contain either x_i oo x_i' for some variable x_i. Then $x_i x_i'$ may be added to this tem without changing the function. Continuing this process for each missing variable in each factors in f will give an equivalent function whose factors contain x_i or x_i' for each i = 1, 2, n. Finally, using aa=a, we can eliminate the duplicate terms and with this the proof is complete.

Example.4: Write the function $(xy'+xz)'+x'$ in CN form.

Solution: Let $f = (xy'+xz)'+x'$

$= (xy')(xz)'+x'$

$=(x'+y)(x'+z')+x'$

$= x'+(x'+y)(x'+z')$

$= x' + x' + y z'$

$= (x'+y)(x'+z')$

$= (x'+y+zz')(x'+z'+yy')$

$= (x'+y+z)\ (x'+y+z')\ (x'+y+z')\ (x'+y'+z')$

$= (x'+y+z)\ (x'+y+z')\ (x'+y'+z')$.

Complete conjunctive normal form

The conjunctive normal form in n variables which contains 2^{nd} factors (max-terms) is called complete conjunctive normal form in n variables.

Theorem 2: Let f be a complete conjunctive normal form in n variable. The exactly one maxterms (factor) will have the value 0 and all other max-terms (factors0 will have the value 1.

Example.5: Find the Boolean function f in CN form that is given by the following table

x	y	z	f
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	0
0	1	1	1
0	1	0	1

0	0	1	1
0	0	0	0

Solution: To get the expression in CN form, we look at the values of f(x, y, z) when it is 0. We see from the table that f takes value 0 at 2nd, 3rd, 4th and 8th row thus

$$f = (x'+y'+z)\,(x'+y+z')\,(x'+y+z)\,(x+y+z).$$

Rule for finding complement of a function given in CN form:

As in disjunctive normal form, we can use the conjunctive normal form to find complements of functions written in this form by inspection. The complement of any function written in CN form is that function whose factors are exactly those factors of the complete conjunctive normal form, which are missing from the given function.

Example.6: Find the complement of $f= (x+y')\,(x'+y)$

Solution: The given function f is written in CN form of two variables. The complete conjunctive normal form in two variables is $(x+y)(x+y')\,(x'+y)\,(x'+y')$

Therefore the complement of f is $(x+y)\,(x' +y')$

Example.7: Find the conjunctive normal form for the function

$$f=xyz+x'yz+xy'z'+x'yz'$$

Solution: We know that $(f')' = f$

$\therefore \quad f = [(xyz + x'yz + xy'z'+x'yz')']'$

$=[(xyz)'(x'yz)'(xy'z')'(x'yz')]'$ by De Morgain's law

$= [(x'+y'+z')(x+y'+z')(x'+y+z)(x+y'+z')]'$ by De Morgain's law

$= (x+y+z)(x'+y+z')(x+y+z')(x'+y'+z)$.

Check your progress:

(1). Express each of the following in CN form in the smallest possible number of variable.(a) $x+x'y$ (b) $(u+v+w)(uv+u'w')$ (c) $(x+y(x+y')(x'+z)$

(2). Write the CN form in three variables x, y anx z (a) $x+y'$ (b) $(x+y)(x'+')$

(3). Write the function of x, y and z which is 0 when any two or more of the variables are 0 otherwise it is 1.

(4). Find by inspection the complement of each of the following

(a) $(x+y)(x'+y)(x'+y')$ (b) $(x+y+z)(x'+y'+z)(x'+y'+z')$

(5). Change the function $f = uv + u'v + u'v'$ from DN from to CN form

(6) .Change the function $f = (x+y')(x'+y)(x'+y)$ from CN form to DN

(7). Let $f(x_1, x_2, x_3,) = [(x_1+x_1)' + x_1' x_1']'$ be a Boolean expression (function) over two valued Boolean algebra. Write $f(x_1, x_2, x_3)$ in both DN and CN form.

(8). Write the expression$(x_1, x_2, x_3) = x_1x_1+ x_1x_3+ x_2 x_3$in both DN and CN forms.

(9). Express the function given by the table below in both DN form and CN form

(x_1, x_2,x_3)	$f(x_1, x_2, x_3)$

(0,0,0)	1
(0,0,1)	0
(0,1,0)	1
(0,1,1)	0
(1,0,0)	0
(1,0,1)	1
(1,1,0)	0
(1,1,1)	1

(10). For any Boolean function $f(x_1, x_2)$, show that

$$f(x_1, x_2)=[x_1+f(0, x_2)][x_1'+f(1, x_2)]$$

(11). Simplify the following Boolean function

(a) ab + abc + bc (b) (ab′+c) (a+b′+c).

Answer

---

1. (a) x + y (b) (u+v+w) (u+v+w′) (u+v′+w′) (u′+v′+w) (u′+v′+w′) (c) (x+z) (x+z′) (x′+z)

2. (a) (x+y′+z) (x+y′+z′) (b) (x+y+z) (x+y+z) (x+y+z′) (x′+y′+z) (x′+y′+z′)

3. (x+y+z) (x+y+z′) (x+y+z′) (x+y′+z) (x′+y+z)

4. (a) (x+y′) (b) (x+y+z′) (x+y′+z) (x+y′+z′) (x′+y+z) (x′+y+z′)

5. u′+v

6. x′y′

7. DN form of $f = x_1x_2x_3 + x_1x_2x_{3'} + x_1x_2'x_3 + x_1x_2'x_3' + x_1'x_2'x_3'$

CN form of $f = (x_1x_2x_3)\ (x_1x_2x_3')(\ x_1x_2'x_3')$

8. DN form of f = $x_1x_2x_3 + x_1x_2x_3' + x_1x_2'x_3 + x_1'x_2'x_3$

CN form of f = $(x_1+x_2+x_3)(x_1+x'_2+x_3)(x_1+x'_2+x'_3)(x'_1+x_2+x_3)$

9. $F(x_1,x_2,x_3) = (x_1+x_2+x'_3)(x_1+x'_2+x'_3)(x'_1+x_2+x_3)(x'_1+x'_2+x_3)$

10. (a) ab+ac (b) ac+b′c

6.6 Rule for finding complement of a function in DN form:

We can find by inspection the complement of any function given in disjunctive normal form. If a function f is given in disjunctive normal from then its complement f′ is obtained by omitting from the complete disjunctive normal form the terms which appear in the function f.

Example.8: Find the complement of the following functions –

(a) f = a′b + ab′

(b) f = a′bc + abc′ + a′b′c+a′b′c′

Solution: Therefore the complement of f = a′b + ab′ is given by f′ = ab + a′b′

(a) The complete disjunctive normal form in three variables a, b, c is

abc+abc′+ab′c+ab′c′+a′bc+a′bc′+a′bc′+a′b′c+a′b′c′

(b) Therefore the complement of f = a′bc + abc′ + a′b′c + a′b′c′ is given by

f′ = abc + ab′c + ab′c + a′ bc′

Example.9: Find the function of three variables x, y and z which is I if either x=y=1 and z=0 or if x = z = 1 and y = 0 and is 0 otherwise.

Solution: By given conditions, the required function takes the value 1 at two points namely when $x = 1, y = 1, z = 0$ and when $x = 1, y = 0 = 1$. The corresponding minterms are xyz' and $xy'z$. Hence the required function f DN form is equal to sum of these two terms, i.e. $f = xyz'+xy'z$

Example.10: Express each of the following in DN from in the smallest possible number of variables

(i) $xy'+xz+xy$

(ii) $(x'+xyz'+xy'z+x'y'z't+t')'$

Solution: (i) Let $f = xy'+xz+xy = xy'+xy+xz$ by commutative law

$= x (y'+y) +xz$ by distributive law $= x+ xz$ by complement law

$=x$ by absorption law. Now f contains only one variable.

Hence $f = x$ is its disjunctive normal form in smallest possible variables.

(ii) Let $f = [x'y+xy'+xy'z+x'y'z't+t']'$

we first consider the expression without complement, i.e. $x'y+xyz'+xy'z+x'y'z't+t'$

and express it in disjunctive normal form in four variables x, y, z and t.

Now $x'y+xyz'+x'y'z't+t'= x'y(z+z') (t+t') + xyz' (t+t')xy'z(t+t') + x'y'z't+t'(x+x') (y+y') (z+z')$

$=x'y(zt+z't+zt'+z't'+)+xyz'(t+t')+xy'z(t+t')$

$+x''z't+t'(zyz+x'yz+xy'z+xyz'+x'y'z+x'yz'+xy'z'+x'y'z')$

Using the law a+a = a, we ge

$t= x'yzt+x'yz't+x'yzt'+x'yz't'+xyz't+xyz't'+xy'zt+ x y'zt'+ x'y'z't+ x'yzt'+ x'y'zt'+ xy'z't'+x'y'z't'$. which contains 13 terms.

Now writing the missing terms from the complete DN form, we get the required function (which is complement of this function) as f = $xyzt+x'y'zt+xy'z't$ in DN form in three variables.

Example.11: Solve $f(x,y,z)=(x+yy')(x'+z)$ using distributive law

Solution: We can write $f(x,y,z)=(x+yy')(x'+z)$ using distributive law

$$= x(x'+z)$$

$$= xz$$

$$= x(y+y')z$$

$$= xyz+xy'z$$

The table for above function is given DN from of the function is $xyz+xy'z$

x	y	z	f(x,y,z)
1	1	1	1
1	1	0	0
1	0	1	1
1	0	0	0
0	1	1	0
0	1	0	0

0	0	1	0
0	0	0	0

Example.12: In a Boolean algebra, show that f(x,y) = xf(1, y) +x′f(0,y)

Solution: The complete DN form of f(x,y) is

$$f(x,y) = xy + xy'+x'y+x'y'=x(y+y')+x'(y+y') \qquad \text{....(1)}$$

Putting x = 1 and therefore x′ = 0, we get f (1, y) = y + y′

Again putting x = 0 and therefore x′ = 1, we get

$$f(0, y) = y + y'$$

$$f(x,y)= x\, f(1, y) +x'f(0, y) \qquad \text{by (1)}$$

Example.13: Write all 16 possible functions of two variables x and y.

Solution: All possible 16 functions are listed in the following table:

x	y	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}	f_{16}
1	1	1	1	1	1	0	1	1	1	0	0	0	1	0	0	0	0
1	0	1	1	1	0	1	1	0	0	1	0	1	0	0	1	0	0
0	1	1	1	0	1	1	0	1	0	0	1	1	0	1	0	0	0

0	0	1	0	1	1	1	0	0	1	1	1	0	0	0	0	1	0

Therefore, the 16 functions are:

$f_1 = xy + x'y + xy' + x'y' = 1$,

$f_2 = xy+x'y+xy' = y+xy'$

$f_3 = xy+xy'+x'y'$, $f_4 = xy+x'y+x'y'$

$f_5 = x'y+xy'+x'y' = x'y+y'$, $f_6 = xy + xy' = x$

$f_7 = xy + x'y = y$, $f_8 = xy + x'y'$

$f_9 = xy' + x'y' = y'$, $f_{10} = x'y+x'y' = x'$

$f_{11} = x'y + xy'$, $f_{12} = xy$, $f_{13} = x'y = x'y$

$f_{14} = xy' = xy'$, $f_{15} = x'y'$, $f_{16} = 0 = 0$.

Check your progress

1. Express each of the following in DN from in the smallest possible number of variables

(a). $x + x'y$ (b) $(u+v+w)(uv+u'w)'$

2. Write complete disjunctive normal form in three variables x, y and z. Determine which term equal 1 if (a) $x = 1, y = z = 0$, (b) $x = z = 1$ and $y = 0$

3. Write disjunctive normal form in the three variables x, y and y of the function $f = x + y'$.

4. Write the function f of x, y and z which is 1 if and only if any two or more of the variables are 1.

5. Find, by inspection, the complement of each of the following:

(a) $xy + x'y$,   (b) $x'y'z' + x'yz + xy'z'$

6. Prove that there are exactly 2^{2n} distinct functions of n variables in a Boolean algebra.

7. Write and simplify the two functions f_1 and f_2 specified by the table

x	y	z	f_1	f_2
1	1	1	0	1
1	1	0	1	1
1	0	1	0	0
1	0	0	1	0
0	1	1	0	0
0	1	0	0	0
0	0	1	0	0
0	0	0	0	1

Answer

1. (a) $xy + xy' + x'y$ (b) $uv'w + u'vw' + uv'w'$

3. $Xyz + xyz' + xy'z + xy'z' + x'y'z + x'y'z'$

4. $xy + yz + zx$

5. (a) $xy' + x'y'$ (b) $xyz + xyz' + xy'z + x'yz'$

7. $f_1 = xz'$, $f_2 = xy + x'y'z'$

Argument

An argument is a process by which a conclusion is formed from a given set of statements called premises.

An argument is said to be **valid argument** if and only if the conjunction of the premises implies the conclusion. That is, the argument which yields a conclusion *r* from the premises p_1, p_2, p_3-------, p_n is valid if and only if the statement is tautology.

An argument which is not valid is called a fallacy. An argument which is derived from the premises or hypothesis $p_1, p_2, -----, p_n$ is written as

$$\begin{array}{c} p_1 \\ p_2 \\ p_3 \\ - \\ - \\ p_n \\ \hline q \end{array}$$

That is, the premise or premises will be listed first and the conclusion will be written beneath a horizontal line.

Example.14: Prove that the following argument is valid:

$$P$$

$$\frac{p \to q}{q}$$

Solution: Here p and $p \to q$ are two premises and q is the conclusion. To show that the argument is valid we show that conjunction of the given premises implies the conclusion is a tautology. That is, we show $[p \wedge (p \to q)] \to q$ is a tautology by constructing truth table.

p	q	$p \to q$	$p \wedge (p \to q)$	$[p \wedge (p \to q)] \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

It follows from the truth table that $[p \wedge (p \to q)] \to q$ is a tautology.

Thus the argument is valid.

6.6 Rule of Detachment or Modus Ponens:

The valid argument

$$P$$

$$\frac{p \to q}{q}$$

is called rule of detachment. Rule of detachment is also known as **modus ponens**.

Law of Syllogism The argument

$$p \to q$$

$$q \to r$$

$$\overline{}$$

$$p \to r$$

is valid argument and is known as the **law of syllogism.**

Example.15: The validity of the law of syllogism is proved by constructing truth table for $[(p \to q) \wedge (q \to r)] \to (p \to r)$.

p	q	r	$p \to q$	$q \to r$	$p \to r$	$(p \to q) \wedge (q \to r)$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T

F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Truth table for $[(p \to q) \wedge (q \to r)] \to (p \to r)$

It follows from the truth table that the law of syllogism is a valid argument.

Given an argument, there are, in general, three methods to check the validity of the argument. These methods are given below.

a. Validity using Truth Table:

In this method, we construct a truth table as follows: $p_1 p_2 ----, p_n$ be all the premise and let q be the conclusion in the given argument. We construct truth table for the statement $(p_1 \wedge p_2 \wedge ---- \wedge p_n) \to q$

If we have all *T*s in the column of this statement then the statement is tautology and so the argument used is valid otherwise the argument is not valid.

b. Validity using Simplification Methods:

In this method, we convert all the implication statement $p \to q$ to the equivalent statement $\sim p \vee q$ in the argument involved and then we simplify the resulting statement using rules of statement (Theorem 1 of § 1.9). If the statement

$$(p_1 \wedge p_2 \wedge p_3 \wedge ---- \wedge p_n) \to q$$

can be reduced to *T*, then we say that the argument is valid.

c. Validity using Rules of Inference:

In this method, we reduce the given argument to a series of arguments each of which is known to be valid. Two of the most frequently used rules of inference (i.e. valid argument) are the rule of detachment and the law of syllogism.

Example.16: Show that the following argument is valid

$$\begin{array}{c} p \\ p \rightarrow q \\ \underline{q \rightarrow r} \\ r \end{array}$$

Solution: We shall show the validity of the argument by all three methods.

First solution: We construct the truth table for the statement

$$f = [p \wedge (p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow r$$

p	q	r	$p \rightarrow q$	$q \rightarrow r$	$p \wedge (p \rightarrow q)$	$p \wedge (p \rightarrow q) \wedge (q \rightarrow r)$	f
T	T	T	T	T	T	T	T
T	T	F	T	F	T	F	T

T	F	T	F	T	F	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	F	F	T
F	T	F	T	F	F	F	T
F	F	T	T	T	F	F	T
F	F	F	T	T	F	F	T

Truth table for $f = [p \wedge (p \to q) \wedge (q \to r)] \to r$

Since the column for f contains only Ts, the argument is valid

Second solution: We shall simplify the statement

$f = [p \wedge (p \to q) \wedge (q \to r)] \to r$

since $p \to q = \sim p \vee q$

$= \sim p \vee (p \wedge \sim q) \vee (q \wedge \sim r) \vee r$ using $\sim (p \wedge q) = \sim p \vee \sim q$

$= (\sim p \vee p) \wedge (\sim p \vee \sim q) \vee (q \wedge \sim r) \vee r$ using distributive law.

$= T \wedge (\sim p \vee \sim q) \vee r \vee (q \wedge \sim r)$ since $\sim p \vee p = T$

$= (\sim p \vee \sim q) \vee \{(r \vee q) \wedge (r \vee \sim r)\}$ since $p \wedge T = p$

$= (\sim p \vee \sim q) \vee \{(r \vee q) \wedge T\}$ since $r \vee \sim r = T$

$= (\sim p \vee \sim q) \vee (r \vee q)$ since $p \wedge T = p$

$= \{(\sim p \vee \sim q) \vee q\} \vee r$ since $p \vee q = q \vee p$

$= \{\sim p \vee (\sim q \vee q)\} \vee r$ by associative law

$= \{\sim p \vee T\} \vee r$ since $\sim p \vee p = T$

$= T$

Since f reduces to T, so the argument is valid

Example.17: Check the validity of the argument

$$p \rightarrow q$$

$$r \rightarrow \sim q$$

$$\overline{}$$

$$p \rightarrow \sim r$$

Solution: Since the statement $r \rightarrow \sim q$ is equal to $q \rightarrow \sim r$, we can replace the premise $r \rightarrow \sim q$ by $q \rightarrow \sim r$. Now $p \rightarrow q$

$$q \rightarrow \sim r$$

$$\overline{}$$

$$p \rightarrow \sim r$$

is valid argument by the law of syllogism. Hence given argument is valid.

6.7 Invalidity of an Argument:

A Short Method for Invalidity of an Argument:

In checking a validity of a given argument, if it is found or suspected that the argument is not valid, a proof of invalidity can be given more easily than by constructing the entire truth table related to the argument. For proving that the argument is invalid, it is sufficient to exhibit a particular set of truth values for the statements involved for which the premises are all true and the conclusion is false. This is equivalent to demonstrating that one row in the truth table would contain *F* and hence the argument is invalid.

Example.18: Show that the following argument is not valid

$$\begin{array}{l} p \\ \sim p \vee r \\ \underline{\sim p \rightarrow q} \\ \quad r \end{array}$$

Solution: If *p* is true, *q* is false and *r* is false then each of the premises is true but the conclusion is false. Hence the argument is invalid.

Example.19: Given the following statements as premises, all referring to an arbitrary meal:

(a) If he takes coffee, he doesn't drink milk.

(b) He eats crackers only if be drinks milk.

(c) He does not take soup unless he eats crackers.

(d) At noon today, he had coffee.

Whether he took soup at noon today? If so, what is the correct conclusion?

Solution: Let *p*: he takes coffee.

q: he drinks milk.

r: he eats crakers.

s: he takes soup.

Then we have, by condition (a) that $p \rightarrow \sim q$

by condition (b), we have $r \rightarrow q$, by condition (c), we have $\sim r \rightarrow \sim s$

and by condition (d), we have p

Since implication $r \to q$ is equivalent to its contrapositive $\sim q \to \sim r$., we have the following chain of argument:

$p \to \sim q$	a premise
$\sim q \to \sim r$	contrapositive of premise (b)
$\overline{p \to \sim r}$	a conclusion by law of syllogism
$\sim r \to \sim s$	a premise
$\overline{p \to \sim s}$	a conclusion by law of syllogism
$p / \sim s$	

Hence ~*s* is the conclusion. That is, he did not take soup at noon today.

6.8 Indirect Method of proof:

An important proof technique called the indirect method follows from the fact that any implication $p \to q$ is equivalent to its contra-positive $\sim q \to \sim p$. Thus to prove $p \to q$ indirectly, we assume that q is false and show that p is then false.

More generally, to prove the validity of an argument with premises $p_1, p_2, p_3, ----, p_n$ and conclusion q by indirect method, we consider second argument with premises $\sim q p_1 p_2 p_3 {}_{-------p_{i-1}} p_{i+1} --- p_n$ and conclusion p_i and prove the validity of this second argument.

Example.20: Show that the following argument is a valid argument.

$$\begin{array}{l} p \\ p \wedge q \rightarrow r \vee s \\ q \\ \underline{\sim s} \\ \quad r \end{array}$$

Solution: We will take as premises for the indirect proof all given premises except $\sim s$ and the negation of the conclusion $\sim r$ *i.e.* we shall show that the following argument is valid:

$$\begin{array}{l} p \\ p \wedge q \rightarrow r \vee s \\ q \\ \sim r / s \end{array}$$

Now, p a premise

q a premise

$\overline{\quad}$ a conclusion because $p \wedge q \rightarrow p \wedge q$ is always a tautology.

$p \wedge q$

Now, $p \wedge q$ a valid conclusion

$\underline{p \wedge q \rightarrow r \vee s}$ a premise

$r \vee s$ a valid conclusion by *modus ponens*

$\sim r$ a valid conclusion because

$(r \vee s) \wedge \sim r \rightarrow s$ is a tautology.

6.9 Proof by Counter-Example:

If a statement claims that a property holds for all objects of a certain type, then to prove it, we must use steps that are valid for all objects of that type. To disprove such a statement, we need only show one counter example. That is one particular object for which the statement is false. Such a proof is called a proof by counter-example.

Example.21: Prove that the statement “if n is an integer, then $n^2 - n + 41$ is a prime number” is false.

Solution: We need only find one example for which the statement is false. If $n = 41$, then $n^2 - n + 41 = 41^2$ which is not a prime. Hence the statement is false.

Example.22: Rohan made the following two statement:

1. I love Vicky

2. If I love Vicky then I also love Vivian

Given that Rohan either told the truth or lied in both cases, determine whether Rohan really loves Vicky.

Solution: Let *p*: Rohan loves Vicky and *q*: Rohan loves Vivian

Consider the following truth table:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

We are given that both p and $p \to q$ are either true or both of them are false. From the table, it is possible for both p and $p \to q$ to be true (row 1) but not possible for both p and $p \to q$ to be false. Hence Rohan must have told the truth and we conclude that Rohan really loves Vicky.

Example.23: An island has two tribes of natives. Any native from first tribe always tells the truth while any native from the second tribe always lies. Suppose you arrive at the island and ask a native if there is gold on the island. He answers "There is a gold on the island if and only if I always tell the truth." Determine whether there is gold on the island.

Solution: Let p&q be the following propositions:

p : He (the native from whom question is being asked) always tells the truth.

q : There is gold on the island.

Then his answer is the statement ' $p \to q$ '. Suppose p is true then $p \leftrightarrow q$ is true. Consequently q must be true. If p is false then his statement $p \leftrightarrow q$ is false. Consequently q must be true. Thus in both cases we can conclude that there is gold on the island.

Example.24: A logician was captured by a certain gang. The leader of the gang blindfolded the logician and placed him in a locked room containing two boxes. He gave the following instruction "one box contains the key to the room and other a poisonous snake. You are the reach into either box you choose and if you find the key, you can use it to go free. To help you, you can ask my assistant a single question requiring a yes or no answer. However, he does not have to answer truely, he may lie if he chooses." After a moment of thought, the logician asked a question, reached in to the box with the key and left. What question did the logician ask so that he was certain to go free?

Solution: Let p be the statement "the box on my left contains the key". Let q be the statement "you are telling the truth." Suppose we desire the answer "yes" if p is true and "no" if p is false. In the table given below, the first three columns represent the possible truth values of p and q and the desired answer. Then the required statement (i.e. a question requiring answer in 'yes' or 'no') must have column 4 as its truth values.

p	q	***Desired answer***	***Truth value of required statement***
T	T	Yes	T
T	F	Yes	F
F	T	Yes	F
F	F	Yes	T

We explain the reasoning used in forming the truth table by considering row 2. In this row truth value of statement p is T while that of q is F. Thus the key is in the left box and the man is lying. Consequently to obtain an affirmative answer the function must have the value F (because the value of the function in row 1 is T) The statement corresponding to this truth table is $p \leftrightarrow q$.

Hence the proper question is “does the box on the left contain the key if and only if you are telling the truth?

Example.25: Given that the value of $p \to q$ is true. Can you determine the value of $\sim p \vee (p \leftrightarrow q)$?

Solution: We shall construct the truth table having columns for $p \to q$ and $\sim p \vee (p \leftrightarrow q)$.

p	q	$p \to q$	$\sim p$	$p \leftrightarrow q$	$\sim p \vee (p \leftrightarrow q)$
T	T	T	F	T	T
T	F	F	F	F	F
F	T	T	T	F	T
F	F	T	T	T	T

From the table it follows that if $p \to q$ is true then the value of $\sim p \vee (p \leftrightarrow q)$ is true.

Note: In the above example, we could determine the value of $\sim p \vee (p \leftrightarrow q)$ because corresponding to each possible choices of *p* and *q* for which the value of $p \to q$ is true, the value of $\sim p \vee (p \leftrightarrow q)$ is same as *T*.

Sentential form:

Consider the following expressions or statements: (1) *x* is mortal (2) *x* is a fraction

They are not sentences, since we do not know their truth values. If we take x to be a number in (1) and x to be a man in (2), then these two statements (1), (2) become meaningless and hence they are not sentences. But if we restrict x in (1) to men and in (2) to numbers, then these statements will be sentences either true or false.

Here x will be called a variable. Such statements which contain variables like x which are not specified are called open sentences or sentential from. Similarly expressions containing pronouns, as for example 'He is prime minister of India', 'It is a prime number' are open sentences, since we do not know their truth value without additional information specifying the unknown pronouns which behave like variables.

The open sentence 'x is mortal' will be denoted by $P(x)$.

6.10 Quantifiers:

In the discussion of logic, some very important statements contain quantifiers. The following are examples of statements which contain quantifiers:

(1) Some people are honest.

(2) No woman is a player.

(3) All Americans are crazy.

The words *some*, *no* and all are known as quantifiers. From quantifiers, we know "how many" of a certain set of things is being considered.

6.10 (a) Universal Quantifier:

Let p be a statement. We define the symbol $\forall_x p$ to mean that for every value of x in the given set, the statement p is true. The symbol $\forall$ is called the universal quantifier. $\forall$ can also be read as 'for all' , ' for every' or 'for any.'

Illustration: The statement "for all natural numbers, $n + 4 > 3$" can be expressed as $\forall_x p$, where x belongs to the set N of natural numbers and p is the statement '$n + 4 > 3$.'

6.10 (b) Existential Quantifier:

Let p be a statement. We define the symbol $\exists_x p$ to mean that for one or more elements x of a certain set, the proposition p is true. The symbol $\exists$ is called existential quantifier and is usually read as "these exists" or "for at least one "for some".

Illustration: (1) The statement "there exists a number x such that $x^2 - 4x = 16$" may be written as $\exists_x (x^2 - 4x = 16)$

(2) The statement $\exists_n (n + 4 < 7)$, where n is in the set of natural numbers is true since there exists a natural number, namely I, such that $n + 4 < 7$ is true.

6.10 (C) Negation of Quantifiers:

It is important to know how the negations of statements having quantifiers are formed. Consider the statement "All Americans are crazy"

The negation of this statement would be

"It is false that all Americans are crazy" or equivalently,

"There exists at least one American who is not crazy."

In general, we have $\sim(\exists_x p) = \forall_x (\sim p)$ and $\sim(\forall_x p) = \exists_x (\sim p)$

Example.26: The function *f* is said to approach the limit *l* near *a* if (1) $\forall \in > 0$, $\exists \delta > 0$ such that $\forall x$ $0 < | x - a| < \delta \Rightarrow | f(x) - l| < \in$.

Putting $P(x)$: $0 < | x - a| < \delta \Rightarrow | f(x) - | < \in$. It can be written as $\forall \in > 0$ $\exists \delta > 0$ such that $\forall (x)$ $P(x)$. *Hence the negation of the above definition will be :*

The function *f* does not approach *l* at a if $\exists \in > 0$ s.t. $\forall \delta > 0$ $\exists x(\sim P(x))$.

That is, if there exists some $\in > 0$, such that *for every* $\delta > 0$, there exists some *x* for which $0 < |x - a| < \delta$ and $| f(x) - l | \not< \in$.

Example.27: Write the negation of 'No teachers are wise'. Putting $P(x)$: x is wise (*x* is a teacher), the symbolic form of the above sentence is $\forall x$ (*x* is a teacher) *x* is not wise or $\forall x$ $(\sim P(x))$ (*x* is a teacher). Hence its negation will be $\exists x P(x)$ that is, there exists a teacher *x* who is wise or' Some teachers are wise'.

6.11 Summary:

A Boolean function in n variables $x_1, x_2, \ldots, x_n$, is said to be in disjunctive normal form (in short, D N form) if it is a sum of minterms. Also 1 and 0 are said to be in disjunctive normal form.

A disjunctive normal form in n variables, which contains all the 2^n minterms is called the complete disjunctive normal form. Conjunctive normal form is a dual of disjunctive normal form.

A Boolean function is said to be in conjunctive normal form (in short, CN form) in n variables x_1, $x_2, \ldots, x_n$, for $n > 0$ if the $f_1(x_1)f_2(x_2)+\ldots+f_n(x_n)$. Where $f_i(x_i) = x_i$ or x_i' for each $i = 1, 2, \ldots, n$, and no two terms are same. Moreover 0 and 1 are also said to be in conjunctive normal form.

The conjunctive normal form in n variables which contains 2^{nd} factors (max-terms) is called complete conjunctive normal form in n variables.

6.12 Terminal Questions:

(1). Given P is true, Q is false and R is true, find, find the truth values of:

(a) $(P \vee Q) \wedge (Q \vee R)$.

(b) $(P \Rightarrow Q) \Rightarrow (P \wedge - Q)$

(c) $[(P \wedge Q) \wedge - R] \Rightarrow (Q \Rightarrow P)$ [Ans. (a) T, (b) T, (c) T]

(2). Write the Negations of the following

(a) $(P \vee Q) \wedge R$,

(b) $P \wedge (Q \Rightarrow -R)$,

(c) $P \Rightarrow (Q \Rightarrow R)$.

(d) $P \wedge - Q \Leftrightarrow R$,

(e) $\forall^x (x \neq 1, x \neq 2)$,

(f) $\exists^x (x^2 < 0)$

(g) $\forall x (x \neq 0) \Rightarrow (x^2 > 0)$,

(h) $\exists x (x^2 = 1 \text{ and } x^2 - 2x + 3 = 0)$

(i) Every Indian is honest. (j)If there is a will then there is a way.

(3). State if the following are sentence, giving reasons of your answer.

(a) Do you think you will pass in the examination?

(b) Mathematics is black .

(c) Walk right in.

(d) He is a President of India.

(e) 2/5 is a integer .

(f) If you pass in the examination, then the sun will revolve about the earth.

(g) Oh! How sand he is.

(4). By constructing Truth-tables shows that the following are tautologies:

(a) $(P \wedge Q) \Rightarrow P$

(b) $(P \vee Q) \wedge - Q \Rightarrow P$

(c) $[(P \Rightarrow Q) \wedge (Q \Rightarrow R)] \Rightarrow (P \Rightarrow R)$,

(d) $(- P \Rightarrow Q) \wedge - Q) \Rightarrow P$

(5). Prove the following tautological equivalences:

(a) $(P \Rightarrow Q) \vee (P \Rightarrow R) \equiv P \Rightarrow Q \vee R$.

(b) $(P \Rightarrow R) \wedge (Q \Rightarrow R) \equiv P \vee Q \Rightarrow R$

(c) $(P \Rightarrow Q) \wedge (P \Rightarrow R) \equiv P \Rightarrow Q \wedge R$,

(d) $(P \Rightarrow R) \vee (Q \Rightarrow R) \equiv \text{P} \wedge Q \Rightarrow R$.

(6). Prove that the following are tautologies

(a) $[(P \Rightarrow Q) \wedge (R \Rightarrow S)] \Rightarrow (P \wedge R \Rightarrow Q \wedge S)$.

(b) $[P \Rightarrow Q) \wedge (R \Rightarrow S) \Rightarrow (P \vee R \Rightarrow Q \vee S)$.

(7). Find the dual of the following:

(a) $(p \vee q) \wedge r$ (b) $(p \wedge q) \vee T$

(c) $\sim (p \vee q) \wedge (p \vee \sim (p \wedge s))$

(8). Form the negation of each of the following:

(b) "For all positive integers *x*, we have $x + 2 > 8$"

(c) "All men are honest or some man is a thief."

(d) "There is at least one person who is happy all the time."

(e) "The sum of any two integers is an even integer."

(f) At least one student does not live in the dormitories.

Solution: (a) Interchanging $\vee$ and $\wedge$, we have, dual as $(p \wedge q) \vee r$.

(b) Dual is $(p \vee q) \wedge F$

(8). Dual is Negation of the statement are:

(a) There exists a positive integer x such that $x+2>8$.

(b) There exists a man who is not honest and all men are not thief.

(c) No person is happy all the time.

(d) There exists two integers such that their sum is not an even integer.

(e) All students live in the dormitories.

UNIT-7 : Mathematical Induction

Structure

7.1 Introduction

This is most basic unit of this block as it introduces the concept of The principle of Mathematical induction is of great help in proving results involving a natural member for every n or for every $n \geq$ some positive integer m. If $P(n)$ is a statement involving a positive integer n.If $P(l)$ is true $\Rightarrow$truth of $P(l+1)$ $\forall l \geq$ m.

Then $P(n)$ is true for every $n \geq m$. The particular case of this result for $m = 1$ is usually referred to as the principle of mathematical induction and in fact the general version stated above can be obtained from version stated above can be obtained from this particular case. The above principle is popularly stated as if a statement holds for $n = 1$ and whenever it is true for $n = t$, it holds for

$n = t + 1$, then it holds for all natural numbers n. There was a reason for looking the further generalization, apart from mathematical interest. The reason was the many applications. Apart from the ones we mentioned at the beginning, the binomial theorem has several applications in probability theory, calculus and approximating numbers like$(\mathbf{1.02})^{\mathbf{15}}$**.** We shall discuss a few of them in this unit.

7.2 Objectives

After reading this unit the learner should be able to understand about:

- the Principle of Mathematical Induction
- Second Principle of Induction
- Well ordering property.

7.3 Principle of Mathematical Induction

The principle of Mathematical induction is of great help in proving results involving a natural member for every n or for every $n \geq$ some positive integer m.

If $P(n)$ is a statement involving a positive integer n for which

(1) (1) is true.

(2) $P(m)$ is true for some integer m.

(3) Truth of $P(m) \Rightarrow$ Truth of $P(m+1)\ \forall\ m+1 \geq$ m.

Then $P(n)$ is true for every $n \geq m$. The particular case of this result for $m = 1$ is usually referred to as the principle of mathematical induction and in fact the general version stated above can be obtained from version stated above can be obtained from this particular case. The above principle is popularly stated as if a statement holds for $n = 1$ and whenever it is true for $n = t$, it holds for $n = t + 1$, then it holds for all natural numbers n.

Example 1: $2^n > n^2$ for all $n \geq 5$. Clearly the statement does not hold for $n = 2, 3, 4$.

$2^5 = 32 > 25 = 5^2$ & so if holds for n = 5.

Take any $l \geq 5$ and assume that $2^l > l^2$.

Then $2^{l+1} = 2.2^l = 2^l + 2^l. > l^2 + l^2$ (by hypothesis)

$\geq l^2 + 5l \qquad (\because l \geq 5)$

$= l^2 + 2l + 3l \leq l^2 + 2l + 3\times5\ (\because l \geq 5)$

$> l^2 + 2l + 1\ (\because\ 15 > 1)$

$= (l + 1)^2$ i.e. $2^{l+1} > (l + 1)^2 \forall l \geq 5$.

∴ Assumption of truth of the statement for $l \geq 5$ implies its truth for $l + 1$. ∴ the above statement is true for every $l \geq 5$ by the above result.

Sometimes the statements $P(n)$ do not imply $P(n+1)$ and in this case the above principle cannot be used. For such situations we have the stronger.

7.4. Second Principle of Induction:

If $P(n)$ is a statement involving a natural number n and Truth of $P(l)$ $\forall l < m \Rightarrow$ Truth of $P(m)$, then the statement is true for all natural numbers n. We shall illustrate its uses later in this unit. The second principle of induction is a consequence of the well ordering property of the set $\mathbb{N}$ of natural number or of $\mathbb{N} \cup \{0\}$.

7.5. Well ordering property:

Every non empty subset A of $\mathbb{N}$ (or of $\mathbb{N} \cup \{0\}$ has a least element i.e. there is an element $l \in A$ for which $l \leq a$ for every $a \in A$.

We are omitting the proof but the reader may satisfy himself by considering various subsets of $\mathbb{N}$ and obtain least elements of them.

This result does not hold for $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$.

Example 2: Prove that the sum of first n odd numbers is n^2.

Proof: Let us write $P(n) = 1 + 3 + 5 + \cdots + (2n - 1) = n^2$

We wish to prove that $P(n)$ is true for all n.

We firstly to prove $P(1) = 1^2 = 1$ is true.

Now suppose it is true for $n = k$ that is it is true for $P(k) = 1 + 3 + 5 + \cdots . +(2k - 1) = k^2$

We need to prove that $P(k + 1)$ is true.

We consider $1 + 3 + 5 + \cdots + (2k - 1) + (2(k + 1) - 1) = k^2 + (2k + 2 - 1) = k^2 + (2k + 1) = (k + 1)^2$ is true.

Hence, it is true for $n = k + 1$ and so by inductive proof it is true for all n.

Example 3: Prove that $1^2 + 2^2 + \cdots .. + n^2 = \frac{n(n+1)(2n+1)}{6}$ $\forall$ n ≥ 1

Proof: Let us write $P(n) = 1^2 + 2^2 + \cdots .. + n^2 = \frac{n(n+1)(2n+1)}{6}$ $\forall$ n ≥ 1

We wish to prove that $P(n)$ is true for all n.

We firstly to prove $P(1) = 1^2 = \frac{1(2+1)(2+1)}{6} = 1$ is true.

Now suppose it is true for $n = k$ that is it is true for $1^2 + 2^2 + \cdots .. + k^2 = \frac{k(k+1)(2k+1)}{6}$

We need to prove that $P(k + 1)$ is true.

We consider $1^2 + 2^2 + \cdots .. + k^2 + (k + 1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$ is true. For this we take $\frac{k(k+1)(2k+1)}{6} + (k + 1)^2 = \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} = \frac{(k+1)[k(2k+1)+6(k+1)]}{6}$

$$= \frac{(k + 1)[(2k^2 + 7k + 6)]}{6} = \frac{(k + 1)(k + 1 + 1)(2(k + 1) + 1)}{6}$$

Hence, it is true for $n = k + 1$ and so by inductive proof it is true for all $n \geq 1$.

Example 4: For all $n \geq 1$, prove that $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$

Proof: Let us write $P(n) = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ ... + \frac{1}{n(n+1)} = \frac{n}{n+1} \quad \forall\ n \geq 1$

We wish to prove that $P(n)$ is true for all n.

We firstly to prove $P(1) = \frac{1}{1.2} = \frac{1}{2} = \frac{1}{2}$ is true.

Now suppose it is true for $n = k$ that is it is true for $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ ... + \frac{1}{k(k+1)} = \frac{k}{k+1}$

We need to prove that $P(k+1)$ is true.

We consider $P(k+1) = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ ... + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{(k+1)}{(k+1)+1}$

Now, $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ ... + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{1}{k+1}[k + \frac{1}{k+2}]$

$$= \frac{1}{k+1}\left[\frac{k^2 + 2k + 1}{k+2}\right] = \frac{1}{k+1}\left[\frac{(k+1)^2}{k+2}\right] = \frac{(k+1)}{(k+1)+1}$$

Hence, it is true for $n = k + 1$ and so by inductive proof it is true for all $n \geq 1$.

Example 5: For every positive integer n, prove that $7^n - 3^n$ is divisible by 4.

Proof: Let us write $P(n) = 7^n - 3^n$ is divisible by 4

We wish to prove that $P(n)$ is true for all n.

We firstly to prove $P(1) = 7^1 - 3^1 = 7 - 3 = 4$ is divisible by 4 is true.

Now suppose it is true for $n = k$ that is it is true for $7^k - 3^k$ is divisible by 4 i.e. $7^k - 3^k = 4d$

We need to prove that $P(k+1)$ is true.

Now $P(k+1) = 7^{k+1} - 3^{k+1}$ is divisible by 4

Consider, $P(k+1) = 7^{k+1} - 3^{k+1} = 7^{k+1} - 73^k + 73^k - 3^{k+1}$

$$= 77^k - 73^k + 73^k - 33^k = 7(7^k - 3^k) + 3^k(7 - 3)$$

$= 7.4d + 3^k 4 = 4(7d + 3^k)$ is divisible by 4.

We see that $P(k+1) = \ 7^{k+1} - 3^{k+1}$ is divisible by 4 is true. Hence it is true for every positive integer n.

Example 6: Prove that $(1+x)^n \geq (1+nx)$ for all natural number $n,$ where $x > -1$.

Proof: Let us write $P(n) = (1+x)^n \geq (1+nx)$ for all natural number $n,$ where $x > -1$.

We wish to prove that $P(n)$ is true for all n.

We firstly to prove $P(1) = (1+x)^1 \geq (1+1x) = (1+x)$ is true.

Now suppose it is true for $n = k$ that is it is true for $(1+x)^k \geq (1+kx)$.

We need to prove that $P(k+1)$ is true.

Now $P(k+1) = \ (1+x)^{k+1} \geq (1+(k+1)x)$

Consider,

$$(1+x)^{k+1} = (1+x)(1+x)^k \geq (1+x)(1+kx) = (1+x+kx+kx^2)$$

Since x is a natural number and so $x^2 \geq 0$, hence, $kx^2 \geq 0$

Therefore, $(1+x+kx+kx^2) \geq (1+x+kx)$

i.e. $(1+x)^{k+1} = (1+x+kx+kx^2) \geq (1+x+kx)$

or, $(1+x)^{k+1} \geq (1+x+kx)$

Hence, it is true for all n. i.e. $(1+x)^n \geq (1+nx)$ for all natural number $n,$ where $x > -1$.

Example 7: Prove that $2.7^n + 3.5^n - 5$ is divisible by 24 for all natural number n.

Proof: Let us write $P(n) = 2.7^n + 3.5^n - 5$ is divisible by 24 for all natural number n.

We wish to prove that $P(n)$ is true for all natural numbers n.

We firstly to prove $P(1) = 2.7^n + 3.5^n - 5 = 2.7 + 3.5 - 5 = 24$ is divisible by 24 is true.

Now suppose it is true for $n = k$ that is it is true for $2.7^k + 3.5^k - 5$ is divisible by 24 is true.

i.e. $2.7^k + 3.5^k - 5 = 24d$

We need to prove that $P(k+1)$ is true.

Now $P(k+1) = 2.7^{k+1} + 3.5^{k+1} - 5$ is divisible by 24

Consider, $P(k+1) = 2.7^{k+1} + 3.5^{k+1} - 5$

$$= 2.7^k.7 + 3.5^k.5 - 5 = 14.7^k + 15.5^k - 5$$
$$= 7[2.7^k + 3.5^k - 3.5^k - 5 + 5] + 3.5^k.5 - 5$$
$$= 7.24d - 21.5^k + 35 + 15.5^k - 5$$

$$= 7 \times 24d - 21 \times 5^k + 35 + 15 \times 5^k - 5$$

$$= 7 \times 24d - 6 \times 5^k + 30 = 7 \times 24d - 6(5^k - 5)$$

We know that $(5^k - 5)$ is divisible by 4 i.e. $(5^k - 5) = 4p$

$7 \times 24d - 6(5^k - 5) = 7 \times 24d - 6 \times 4p = 24(7d - p)$.

Hence, it is true that $P(k+1) = 2.7^{k+1} + 3.5^{k+1} - 5$ is divisible by 24.

Hence, it is true that $2.7^n + 3.5^n - 5$ is divisible by 24 for all natural number n.

Example 8: Prove that $1^2 + 2^2 + \cdots .. + n^2 > \frac{n^3}{3}$, for all natural number n.

Proof: Let us write $P(n) = 1^2 + 2^2 + \cdots .. + n^2 > \frac{n^3}{3}$, for all natural number n

We wish to prove that $P(n)$ is true for all n.

We firstly to prove $P(1) = 1^2 > \frac{1^3}{3} = \frac{1}{3}$ is true.

Now suppose it is true for $n = k$ that is it is true for $1^2 + 2^2 + \cdots .. + k^2 > \frac{k^3}{3}$

We need to prove that $P(k+1)$ is true.

We consider $1^2 + 2^2 + \cdots .. + k^2 + (k+1)^2 > \frac{(k+1)^3}{3}$

Now we have $1^2 + 2^2 + \cdots .. + k^2 + (k+1)^2 > \frac{k^3}{3} + (k+1)^2 = \frac{1}{3}[k^3 + 3(k+1)^2]$

$$> \frac{1}{3}[k^3 + 3(k^2 + 2k + 1)]$$

$$= \frac{1}{3}[k^3 + 3k^2 + 6k + 3]$$

$$= \frac{1}{3}[(k+1)^3 + 3k + 2]$$

$$> \frac{1}{3}(k+1)^3$$

Hence, it is true that $P(k+1) = 1^2 + 2^2 + \cdots .. + k^2 + (k+1)^2 > \frac{(k+1)^3}{3}$.

Hence, it is true that $1^2 + 2^2 + \cdots .. + n^2 > \frac{n^3}{3}$, for all natural number n.

Example 9: Prove that $(ab)^n = \mathrm{a^n b^n}$ for all natural number n.

Proof: Let us write $P(n) = (ab)^n = \mathrm{a^n b^n}$ for all natural number n

We wish to prove that $P(n)$ is true for all n.

We firstly to prove $P(1) = (ab)^1 = \mathrm{a}^1\mathrm{b}^1$ is true.

Now suppose it is true for $n = k$ that is it is true for $(ab)^k = \mathrm{a}^\mathrm{k}\mathrm{b}^\mathrm{k}$

We need to prove that $P(k+1)$ is true.

We consider $(ab)^{k+1} = \mathrm{a}^{\mathrm{k+1}}\mathrm{b}^{\mathrm{k+1}}$

Now$(ab)^{k+1} = (ab)^k(ab) = \mathrm{a}^\mathrm{k}\mathrm{b}^\mathrm{k}.\mathrm{ab} = (\mathrm{a}^\mathrm{k}.\mathrm{a}).(\mathrm{b}^\mathrm{k}.\mathrm{b}) = \mathrm{a}^{\mathrm{k+1}}\mathrm{b}^{\mathrm{k+1}}$

Hence, it is true for $P(k+1) = (ab)^{k+1} = \mathrm{a}^{\mathrm{k+1}}\mathrm{b}^{\mathrm{k+1}}$

Hence, it is true that $(ab)^n = \mathrm{a}^\mathrm{n}\mathrm{b}^\mathrm{n}$, for all natural number n.

Example 10: For all $n \geq 2$, prove that $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{5^2}\right) \ldots \left(1-\frac{1}{n^2}\right) = \frac{n+1}{(2n)}$.

Proof: Let us write

$P(n) = \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{5^2}\right) \ldots \left(1-\frac{1}{n^2}\right) = \frac{n+1}{(2n)}$ for all natural number $n \geq 2$

We wish to prove that $P(n)$ is true for all n.

We firstly to prove $P(2) = \left(1-\frac{1}{2^2}\right) = \frac{2+1}{(2.2)} = \frac{3}{4}$ is true.

Now suppose it is true for $n = k$ that is it is true for

$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{5^2}\right) \ldots \left(1-\frac{1}{k^2}\right) = \frac{k+1}{(2k)}$

We need to prove that $P(k+1)$ is true.

Consider $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{5^2}\right) \ldots \left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right) = \frac{k+2}{(2k+2)}$

Now we have

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{5^2}\right)\ \dots\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right)$$

$$=\frac{k+1}{(2k)}\left(1-\frac{1}{(k+1)^2}\right)$$

$$=\frac{k+1}{(2k)}\left(\frac{(k+1)^2\ -1}{(k+1)^2}\right)$$

$$=\ \frac{2k+k^2}{(2k)(k+1)}=\frac{[(k+1)+1]}{2(k+1)}$$

Hence, it is true for

$P(k+1)=\ \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{5^2}\right)\ \dots\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right)=\frac{k+2}{(2k+2)}$

Hence, it is true that $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{5^2}\right)\ \dots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{(2n)}$ for all natural number $n\geq 2$.

Example 11: prove that $2^2 3+2^3.3^2+2^4.3^3\ +\cdots.+2^{n+1}.3^n=\frac{12}{5}(6^n-1)$ all natural number n.

Proof: Let us write $P(n)=2^2 3+2^3.3^2+2^4.3^3\ +\cdots.+2^{n+1}.3^n=\frac{12}{5}(6^n-1)$

We wish to prove that $P(n)$ is true for all n.

We firstly to prove $P(1)=2^{1+1}.3^1=\frac{12}{5}(6^1-1)=12$ is true.

Now suppose it is true for $n=k$ that is it is true for $2^2 3+2^3.3^2+2^4.3^3\ +\cdots.+2^{k+1}.3^k=\frac{12}{5}(6^k-1)$

We need to prove that $P(k+1)$ is true.

Consider $2^2 3+2^3.3^2+2^4.3^3\ +\cdots.+2^{k+1}.3^k+2^{k+2}.3^{k+1}=\frac{12}{5}(6^{k+1}-1)$

Now we have

$$2^2 3 + 2^3 . 3^2 + 2^4 . 3^3 \ + \cdots . + 2^{k+1} . 3^k + 2^{k+2} . 3^{k+1} = \frac{12}{5}(6^k - 1) + 2^{k+2} . 3^{k+1}$$

$$= \frac{12}{5}(6^k - 1) + 2.2^{k+1} . 3^{k+1}$$

$$= \frac{2}{5} 6.6^k - \frac{12}{5} + 2.2^{k+1} . 3^{k+1}$$

$$= \ \frac{2}{5} 6^{k+1} - \frac{12}{5} + 2.6^{k+1}$$

$$= \left(\frac{2}{5} + 2\right) 6^{k+1} - \frac{12}{5} = \frac{12}{5} \ 6^{k+1} - \frac{12}{5}$$

$$= \frac{12}{5}(6^{k+1} - 1)$$

Hence, it is true for

$$P(k+1) = \ 2^2 3 + 2^3 . 3^2 + 2^4 . 3^3 \ + \cdots . + 2^{k+1} . 3^k + 2^{k+2} . 3^{k+1} = \frac{12}{5}(6^{k+1} - 1)$$

Hence, it is true that $2^2 3 + 2^3 . 3^2 + 2^4 . 3^3 \ + \cdots . + 2^{n+1} . 3^n = \frac{12}{5}(6^n - 1)$ all natural number n.

We firstly to prove $P(1) = 2^{1+1} . 3^1 = \frac{12}{5}(6^1 - 1) = 12$ is true.

Example 12: Prove that $5^{n+1} + 4.6^n$ when divisible by 20, leaves the remainder 9 for all natural number n.

Proof: Let us write $P(n) = \ 5^{n+1} + 4.6^n$ when divisible by 20, leaves the remainder 9 for all natural number n. i.e. $P(n) = 5^{n+1} + 4.6^n - 9$ is divisible by 20.

We wish to prove that $P(n)$ is true for all natural numbers n.

We firstly to prove $P(1) = 5^{1+1} + 4.6^1 - 9 \ = 25 + 24 - 9 = 40$ is divisible by 20 is true.

Now suppose it is true for $n = k$ that is it is true for $5^{k+1} + 4.6^k - 9$ is divisible by 20.

i.e. $5^{k+1} + 4.6^k - 9 = 20d$

We need to prove that $P(k+1)$ is true.

Now $P(k+1) = 5^{k+1+1} + 4.6^{k+1} - 9$ is divisible by 20

Consider, $P(k+1) = 5^{k+2} + 4.6^{k+1} - 9$

$$= .5^k.25 + 4.6^k.6 - 9$$

$$= 5.5^{k+1} + 4.6.6^k - 9$$

$$= 5.5^{k+1} + 6(20d - 5^{k+1} + 9) - 9$$

$$= 120d - 5^{k+1} + 45$$

$$= 120d - 5(5^k - 5) + 20$$

$$= 120d - 5.4p + 20$$

$$= 20(6d - p + 1)$$

We know that $(5^k - 5)$ is divisible by 4 i.e. $(5^k - 5) = 4p$

Hence, it is true that $P(k+1) = 5^{k+1+1} + 4.6^{k+1} - 9$ is divisible by 20.

Hence, it is true that $P(n) = 5^{n+1} + 4.6^n - 9$ is divisible by 20 for all natural number n.

Example 13: Prove that $n(n+1)$ is an even natural number for all natural number n.

Proof: Let us write $P(n) = n(n+1)$ is an even natural number for all natural number n.

We wish to prove that $P(n) = n(n+1)$ is an even natural number is true for all natural numbers n.

We firstly to prove $P(1) = 1(1+1) = 1.2 = 2$ is an even natural number is true.

Now suppose it is true for $n = k$ that is it is true for $k(k+1)$ is an even natural number

We need to prove that $P(k+1)$ is true.

Now $P(k+1) = (k+1)(k+1+1) = (k+1)(k+2) = k(k+1) + 2(k+1)$

$= 2d + 2(k+1) = 2(d+k+1)$, which is an even natural number.

Hence, it is true that $P(k+1) = (k+1)(k+2)$ is an even natural number.

Hence, it is true that $P(n) = n(n+1)$ is an even natural number for all natural numbers n.

Example 14: Prove that $2^n > n$ for all natural numbers.

Proof: Let us write $(n) = 2^n > n$.

We wish to prove that $2^n > n$ for all natural numbers is true .

We firstly to prove $P(1) = 2^1 > 1 = 2 > 1$ is true.

Now suppose it is true for $n = k$ that is it is true for $2^k > k$

We need to prove that $P(k+1)$ is true.

Now $P(k+1) = 2^{k+1} > k+1 = 22^k > 2(k) > (k+1)$

Hence, it is true that $P(k+1) = 2^{k+1} > k+1$.

Hence, it is true that $P(n) = 2^n > n$ for all natural numbers

Example 15: Prove that $(1+x)^n \geq 1 + nx$ for all natural numbers n.

Proof: Let us write $P(n) = (1+x)^n \geq 1 + nx$ for all natural numbers . .

We wish to prove that $(1+x)^n \geq 1+nx$ for all natural numbers n. is true .

We firstly to prove $P(1) = (1+x)^1 \geq 1+1x$ is true.

Now suppose it is true for $n=k$ that is it is true for $(1+x)^k \geq 1+kx$

We need to prove that $P(k+1)$ is true.

Now $P(k+1) = (1+x)^{k+1} \geq 1+(k+1)x$

$$(1+x)^{k+1} = (1+x)(1+x)^k > (1+x)(1+kx) = 1+x+kx+kx^2 \geq 1+(k+1)x$$

Hence, it is true that $P(k+1) = (1+x)^{k+1} \geq 1+(k+1)x$.

Hence, it is true that $P(n) = (1+x)^n \geq 1+nx$ for all natural numbers n.

Example 16: if x is not a multiple of 2π, then prove that

$$sinx + sin2x + sin3x + \cdots + sinnx = \frac{[\sin\frac{n+1}{2}x.\sin\frac{nx}{2}]}{(\sin\frac{x}{2})}$$

Proof: Let us write $P(n) = sinx + sin2x + sin3x + \cdots + sinnx = \frac{[\sin\frac{n+1}{2}x.\sin\frac{nx}{2}]}{(\sin\frac{x}{2})}$

We wish to prove that $sinx + sin2x + sin3x + \cdots + sinnx = \frac{[\sin\frac{n+1}{2}x.\sin\frac{nx}{2}]}{(\sin\frac{x}{2})}$

for all natural numbers n. is true .

We firstly to prove $P(1) = sinx = \frac{[\sin\frac{1+1}{2}x.\sin\frac{1x}{2}]}{(\sin\frac{x}{2})} = sinx$ is true.

Now suppose it is true for $n=k$ that is it is true for

$$sinx + sin2x + sin3x + \cdots + sinkx = \frac{[\sin\frac{k+1}{2}x.\sin\frac{kx}{2}]}{(\sin\frac{x}{2})}$$

We need to prove that $P(k+1)$ is true.

Now $P(k+1) = sinx + sin2x + sin3x + \cdots + sinkx + \sin(k+1)x = \frac{[\sin\frac{k+2}{2}x.\sin\frac{(k+1)x}{2}]}{(\sin\frac{x}{2})}$

$sinx + sin2x + sin3x + \cdots + sinkx + \sin(k+1)\,x = \frac{[\sin\frac{k+1}{2}x.\sin\frac{kx}{2}]}{(\sin\frac{x}{2})} + sin(k+1)x$

$$= \frac{[\sin\frac{k+1}{2}x.\sin\frac{kx}{2}]}{(\sin\frac{x}{2})} + 2\sin\frac{k+1}{2}x\cos\frac{k+1}{2}x$$

$$= \sin\frac{k+1}{2}x\left[\frac{\left[.\sin\frac{kx}{2}\right]}{(\sin\frac{x}{2})} + 2\cos\frac{k+1}{2}x\right]$$

$$= \sin\frac{k+1}{2}x\left[\frac{.\sin\frac{(k)x}{2} + 2\cos\frac{k+1}{2}x\,(\sin\frac{x}{2})}{(\sin\frac{x}{2})}\right]$$

$$= \sin\frac{k+1}{2}x\left[\frac{.\sin\frac{(k)x}{2} + \sin\frac{k+2}{2}x - (\sin\frac{kx}{2})}{(\sin\frac{x}{2})}\right]$$

$$= \frac{[\sin\frac{k+2}{2}x.\sin\frac{(k+1)x}{2}]}{(\sin\frac{x}{2})}$$

Hence, it is true that

$P(k+1) = sinx + sin2x + sin3x + \cdots + sinkx + \sin(k+1)x = \frac{[\sin\frac{k+2}{2}x.\sin\frac{(k+1)x}{2}]}{(\sin\frac{x}{2})}$.

Hence, it is true that $P(n) = sinx + sin2x + sin3x + \cdots + sinnx = \frac{[\sin\frac{n+1}{2}x.\sin\frac{nx}{2}]}{(\sin\frac{x}{2})}$ for all natural numbers n.

7.6 Summary:

The principle of Mathematical induction is of great help in proving results involving a natural member for every n or for every $n \geq$ some positive integer m.

If $P(n)$ is a statement involving a positive integer n for which

(1) (1) is true.

(2) $P(m)$ is true for some integer m.

(3) Truth of $P(m) \Rightarrow$ Truth of $P(m+1)$ $\forall$ $m+1 \geq$ m.

Then $P(n)$ is true for every $n \geq m$. The particular case of this result for $m = 1$ is usually referred to as the principle of mathematical induction and in fact the general version stated above can be obtained from version stated above can be obtained from this particular case.

If $P(n)$ is a statement involving a natural number n and Truth of $P(l)$ $\forall l <$ m $\Rightarrow$ Truth of $P(m)$, then the statement is true for all natural numbers n. We shall illustrate its uses later in this unit. The second principle of induction is a consequence of the well ordering property of the set $\mathbb{N}$ of natural number or of $\mathbb{N} \cup \{0\}$.

Every non empty subset A of $\mathbb{N}$ (or of $\mathbb{N} \cup \{0\}$ has a least element i.e. there is an element $l \in$ A for which $l \leq$ a for every a $\in$ A.

7.7 Terminal Questions

1. Prove that $|m + n| = |m| + |n|$ occurs if and only if m and n have same sign (positive or negative) or one of them at least in zero and that

$|m + n| < |m| + |n|$ if and only if they are of opposite sings.

2. Prove that $||a| - |b|| \leq |a - b|$ for any $a, b \in \mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$.

3. Prove that $3^n > 2^n + 1$ for all $n \geq 2$.

4. Prove that $1^3+2^3+\ldots\ldots n^3 = \dfrac{n^2(n+1)^2}{4}$ $\forall$ $n \geq 1$.

5. Prove that for any real number $x > -1$, $(1 + x)^n \geq (1 + nx)$ $\forall$ $n \geq 1$.

6. Prove that $n! > 2^n$ $\forall$ $n \geq 4$.

7. Prove that $n! > 4^n$ $\forall$ $n \geq 9$.

8. Let $a_1 = 1$ and $a_n = \sqrt{3a_{n-1} + 1}$ $\forall$ $n \geq 2$. Prove that $a_n < \dfrac{7}{2}$ $\forall$ integer $n \geq 1$.

9. Prove that $2^n > n^3$ for all $n \geq 10$.

10. Prove that $n(n + 1)(n + 5)$ is a multiple of 3 for all natural number n.

11. Prove that $10^{2n-1} + 1$ is divisible by 11 for all natural number n.

12. Prove that $3^{2n+2} - 8n - 9$ is divisible by 8 for all natural number n.

13. Prove that $41^n - 14^n$ is a multiple of 27 for all natural number n.

14. Prove that $x^{2n} - y^{2n}$ is divisible by $(x + y)$ for all natural number n.

15. Prove that $1 + 2 + 3 + \cdots + n < \frac{1}{8}(2n + 1)^2$ for all natural number n.

16. For all $n \geq 1$, prove that $\frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \ \ldots + \frac{1}{(2n+1)(2n+3)} = \frac{n}{3(2n+3)}$

17. For all $n \geq 1$, prove that $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \ \ldots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{(3n+1)}$

18. Prove that $1^2 + 3^2 + 5^2 + \cdots .. + (2n - 1)^2 = \frac{n(2n-1)(2n+1)}{3}$, for all natural number n.

19. For all $n \geq 1$, prove that $\frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \ \ldots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{(6n+4)}$

20. For all $n \geq 1$, prove that $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \ \ldots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$

21. For all $n \geq 1$, prove that $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ \ldots + \frac{1}{2^n} = (1 - \frac{1}{2^n})$

22. prove that $1 + \frac{1}{(1+2)} + \frac{1}{(1+2+3)} + \frac{1}{(1+2+3+4)} + \ \ldots + \frac{1}{(1+2+3+\cdots+n)} = \frac{2n}{n+1}$

23. prove that $1.2.3 + 2.3.4 + 3.4.5 + \cdots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$

24. prove that $1.3 + 3.5 + 5.7 + \cdots + (2n-1)(2n+1) = \frac{n(4n^2+6n-1)}{3}$

25. prove that $1.3 + 2.3^2 + 3.3^3 \ + \cdots . + n.3^n = \frac{[(2n-1)3^{n+1}+3]}{4}$

26. Prove that $(n+3)^2 > (2n+7)$ for all $n \geq 10$

27. if x is not a multiple of 2π, then prove that

$$cosx + cos2x + cos3x + \cdots + cosnx = \frac{[\cos\frac{n+1}{2}x.\sin\frac{nx}{2}]}{(\sin\frac{x}{2})}$$

28. Prove that $7 + 77 + 777 + \cdots +$to n terms $= \frac{7}{81}(10^{n+1} - 9n - 10)$ for all natural number n

29. Prove that $1 + 4 + 7 + \ \ldots + \ (3n-2) = \frac{1}{2}n(3n-1)$for all natural number n.

30. Prove that $\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$ is a positive integer for all natural number n.

UNIT-8: Recurrence Relations

Structure

8.1 Introduction

A recurrence relation of the sequence $\{a_n\}$ is an equation that expresses an in terms of one or more of the previous terms of the sequence namely a_0, a_1, a_2, ….a_{n-1} for all integers n with $n \geq n_0$ is non negative integers.

A recurrence relation are also called difference equations solution of recurrence relation = A sequence is called a solution of recurrence relation of its term satisfy of the recurrence relation.

This is most basic unit of this block as Recursively defined functions should be well defined. It means for every positive integer, the value of the function at this integer is determined in an unambiguous way.

Assume a is a nonzero real number and n is a non-negative integer. Give a recursive definition of a_n, n some recursive functions. The values of the function at the first k positive integers are specified A rule is given to determine the value of the function at larger integer from its values at some of the preceding k integers.

8.2 Objectives

After reading this unit the learner should be able to understand about:

- Generating functions
- the properties of generating functions
- the Recursive functions
- the solution of the recurrence relation

8.3. Generating functions:

If { a_1, a_2, a_3, ….. a_r,……} is a sequence of real or complex numbers, then the power series given by $A(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_r z^r + \cdots$

$A(z) = \sum_{r=0}^{\infty} a_r z^r$ is called generating function for the given sequence.

Example 1: Find generating function of the numeric function a_r = {1, 3, 9, 27, ……..}.

Solution: Here $a_0 = 1$, $a_1 = 3$, $a_2 = 9$, $a_3 = 27$

Suppose for generating function is

$$A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_r z^r + \cdots$$

$$A(z) = 1 + 3z + 9z^2 + 27z^3 + \cdots$$

$$A(z) = 1 + 3z + 3^2 z^2 + 3^3 z^3 + \cdots$$

$$A(z) = 1 + (3z) + (3z)^2 + (3z)^3 + \cdots$$

$$(1 - x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$

$$\Rightarrow A(z) = (1 - 3z)^{-1} \quad , x = 3z.$$

8.4 Properties of generating function (operations on generating function)

Suppose A(z) and C(z) represented the generating function of the numeric functions a_r, b_r and C_r respectively then

1. Sum of A(z) and B(z)

If $C_r = a_r + b_r$ then $C(z) = A(z) + B(z)$

$$C_r = \sum_{r=0}^{\infty} (a_r + b_r)\, z^r$$

2. Scalar multiplication of k in A(z)

$$= \sum_{r=0}^{\infty} a_r\, z^r$$

3. Product of A(z) and B(z)

If $C_r = a_r . b_r$ then $C(z) = A(z). B(z) = C_r z^r$

where $C_r = \sum_{k=0}^{r} a_r b_{r-k}$

4. Multiplication of α^r with a_r

If $b_r = \alpha^r a_r$ then $B(z) = A(\alpha z) = \sum_{r=0}^{\infty} a_r (\alpha z)^r$

	Numeric function	Generating function A(z)
1.	$a_r = 1$	$A(z) = 1 + z + z^2 + \cdots + z^r + \cdots = (1-z)^{-1} = \frac{1}{(1-z)}$
2.	$a_r = k$	$A(z) = k + kz + kz^2 + \cdots + kz^r + \cdots = k(1-z)^{-1} = \frac{k}{(1-z)}$
3.	$a_r = \alpha^r$	$A(z) = 1 + \alpha z + \alpha^2 z^2 + \cdots + \alpha^r z^r + \cdots = (1-\alpha z)^{-1} = \frac{1}{(1-\alpha z)}$
4.	$a_r = r$	$A(z) = z + 2z^2 + 3z^3 \ldots + rz^r + \cdots = z(1-z)^{-2} = \frac{z}{(1-z)^2}$

5.	$a_r = r^2$	$A(z) = z + 2^2z^2 + 3^2z^3 \ldots + r^2z^r + \cdots = z(z+1)(1-z)^{-3}$ $= \frac{z(z+1)}{(1-z)^3}$
6.	$a_r = r(r+1)$	$A(z) = 1.2z + 2.3z^2 + 3.4z^3 \ldots + r(r+1)z^r + \cdots = \frac{2z}{(1-z)^3}$
7.	$a_r = {}^n_rC$	$A(z) = (1+z)^n$

Example 2: Find the generating functions of the following numeric function $a_r = 7.3^r$

Solution: The required generating function of given series $a_r = 7.3^r$

$$A(z) = \sum_{r=0}^{\infty} a_r z^r = \sum_{r=0}^{\infty} 7.3^r z^r$$

$$A(z) = 7\{1 + (3z) + (3z)^2 + (3z)^3 + \cdots\}$$

by infinite sum of G.P. = $\frac{a}{1-r}$, a = 1, r = 3z

$$\therefore A(z) = 7.\frac{1}{1-3z} = \frac{7}{1-3z}$$

Example 3: Find the generating functions of the following numeric function $a_r = 2^r$

Solution: The required generating function of given series $a_r = 2^r$

$$A(z) = \sum_{r=0}^{\infty} a_r z^r = \sum_{r=0}^{\infty} 2^r z^r$$

$$A(z) = \{1 + (2z) + (2z)^2 + (2z)^3 + \cdots\}$$

by infinite sum of G.P. = $\frac{a}{1-r}$,   a = 1, r = 2z

$\therefore A(z) = \frac{1}{1-2z}$

Example 4: Find the generating functions of the following numeric function $a_r = (r+1)3^r$

Solution: The required generating function of given series $a_r = (r+1)3^r$

$$A(z) = \sum_{r=0}^{\infty} a_r z^r = \sum_{r=0}^{\infty} (r+1)3^r z^r = \sum_{r=0}^{\infty} r3^r z^r = \sum_{r=0}^{\infty} 3^r z^r$$

$$A(z) = \{0 + 1.(3z) + 2.(3z)^2 + 3.(3z)^3 + \cdots\} + \{1 + (3z) + (3z)^2 + (3z)^3 + \cdots\}$$

$$= 3z\{1 + 2.(3z) + 3.(3z)^2 + \cdots\} + \{1 + (3z) + (3z)^2 + (3z)^3 + \cdots\}$$

$$= 3z\{(1-3z)^{-2}\} + (1-3z)^{-1}$$

$$= \frac{3z}{(1-3z)^2} + \frac{1}{1-3z} = \frac{1}{(1-3z)^2}$$

Example.5: Find the generating functions of the following numeric functions

$$a_r = 5^r + (-1)^r 3^r + 8^r + {}^3_rC$$

Solution: The required generating function of given series ar

$$A(z) = \sum_{r=0}^{\infty} a_r z^r = \sum_{r=0}^{\infty} \{5^r + (-1)^r 3^r + 8^r + {}^3_rC\} z^r$$

$$A(z) = \{1 + (5z) + (5z)^2 + (5z)^3 + \cdots\} + \{1 - (3z) + (3z)^2 - (3z)^3 + \cdots\}$$

$$+\{1 + (8z) + (8z)^2 + (8z)^3 + \cdots\} + \{{}^3_0C + {}^3_1Cz + {}^3_2Cz^2 + \cdots\}$$

$$A(z) = (1-5z)^{-1} + (1+3z)^{-1} + (1-8z)^{-1} + (1+z)^3$$

$$A(z) = \frac{1}{1-5z} + \frac{1}{1+3z} + \frac{1}{1-8z} + (1+z)^3$$

$$A(z) = \frac{120z^6 + 361z^5 + 367z^4 + 94z^3 - 25z^2 - 27z + 4}{(1-5z)(1+3z)(1-8z)}$$

Example 6: Find the generating function for the finite sequence 3, 3, 3, 3, 3, 3, 3

Solution: Here $a_0 = 3$, $a_1 = 3$, $a_2 = 3$, $a_4 = 3$, $a_5 = 3$, $a_6 = 3$

The required function is $A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + a_6 z^6$

$$A(z) = 3 + 3z + 3z^2 + 3z^3 + 3z^4 + 3z^5 + 3z^6$$

$$A(z) = 3\{1 + z + z^2 + z^3 + z^4 + z^5 + z^6\}$$

By sum of nth term of G.P. = $\frac{a(1-r^n)}{1-r}$

$\therefore A(z) = \frac{3(1-z^7)}{1-z}$

Example 7: Find the generating function for the sequence $a_r = C(m, r)$ or ${}^m_rC, \ r \geq 0$

Solution: Here $a_0 = C(m, 0)$, $a_1 = C(m, 1)$, $a_2 = C(m, 2)$ …….

Hence the required generating function is

$$A(z) = \sum_{r=0}^{\infty} a_r z^r = a_0 + a_1 z + a_2 z^2 + \cdots$$

$$= \{{}^m_0C + {}^m_1Cz + {}^m_2Cz^2 + \cdots\}$$

$$A(z) = (1+z)^m$$

Example 8: Determine the generating function of a numeric function a_r where

$$\boldsymbol{a_r} = \begin{cases} \boldsymbol{2^r} & \boldsymbol{if\ r\ is\ even} \\ \boldsymbol{-2^r} & \boldsymbol{if\ r\ is\ odd} \end{cases}$$

Solution: The required generating function is $A(z) = \sum_{r=0}^{\infty} a_r z^r$

$$A(z) = \sum_{r=0}^{\infty} (-1)^r 2^r z^r$$

$$=2^0 z^0 - 2z + 2^2 z^2 - 2^3 z^3 + \cdots$$

$$= 1 - (2z) + (2z)^2 - (2z)^3 + \cdots$$

$$= (1 + 2z)^{-1}$$

Hence $A(z) = \frac{1}{1+2z}$

Example 9: Obtain the generating function of the numeric function $\boldsymbol{a_r = 2^r + 3^r,\ r \geq 0}$

Solution: The required generating function $A(z) = \sum_{r=0}^{\infty} a_r z^r$

$$A(z) = \sum_{r=0}^{\infty} (2^r + 3^r) z^r = \sum_{r=0}^{\infty} 2^r z^r + \sum_{r=0}^{\infty} 3^r z^r$$

$$= [1 + (2z) + (2z)^2 + \cdots] + [1 - (3z) + (3z)^2 + \cdots]$$

$$= (1 - 2z)^{-1} + (1 - 3z)^{-1}$$

$$= \frac{1}{1-2z} + \frac{1}{1-3z}$$

$$A(z) = \frac{2 - 5z}{6z^2 - 5z + 1}$$

Example 10: Determine the numeric function corresponding of the following generating functions.

Solution: $A(z) = \frac{2}{1-4z^2}$

$$\Rightarrow A(z) = \frac{2}{(1-2z)(1+2z)}$$

$$\Rightarrow A(z) = \frac{A}{1-2z} + \frac{B}{1+2z}$$

$$\Rightarrow A(z) = \frac{1}{1-2z} + \frac{1}{1+2z}$$

The required numeric function

$$a_r = 2^r + (-1)^r, \ \ r \geq 0$$

Example 11: Determine the discrete numeric function corresponding the generating function $\boldsymbol{A(z) = \frac{7z^2}{(1-2z)(1+3z)}}$

Solution: Given the generating function is

$$A(z) = \frac{7z^2}{(1-2z)(1+3z)} = 7\left[\frac{z^2}{1+z-6z^2}\right]$$

$$A(z) = 7\left[-\frac{1}{6} + \frac{\frac{z}{6}+\frac{1}{6}}{1+z-6z^2}\right]$$

$$= \frac{-7}{6} + \frac{7}{6}\left[\frac{z+1}{(1-2z)(1+3z)}\right]$$

$$= \frac{-7}{6} + \frac{7}{6}\left[\frac{\frac{1}{2}+1}{(1-2z)\left(1+\frac{3}{2}\right)} + \frac{\frac{-1}{3}+1}{\left(1-\frac{2}{3}\right)(1+3z)}\right]$$

$$= \frac{-7}{6} + \frac{7}{6}\left[\frac{3}{5(1-2z)} + \frac{2}{5(1+3z)}\right]$$

$$A(z) = \frac{7}{6}\left[\frac{3}{5}(1-2z)^{-1} + \frac{2}{5}(1+3z)^{-1} - 1\right]$$

$(1-2z)^{-1} = 2^r$, $(1+3z)^{-1} = (-3)^r$

$$a_r = \begin{cases} \frac{7}{6}\left(\frac{3}{5}+\frac{2}{5}-1\right) & ; r = 0 \\ \frac{7}{6}\left(\frac{3}{5}.2^r+\frac{2}{5}.(-3)^r\right) & ; r \geq 1 \end{cases}$$

$$\Rightarrow a_r = \begin{cases} 0 & ; r = 0 \\ \frac{7}{30}(3.2^r+2.(-3)^r) & ; r \geq 1 \end{cases}$$

Example 12: Determine the discrete numeric function corresponding the generating function $\boldsymbol{A(z) = \frac{z^5}{5-6z+z^2}}$

Solution: Given the generating function is

$$A(z) = \frac{z^5}{5-6z+z^2} = \frac{z^5}{(1-z)(5-z)} = z^5\left[\frac{1}{(1-z)(5-z)}\right]$$

$$A(z) = z^5\left[\frac{1}{(1-z)(5-1)}+\frac{1}{(1-5)(5-1)}\right] = z^5\left[\frac{1}{4}\left(\frac{1}{1-z}\right)-\frac{1}{4}\left(\frac{1}{5-z}\right)\right]$$

$$\Rightarrow A(z) = z^5\left[\frac{1}{4}(1-z)^{-1}-\frac{1}{20}\left(1-\frac{z}{5}\right)^{-1}\right]$$

$$\Rightarrow A(z) = z^5\left[\frac{1}{4}(1+z+z^2+z^3+\cdots)-\frac{1}{20}\left(1+\frac{z}{5}+\frac{z^2}{5^2}+\frac{z^3}{5^3}+\cdots\right)\right]$$

$$\Rightarrow A(z) = \frac{1}{4}(z^5+z^6+\cdots+z^r+\cdots)-\frac{1}{20}\left(z^5+\frac{z^6}{5}+\cdots+\frac{z^r}{5^{r-5}}+\cdots\right)$$

The numeric function is

$$a_r = \begin{cases} 0 & ; \quad 0 \leq r \leq 4 \\ \frac{1}{4}-\frac{1}{20}\left(\frac{1}{5^{r-5}}\right) & ; \quad r \geq 5 \end{cases}$$

$$A(z) = \frac{1}{4}(z^5+z^6+\cdots+z^r+\cdots)-\frac{1}{20}\left(z^5+\frac{z^6}{5}+\cdots+\frac{z^r}{5^{r-5}}+\cdots\right)$$

$$a_r = \begin{cases} 0 & ; \quad 0 \le r \le 4 \\ \frac{1}{4}\left(1 - \frac{1}{5^{r-5}}\right) & ; \quad r \ge 5 \end{cases}$$

Example 13: Determine the discrete numeric function corresponding the generating function $A(z) = \frac{1}{5-6z+z^2}$

Solution: The generating function is $A(z) = \frac{1}{5-6z+z^2}$

$$A(z) = \frac{1}{(5-z)(1-z)} = \frac{A}{5-z} + \frac{B}{1-z}$$

$$A(z) = \frac{1}{(5-z)(1-5)} + \frac{1}{(5-1)(1-z)}$$

$$A(z) = \frac{-1}{4(5-z)} + \frac{1}{4(1-z)}$$

$$A(z) = \frac{1}{4}(1-z)^{-1} - \frac{1}{20}\left(1 - \frac{z}{5}\right)^{-1} \therefore \left(\frac{1}{1-\alpha z}\right) = \alpha^r$$

The numeric function is

$$a_r = \frac{1}{4}(1)^r - \frac{1}{20}\left(\frac{1}{5}\right)^r; \quad r \ge 0$$

8.5 Recursively defined functions:

Assume *f* is a function with the set of nonnegative integers as its domain We use two steps to define *f*.

Basis step- Specify the value of f(0).

Recursive step- Give a rule for f(x) using f(y) where y<0<x.

Example 14: Suppose f(0) = 3 f(n+1) = 2f(n)+2, n = 0. Find f(1), f(2) and f(3).

Solution: f(1) = 2f(0) + 2

= 2(3) + 2 = 8

f(2) = 2f(1) + 2 = 2(8) + 2 = 18

f(3) = 2f(2) + 2 = 2(18) + 2 = 38

Example 15: Give an inductive definition of the factorial function F(n) = n!.

Solution: Basis step: (Find F(0).) F(0)=1

Recursive step: (Find a recursive formula for F(n+1).)

F(n+1) = (n+1) F(n)

Example 16: What is the value of F(5)?

Solution-F(5) = 5F(4)

= 5 . 4F(3)

= 5 . 4 . 3F(2)

= 5 . 4 . 3 . 2F(1)

= 5 . 4 . 3 . 2 . 1F(0)

= 5 . 4 . 3 . 2 . 1 . 1 = 120.

8.6 Recursive functions:

Recursively defined functions should be well defined. It means for every positive integer, the value of the function at this integer is determined in an unambiguous way.

Example.17: Assume a is a nonzero real number and n is a nonnegative integer. Give a recursive definition of a_n.

Solution: Basic step: (Find F(0).) $F(0) = a_0 = 1$

Recursive step: (Find a recursive formula for F(n+1).)

$$F(n+1) = a \,.\, a_n = a \,.\, F(n)$$

Recursive functions:

In some recursive functions, the values of the function at the first k positive integers are specified. A rule is given to determine the value of the function at larger integer from its values at some of the preceding k integers.

Example 18: f(0) = 2 and f(1) = 3

$$f(n+2) = 2f(n) + f(n+1) + 5, n = 0.$$

Fibonacci numbers:

The Fibonacci numbers, f_0, f_1, f_2, …, are defined by the equations

$$f_0 = 0, \; f_1 = 1$$

$$f_n = f_{n-1} + f_{n-2} \text{ for } n = 2,3,4,\ldots .$$

Example 19: Find the Fibonacci number f_4.

Solution: $f_4 = f_3 + f_2$

$$f_2 = f_0 + f_1 = 0 + 1 = 1$$

$f_3 = f_1 + f_2 = 1 + 1 = 2$

$f_4 = f_3 + f_2 = 1 + 2 = 3$

The Fibonacci Series:

The Fibonacci series f_n=0 is a famous series defined by:

$f_0 = 0$, $f_1 = 1$,

$$\boldsymbol{f_n = f_{n-1} + f_{n-2}\,,\ n \geq 2.}$$

8.7 Discrete Numeric function:

If R is a set of real numbers. A function whose domain is the set {0, 1, 2, 3, …..} of non-negative integers and whose range is a subset of R, is called a discrete numeric function or numeric function and it is denoted by a_r or a.

$$a = \{a_0, a_1, a_2, a_3, \ldots\ldots a_r \ldots.\}$$

In this we discussed about:

1. Concept of Discrete Numeric function.

2. Manipulation of numeric function (or operation on numeric function)

(i) Sum of numeric function

(ii) Multiplication of numeric function by a real number.

(iii) Product of numeric function

(iv) Value of numeric function

(v) Forward and backward difference of numeric functions.

(vi) Convolution of numeric function

For example: The rth form of discrete numeric function {1, 9, 28, ……}

Or, {1, 2^2 + 1, 3^3 + 1, …….} is r^3 + 1

For example: The rth form of discrete numeric function {0, 3, 6, 7, 15, 31, ……}

$$a_r = \begin{cases} 3r & ; \quad 0 \le r \le 2 \\ 2^r - 1 & ; \quad r \ge 3 \end{cases}$$

For example: If the implicit form of discrete numeric function is

$$a_r = \begin{cases} 3r & ; \quad 0 \le r \le 5 \\ r - 1 & ; \quad r \ge 6 \end{cases}$$

The function can be written as {0, 3, 6, 9, 12, 15, 7, 8, 9, …..}.

8.8. Manipulation of Numeric Function (Operations on Numeric functions):

Suppose the numeric functions are

$$a_r = \begin{cases} 0 & ; \quad 0 \le r \le 2 \\ 2^r + 5 & ; \quad r \ge 3 \end{cases}$$ and

$$b_r = \begin{cases} 3 - 2^r & ; \quad 0 \le r \le 1 \\ r + 2 & ; \quad r \ge 2 \end{cases}$$

a_r = {0, 0, 0, 2^{-3} + 5, 2^{-4} + 5, 2^{-5} + 5, ………..}

and b_r = {3 – 2^0, 3 – 2^1, 2 + 2, 3 + 2, 4 + 2, 5 + 2, ……}

1. Sum of numeric functions:- The sum of two numeric functions is a function whose value at r is equal to the sum of the values of the two numeric function at r.

For example:

	r	0	1	2	3	4	5
	a_r	0	0	0	$2^{-3} + 5$	$2^{-4} + 5$	$2^{-5} + 5$
+	b_r	$3 – 2^0$	$3 – 2^1$	2 + 2	3 + 2	4 + 2	5 + 2
C_r	$a_r + b_r$	$3 – 2^0$	$3 – 2^1$	2 + 2	$2^{-3} + 3 + 7$	$2^{-4} + 4 + 7$	$2^{-5} + 17$

The required sum

$$C_r = a_r + b_r = \begin{cases} 3 - 2^r & ; \quad 0 \le r \le 1 \\ 4 & ; \quad r = 2 \\ 2^{-r} + r + 7 & ; \quad r \ge 3 \end{cases}$$

2. Multiplication of a numeric function by a real number:- Suppose a_r is a numeric function and α is any real number, then multiplication of ar by α is a numeric function and it is denoted by $\alpha.a_r$.

For example: Suppose the numeric function is

$$a_r = \begin{cases} 0 & ; \quad 0 \le r \le 2 \\ 2^{-r} + 5 & ; \quad r \ge 3 \end{cases}$$

the value of $5a_r$ is $5a_r = \begin{cases} 0 & ; \quad 0 \le r \le 2 \\ 5.2^{-r} + 25 & ; \quad r \ge 3 \end{cases}$

3. Product of numeric functions:-

The product of two numeric functions is a numeric function whose value at r is equal to the product of the values of the two numeric function at r. i.e. $C_r = a_r . b_r$

For example:- Suppose the numeric functions are

$$a_r = \begin{cases} 0 & ; \quad 0 \le r \le 2 \\ 2^{-r} + 5 & ; \quad r \ge 3 \end{cases} \text{ and}$$

$$b_r = \begin{cases} 3 - 2^r & ; \quad 0 \le r \le 1 \\ r + 2 & ; \quad r \ge 2 \end{cases}$$

The product of two numeric functions of a_r and b_r is defined as C_r is $C_r = a_r . b_r$

	r	0	1	2	3	4	5
	a_r	0	0	0	$2^{-3} + 5$	$2^{-4} + 5$	$2^{-5} + 5$
+	b_r	$3 – 2^0$	$3 – 2^1$	2 + 2	3 + 2	4 + 2	5 + 2
C_r	$a_r. b_r$	0	0	0	$(2^{-3} +5)(3+2)$	$(2^{-4} +5)(4+2)$	$(2^{-5} +5)(5+2)$

The required product is

$$C_r = a_r . b_r = \begin{cases} 0 & ; \quad 0 \le r \le 2 \\ (2^{-r} + 5)(r + 2) & ; \quad r \ge 3 \end{cases}$$

4. **Value of a numeric functions**:- If a_r is a numeric function then $|a_r|$ denotes a value of numeric function whose value at r is a_r is non-negative.

For example: $a_r = (-1)^r \left(\frac{7}{r^3}\right), \ r \ge 0$

Then, if $|a_r|$ is denoted by b_r, we have $b_r = \frac{7}{r^3}, \quad r \ge 0$

5. Forward and backward difference of numeric functions:- If a_r is a numeric function, then forward difference of a_r is denoted by Δa_r and defined as

$$\Delta a_r = a_{r+1} - a_r$$

The backward difference of a_r is denoted by ∇a_r and defined as

$$\nabla a_r = a_r - a_{r-1}; \qquad r \geq 1$$

such that $\nabla a_0 = a_0$

8.9 Convolution of numeric functions:

If a_r and b_r are two numeric functions, then the convolution of a_r and b_r is denoted by a_r*b_r and is a numeric function C_r defined as

$$C_r = a_r * b_r = \sum_{k=0}^{r} a_k . b_{r-k}$$

such that $C_0 = a_0 b_0$

Example 20:- If a_r and b_r are two numeric functions find a_r*b_r where

$$\boldsymbol{a_r} = \begin{cases} \mathbf{1} & ; \quad \mathbf{0 \leq r \leq 2} \\ \mathbf{0} & ; \quad \mathbf{r \geq 3} \end{cases} \quad \textbf{and} \quad \boldsymbol{b_r} = \begin{cases} \mathbf{1} & ; \quad \mathbf{0 \leq r \leq 2} \\ \mathbf{0} & ; \quad \mathbf{r \geq 3} \end{cases}$$

Solution: Suppose $C_r = a_r * b_r$ and defined as

$$C_r = a_r * b_r = \sum_{k=0}^{r} a_k . b_{r-k}$$

Putting r = 0, 1, 2, 3, 4, …..

$$C_0 = a_0 b_0 = 1.1 = 1 ,$$

$$C_1 = \sum_{k=0}^{1} a_k . b_{1-k} = a_0 b_1 + a_1 b_0 = 1.1 + 1.1 = 2$$

and $C_2 = \sum_{k=0}^{2} a_k . b_{2-k} = a_0 b_2 + a_1 b_1 + a_2 b_0 = 1.1 + 1.1 + 1.1 = 3$

$$C_3 = \sum_{k=0}^{3} a_k . b_{3-k} = a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0 = 1.0 + 1.1 + 1.1 + 0.1 = 2$$

$$C_4 = \sum_{k=0}^{4} a_k . b_{4-k} = a_0 b_4 + a_1 b_3 + a_2 b_2 + a_3 b_1 + a_4 b_0$$

$$= 1.0 + 1.0 + 1.1 + 0.1 + 0.0$$

$$= 1$$

$$C_5 = \sum_{k=0}^{5} a_k . b_{5-k} = a_0 b_5 + a_1 b_4 + a_2 b_3 + a_3 b_2 + a_4 b_1 + a_5 b_0$$

$$= 1.0 + 1.0 + 1.0 + 0.1 + 0.1 + 0.1 = 0$$

Example 21: If a_r is a numeric function as

$$\boldsymbol{a_r} = \begin{cases} \boldsymbol{0} & ; \quad \boldsymbol{0} < r < 2 \\ \boldsymbol{2}^{-r} + \boldsymbol{7} & ; \quad \boldsymbol{r} > 3 \end{cases}$$

find $\Delta \boldsymbol{a_r}$ and $\boldsymbol{\nabla a_3}$

Solution: The values of numeric functions are

$a_0 = 0,\ a_1 = 0,\ a_2 = 0,\ a_3 = 2^{-3} + 7 = \frac{1}{8} + 7 = \frac{57}{8},\quad a_4 = 2^{-4} + 7 = \frac{1}{16} + 7 = \frac{113}{16},$

$$a_5 = 2^{-5} + 7 = \frac{1}{32} + 7 = \frac{225}{32}$$

(i) Suppose $b_r = \Delta a_r = a_{r+1} - a_r$

putting r = 0, 1, 2, 3, 4, ………..

$$b_0 = a_1 - a_0 = 0 - 0 = 0$$

$$b_1 = a_2 - a_1 = 0 - 0 = 0$$

$$b_2 = a_3 - a_2 = \frac{57}{8} - 0 = \frac{57}{8}$$

$$b_3 = a_4 - a_3 = \frac{113}{16} - \frac{57}{8} = \frac{-1}{16} = \frac{1}{2^{3+1}}$$

$$b_4 = a_5 - a_4 = \frac{225}{32} - \frac{113}{16} = \frac{-1}{32} = \frac{1}{2^{4+1}}$$

$b_r = -\frac{1}{2^{r+1}}$

(ii) Suppose $C_r = \nabla a_r = a_r - a_{r-1}$ for $r \geq 1$

putting r = 1, 2, 3, 4, ……

$$C_0 = a_0 = 0 - 0 = 0$$

$$C_1 = a_1 - a_0 = 0 - 0 = 0$$

$$C_2 = a_2 - a_1 = 0 - 0 = 0$$

$$C_3 = a_3 - a_2 = \frac{57}{8} - 0 = \frac{57}{8}$$

$$C_4 = a_4 - a_3 = \frac{113}{16} - \frac{57}{8} = \frac{-1}{16} = \frac{1}{2^4}$$

$$C_5 = a_5 - a_4 = \frac{225}{32} - \frac{113}{16} = \frac{-1}{32} = \frac{1}{2^5}$$

Therefore, $C_r = \nabla a_r = -\frac{1}{2^r}$

Hence, $C_r = \begin{cases} 1 & ; \quad r = 0,4 \\ 2 & ; \quad r = 1,3 \\ 3 & ; \quad r = 2 \\ 0 & ; \quad r \geq 5 \end{cases}$

Example 22: If $a_r . b_r$ and C_r are three numeric functions such that $C_r = a_r * b_r$ given that

$$a_r = \begin{cases} 1 & ; \quad r = 0 \\ 2 & ; \quad r = 1 \\ 3 & ; \quad r \geq 2 \end{cases} \quad \text{and} \quad C_r = \begin{cases} 1 & ; \quad r = 0 \\ 0 & ; \quad r \geq 1 \end{cases}$$

Solution: Given that the numeric functions are

$$a_r = \begin{cases} 1 & ; \quad r = 0 \\ 2 & ; \quad r = 1 \\ 3 & ; \quad r \geq 2 \end{cases} \quad \text{and}$$

$$C_r = \begin{cases} 1 & ; \quad r = 0 \\ 0 & ; \quad r \geq 1 \end{cases}$$

Therefore, $a_0 = 1$, $a_1 = 2$, $a_2 = 0$, $a_3 = 0$, $a_4 = 0$,, ….. and

$C_0 = 1$, $C_1 = 0$, $C_2 = 0$, $C_3 = 0$, $C_4 = 0$, ……..

Given that $C_r = a_r * b_r$ and since

$C_r = a_r * b_r = \sum_{k=0}^{r} a_k . b_{r-k}$

Putting r = 0, 1, 2, 3, 4, …..

$C_0 = a_0 . b_0 \Rightarrow 1 = 1 \times b_0 \Rightarrow b_0 = 1$

and $C_1 = \sum_{k=0}^{1} a_k . b_{1-k} = a_0 b_1 + a_1 b_0 \Rightarrow 0 = 1. b_1 + 2.1 \Rightarrow b_1 = -2$

and $C_2 = \sum_{k=0}^{2} a_k . b_{2-k} = a_0 b_2 + a_1 b_1 + a_2 b_0 = 1. b_2 + 2.(-2) + 0.1$

$\Rightarrow 0 = 1. b_2 + 2.(-2) + 0.1$

$\Rightarrow b_2 = 4 = (-2)^2$

$$C_3 = \sum_{k=0}^{3} a_k . b_{3-k} = a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0 = 1. b_3 + 2.4 + 0.(-2) + 0.1$$

$\Rightarrow 0 = 1. b_3 + 2.4 + 0.(-2) + 0.1$

$0 \Rightarrow b_3 + 8 + 0$

$\Rightarrow b_3 = -8 = (-2)^3$

Similarly $b_r = (-2)^r$

Hence, the value of b_r is

$b_r = (-2)^r; \qquad r \geq 0.$

8.10 Recurrence Relation:

A recurrence relation of the sequence $\{a_n\}$ is an equation that expresses an in terms of one or more of the previous terms of the sequence namely $a_0, a_1, a_2, \ldots a_{n-1}$ for all integers n with $n \geq n_0$ is non negative integers.

A recurrence relation are also called difference equations solution of recurrence relation = A sequence is called a solution of recurrence relation of its term satisfy of the recurrence relation.

Example 23:- Let $\{a_n\}$ be a sequence that satisfy that recurrence relation $a_n = a_{n-1} - a_{n-2}$ for n = 2, 3,…....; $a_0 = 3$, $a_1 = 5$ what are $a_2 + a_3 = ?$

Sol.:- $a_n = a_{n-1} - a_{n-2}$ putting n = 2, 3, …..

$a_2 = a_1 - a_0 = 5 - 3 = 2$

$a_3 = a_2 - a_1 = 2 - 5 = -3$

Linear recurrence relation with constant coefficients → A recurrence relation of the form

$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \cdots + C_k a_{r-k} = f(r)$ ……………………………….(1)

Where C_i's are constant is called linear recurrence with constant coefficients.

The recurrence relation in equation (1) is known as a kth order recurrence relation. Provided that both $C_0 \neq 0$ and $C_k \neq 0$

Second order recurrence relation → $C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} = f(r)$

Third order recurrence relation → $C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + C_3 a_{r-3} = f(r)$

The solution of equation (1) is $a_r = a_r(homogeneous) + a_r(particular)$.

8.11 Homogeneous recurrence relation:

If f(r) = 0 then equation (1) is called Homogenous recurrence relation.

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \cdots + C_k a_{r-k} = 0$$

Non-Homogeneous recurrence relation:-

If f(r) ≠ 0 then equation (1) is called non homogeneous recurrence relation

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \cdots + C_k a_{r-k} = f(r)$$

Homogeneous linear recurrence relation with constant coefficients.

Suppose the second order homogeneous linear recurrence relation is

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} = 0$$

The characteristic equation (A.E.) is

$$C_0 m^2 + C_1 m + C_2 = 0$$

Case I:- If the roots of A. E. are real and unequal Let $m_1 \neq m_2$

∴ The general solution is $a_r = C_1 m_1^r + C_2 m_2^r$

Case II:- If the roots of A. E. are real and equal let $m_1 = m_2 = m$

∴ The general solution is $a_r = (C_1 + rC_2)m^r$

Case III:- If the roots of complex numbers Let m = α ± iB

The general solution is $a_r = (C_1 cos\, r\theta + C_2 sin\, r\theta)\rho^r$

Where $\rho = \sqrt{\alpha^2 + \beta^2}$ and $\theta = tan^{-1}\left(\frac{\beta}{\alpha}\right)$

Example 24: Solve the recurrence relation $a_r + 5a_{r-1} + 6a_{r-2} = 0$

Solution: The recurrence relation is

$a_r + 5a_{r-1} + 6a_{r-2} = 0$ …………… (1)

It is second order recurrence relation.

The characteristic equation is

$m^2 + 5m + 6 = 0$

$(m + 2)(m + 3) = 0$

$\Rightarrow \therefore m = -2, -3$

The general solution is $a_r = C_1(-2)^r + C_2(-3)^r$

Example 25: Solve the recurrence relation $a_r - 7a_{r-1} + 10a_{r-2} = 0$ given that $a_0 = 0$, $a_1 = 3$

Solution: We have $a_r - 7a_{r-1} + 10a_{r-2} = 0$ ……………… (1)

and given that $a_0 = 0$, $a_1 = 3$

This is second order recurrence relations.

The characteristic equation is

$m^2 - 7m + 10 = 0$

$(m-2)(m-5)=0$

$\Rightarrow \therefore m=2,5$

The general solution is $a_r = C_1(2)^r + C_2(5)^r$ …………………….. (2)

putting in equation (2) $a_0 = 0$ i.e. $a_r = 0$ and r = 0

$$\Rightarrow 0 = C_1(2)^0 + C_2(5)^0$$

$$\Rightarrow C_1 + C_2 = 0 \qquad \text{……………………. (3)}$$

again putting in equation (2) $a_1 = 3$ i.e. $a_r = 3$ and r = 1

$$\Rightarrow C_1(2)^1 + C_2(5)^1 = 3$$

$$\Rightarrow 2C_1 + 5C_2 = 3 \qquad \text{……………………. (4)}$$

Solving equation (3) and (4) we get

$C_1 = 1$ and $C_2 = 1$ putting equation (2)

The required general solution is

$a_r = 5^r - 2^r$

Example 26: Solve the recurrence relation $a_r - 3a_{r-1} + 3a_{r-2} - a_{r-3} = 0$. Given that $a_0 = 1$, $a_1 = -2$, and $a_2 = -1$

Solution: The recurrence relation is $a_r - 3a_{r-1} + 3a_{r-2} - a_{r-3} = 0$ ………… (1)

and given that $a_0 = 1$, $a_1 = -2$, and $a_2 = -1$

This is third order recurrence relation

The characteristic equation is

$m^3 - 3m^2 + 3m - 1 = 0$

$(m-1)^3 = 0$

$\Rightarrow \therefore m = 1,1,1$

The general solution is

$a_r = (C_1 + C_2 r + C_3 r^2)(1)^r = C_1 + C_2 r + C_3 r^2$ …………………….. (2)

putting in equation (2) $a_0 = 1$ i.e. $a_r = 1$ and r = 0

$$1 = C_1 + C_2.0 + C_3.0 \Rightarrow C_1 = 1$$

putting in equation (2) $a_1 = -2$ i.e. $a_r = -2$ and r = 1

$-2 = C_1 + C_2.1 + C_3.1 \Rightarrow -2 = C_1 + C_2 + C_3 = -2$

$C_2 + C_3 = -3$ ……………. (3)

again putting in equation (2) $a_2 = -1$ i.e. $a_r = -1$ and r = 2

$-1 = C_1 + 2C_2 + 4C_3 \Rightarrow 2C_2 + 4C_3 = -2$ …………… (4)

on solving equation (3) and (4) we get

$C_2 = \frac{-11}{2} C_3 = \frac{5}{2} a_r = 1 + \left(\frac{-11}{2}\right) r + \left(\frac{5}{2}\right) r^2$

Example 27: Solve the recurrence relation $a_r + 2a_{r-1} + 2a_{r-2} = 0$ given that $a_0 = 0$ and $a_1 = -1$

Solution: We have $a_r + 2a_{r-1} + 2a_{r-2} = 0$ and given that $a_0 = 0$ and $a_1 = -1$

This is second order recurrence relation is

$$m^2 + 2m + 2 = 0$$

$m = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)} = \frac{-2 \pm \sqrt{4-8}}{2}$

$$m = \frac{-2 \pm \sqrt{-4}}{2} \Rightarrow m = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$\alpha + i\beta = -1 \pm i \Rightarrow \alpha = -1$ and $\beta = 1$

$$\rho = \sqrt{\alpha^2 + \beta^2} = \sqrt{2}$$

$$\theta = tan^{-1}\left(\frac{\beta}{\alpha}\right) = tan^{-1}\left(\frac{1}{-1}\right) = tan^{-1}(-1)$$

$$\therefore \theta = \pi - tan^{-1}(1) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$\therefore$ The general solution is $a_r = (C_1 cos\, r\theta + C_2 sin\, r\theta)\rho^r$

$$a_r = \left(C_1 cos\, \left(\tfrac{3\pi}{4} r\right) + C_2 sin\, \left(\tfrac{3\pi}{4} r\right)\right)\left(\sqrt{2}\right)^r \qquad \text{………….. (2)}$$

Putting in equation (2) a_0 = - i.e. $a_r = 0$ and r = 0

$$0 = \left(C_1(1) + C_2(0)\right)\left(\sqrt{2}\right)^0 \Rightarrow C_1 = 0$$

Putting in equation (2) a_1 = -1 i.e. a_r = -1 and r = 1

$$-1 = \left(C_1 cos\, \left(\frac{3\pi}{4}\right) + C_2 sin\, \left(\frac{3\pi}{4}\right)\right)\left(\sqrt{2}\right)^1$$

$$\Rightarrow \left\{C_1\left(-\frac{1}{\sqrt{2}}\right) + C_2\left(\frac{1}{\sqrt{2}}\right)\right\}\left(\sqrt{2}\right) = -1$$

$$\Rightarrow \left\{0 + C_2\left(\frac{1}{\sqrt{2}}\right)\right\}\left(\sqrt{2}\right) = -1$$

$$C_2 = -1$$

Putting the value of equation (2) we get

$$a_r = \left(0 + (-1)sin\, \left(\frac{3\pi}{4} r\right)\right)\left(\sqrt{2}\right)^r$$

$\therefore a_r = -sin\ \left(\frac{3\pi}{4}r\right)\left(\sqrt{2}\right)^r$

Non-Homogeneous Linear Recurrence relation with constant coefficients:-

Suppose the kth order non homogeneous linear recurrence relation with constant coefficient is

$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + C_3 a_{r-3} + \cdots + C_k a_{r-k} = f(r)$ ……………….. (1)

The solution of equation (1) is

$a_r = a_r^{(h)} + a_r^{(p)}$ ……………………. (2)

Where $a_r^{(h)}$ = Homogeneous solution,

$a_r^{(p)}$ = Particular solution

Method of finding the particular solution f(r) $\neq 0$ to find the PS of RR, we will consider a trial solution on the basis nature of f(r).

Case I:- When f(r) = b^r and b is not a root of characteristic equation.

Suppose the trial solution is $a_r = A.b^r$

Example 28: Solve the recurrence relation $a_r - 5a_{r-1} + 6a_{r-2} = 5^r$

Solution: The recurrence relation is $a_r - 5a_{r-1} + 6a_{r-2} = 5^r$ ……………………. (1)

This is second order recurrence relation.

The characteristic equation is

$m^2 - 5m + 6 = 0$

$(m-2)(m-3) = 0$

$\Rightarrow \therefore m = 2,3$

The Homogeneous solution is $a_r^{(h)} = C_1(2)^r + C_2(3)^r$

Since f(r) = 5^r and 5 is not root of homogeneous solution then we will assume the particular solution is

$a_r^{(p)} = A.(5)^r$ ………………………….. (2)

Putting the value of a_r in equation (1) we get

$$A.5^r - 5(A.5^{r-1}) + 6(A.5^{r-2}) = 5^r$$

$$\Rightarrow A.5^r\left[1 - 1 + \frac{6}{25}\right] = 5^r$$

$$\Rightarrow A.5^r\left[\frac{6}{25}\right] = 5^r$$

$$A = \frac{25}{6}$$

Putting in equation (2) we get

$$a_r^{(p)} = \frac{25}{6}.(5)^r = \frac{1}{6}.(5)^{r+2}$$

The total solution of equation (1) is

$$a_r = a_r^{(h)} + a_r^{(p)}$$

$$a_r = C_1(2)^r + C_2(3)^r + \tfrac{1}{6}.(5)^{r+2}$$

Example 29: Solve the recurrence relation $a_r - 7a_{r-1} + 10a_{r-2} = 3^r$ given that $a_0 = 0$ and $a_1 = 1$.

Solution: The recurrence relation is

$a_r - 7a_{r-1} + 10a_{r-2} = 3^r$ ………………. (1)

This is second order recurrence relation.

The characteristic equation is

$m^2 - 7m + 10 = 0$

$(m-2)(m-5) = 0$

$\Rightarrow \therefore m = 2,5$

The Homogeneous solution is $a_r^{(h)} = C_1(2)^r + C_2(5)^r$

Since f(r) = 3^r and 3 is not root of homogeneous solution then we will assume the particular solution is

$a_r^{(p)} = A.(3)^r$ .... (2)

Putting the value of a_r in equation (1) we get

$A.3^r - 5(A.3^{r-1}) + 6(A.3^{r-2}) = 3^r$

$\Rightarrow A.3^r \left[1 - \frac{7}{3} + \frac{10}{9}\right] = 3^r$

$\Rightarrow A.3^r \left[\frac{-2}{9}\right] = 3^r$

$A = \frac{-9}{2}$

Putting in equation (2) we get

$a_r^{(p)} = \frac{-9}{2}.(3)^r = \frac{-1}{2}.(3)^{r+2}$

The total solution of equation (1) is

$a_r = a_r^{(h)} + a_r^{(p)}$

$a_r = C_1(2)^r + C_2(5)^r - \frac{1}{2}.(3)^{r+2}$ ………………. (3)

Putting $a_0 = 0$ i.e. $a_r = 0$ and r = 0 in equation (3) we get

$$C_1(2)^0 + C_2(5)^0 - \frac{1}{2}.(3)^{0+2} = 0$$

$C_1 + C_2 - \frac{9}{2}$ ………………….. (4)

Again putting $a_1 = 1$ i.e. $a_r = 1$ and r = 1 in equation (3) we get

$$C_1(2)^1 + C_2(5)^1 - \frac{1}{2}.(3)^{1+2} = 1$$

$2C_1 + 5C_2 = \frac{29}{2}$ …………………… (5)

Solving the equation (4) and (5) we get

$C_1 = \frac{8}{3}$ and $C_2 = \frac{11}{6}$

Putting in equation (3) we get

$$a_r = \frac{8}{3}(2)^r + \frac{11}{6}(5)^r - \frac{1}{2}.(3)^{r+2}$$

Case II:- When f(r) = b^r and b is a root of characteristic equation.

1. If multiplicity of b is one i.e. b is root of multiplicity 1, then suppose the trial solution is $a_r = A.rb^r$

2. If multiplicity of b is two i.e. b is a root of multiplicity 2 then suppose the trial solution is $a_r = A.r^2b^r$

3. If multiplicity of b is three i.e. b is a root of multiplicity 3 then suppose the trial solution is $a_r = A.r^3b^r$

Example 30: Solve the recurrence relation $a_r - 3a_{r-1} + 2a_{r-2} = 2^r$

Solution: The recurrence relation is $a_r - 3a_{r-1} + 2a_{r-2} = 2^r$ …………… (1)

The characteristic equation is

$m^2 - 3m + 2 = 0$

$(m-2)(m-1) = 0$

$\Rightarrow \therefore m = 2,1$

The Homogeneous solution is $a_r^{(h)} = C_1(1)^r + C_2(2)^r$

Since f(r) = 2^r and 2 is characteristic root of multiplicity is 1, then we will assume the particular solution is

$a_r^{(p)} = A.r(2)^r$ ………………………….. (2)

Putting the value of a_r in equation (1) we get

$A.r2^r - 3[A.(r-1).2^{r-1}] + 2[A.(r-2).2^{r-2}] = 2^r$

$\Rightarrow A.r2^r - \frac{3}{2}A.r2^r + \frac{3}{2}A.2^r + \frac{1}{2}A.r2^r - A.2^r = 2^r$

$\Rightarrow -\frac{3}{2}A.r2^r + \frac{3}{2}A.2^r + \frac{3}{2}A.r2^r - A.2^r = 2^r$

$\Rightarrow \frac{3A.2^r - 2A.2^r}{2} = 2^r = \frac{A.2^r}{2}$

$\Rightarrow A = 2$

Putting in equation (2) we get

$a_r^{(p)} = 2.r2^r = r2^{r+1}$

The total solution of equation (1) is given by

$a_r = a_r^{(h)} + a_r^{(p)}$

$\Rightarrow a_r = C_1(1)^r + C_2(2)^r + r2^{r+1}$

Example 31: Solve the recurrence relation $a_r - 4a_{r-1} + 4a_{r-2} = 2^r$

Solution: The recurrence relation is $a_r - 4a_{r-1} + 4a_{r-2} = 2^r$ …………… (1)

The characteristic equation is

$m^2 - 4m + 4 = 0$

$(m-2)(m-2) = 0$

$\Rightarrow m = 2,2$

The Homogeneous solution is $a_r^{(h)} = (C_1 + C_2 r)(2)^r$

Since f(r) = 2^r and 2 is characteristic root of multiplicity is 1, then we will assume the particular solution is

$a_r^{(p)} = A.r^2(2)^r$ ………………………….. (2)

Putting the value of a_r in equation (1) we get

$A.r^2 2^r - 4[A.(r-1)^2.2^{r-1}] + 4[A.(r-2)^2.2^{r-2}] = 2^r$

$A.r^2 2^r - 4[A.(r^2 - 2r + 1).2^{r-1}] + 4[A.(r^2 - 4r + 4).2^{r-2}] = 2^r$

$\Rightarrow A.2^r[r^2 - 2r^2 + 4r - 2 + r^2 - 4r + 4] = 2^r$

$\Rightarrow 2A = 1 \Rightarrow A = \frac{1}{2}$

Putting in equation (2) we get $a_r^{(p)} = \frac{1}{2}.r^2 2^r = r^2 2^{r-1}$

The total solution of equation (1) is given by

$a_r = a_r^{(h)} + a_r^{(p)}$

$\Rightarrow a_r = (C_1 + C_2 r)(2)^r + r^2 2^{r-1}$.

Check your progress

Solve the following recurrence relation:

1. $a_r - 4a_{r-1} + 3a_{r-2} = 5^r$

Ans. $a_r = C_1(1)^r + C_2(3)^r + \frac{5^r}{8}$

2. $a_r - 2a_{r-1} + a_{r-2} = 2^r$ given that $a_0 = 1$, $a_1 = 1$

Ans. $a_r = 1 - 2r + 2^r$

3. $a_r + 5a_{r-1} + 6a_{r-2} = 42.4^r$ given that $a_2 = 278$, $a_3 = 962$

Ans. $a_r = (-2)^r + 2.(-3)^r + 16.4^r$

Case III:- When f(r) is a polynomial in r

1. If f(r) is first degree polynomial in r then trial solution is $a_r^{(p)} = A_0 + A_1 r$

2. If f(r) is second degree polynomial in r then trial solution is $a_r^{(p)} = A_0 + A_1 r + A_2 r^2$

3. If f(r) is third degree polynomial in r then trial solution is $a_r^{(p)} = A_0 + A_1 r + A_2 r^2 + A_3 r^3$

Example 32: Solve the recurrence relation $a_r - 5a_{r-1} + 6a_{r-2} = 2 + r$ given that $a_0 = 1$, $a_1 = 1$

Solution: The recurrence relation is $a_r - 5a_{r-1} + 6a_{r-2} = 2 + r$ ………………….. (1)

The initial condition are $a_0 = 1$, $a_1 = 1$

The characteristic equation is $m^2 - 5m + 6 = 0$

$$(m-2)(m-3) = 0 \quad \Rightarrow m = 2,3$$

The Homogeneous solution is $a_r^{(h)} = C_1(2)^r + C_2(3)^r$

Since f(r) = 2 + r is a polynomial of degree one, then we will assume the particular solution is $a_r^{(p)} = A_0 + A_1 r$ ………………. (2)

Putting the value of a_r in equation (1) we get

$$[A_0 + A_1 r] - 5[A_0 + A_1(r-1)] + 6[A_0 + A_1(r-2)] = 2 + r$$

$$\Rightarrow A_0 + A_1 r - 5A_0 - 5A_1 r + 5A_0 + 6A_0 + 6A_1 r - 12A_0 = 2 + r$$

$$\Rightarrow (2A_0 - 7A_1) + 2A_1 r = 2 + r$$

Equating the coefficient of r^0 and r^1 on both sides we get

$$2A_0 - 7A_1 = 2 \qquad \text{……………………… (3)}$$

and $2A_1 = 1 \Rightarrow A_1 = \frac{1}{2}$

Putting the value of A_1 in equation (3), we get

$$A_0 = \frac{11}{4}$$

Putting the value in equation (2), we get

$$a_r^{(p)} = \frac{11}{4} + \frac{1}{2}r$$

The total solution of equation (1) is given by

$$a_r = a_r^{(h)} + a_r^{(p)}$$

$a_r = C_1(2)^r + C_2(3)^r + \frac{11}{4} + \frac{1}{2}r$ …………………….. (4)

Putting $a_0 = 1$ i.e. $a_r = 1$ and r = 0 in equation (4), we get

$$C_1(2)^0 + C_2(3)^0 + \frac{11}{4} + \frac{1}{2}0 = 1$$

$C_1 + C_2 = -\frac{7}{4}$ …………………………………….. (5)

again putting $a_1 = 1$ i.e. $a_r = 1$ and r = 1 in equation (4) we get

$$C_1(2)^1 + C_2(3)^1 + \frac{11}{4} + \frac{1}{2}.1 = 1$$

$2C_1 + 3C_2 = -\frac{9}{4}$ ………………………… (6)

Solving the equation (5) and (6), we get

$C_1 = -3$ and $C_2 = \frac{5}{4}$

Putting in equation (4), we get

$$a_r = -3(2)^r + \frac{5}{4}(3)^r + \frac{11}{4} + \frac{1}{2}r$$

Example 33: Solve the recurrence relating $a_r - 4a_{r-1} + 4a_{r-2} = (r+1)^2$.

Solution: The recurrence relation is

$$a_r - 4a_{r-1} + 4a_{r-2} = (r+1)^2 \quad \text{…………… (1)}$$

The characteristic equation is $m^2 - 4m + 4 = 0$

$$(m-2)^2 = 0 \quad \Rightarrow m = 2,2$$

The Homogeneous solution is $a_r^{(h)} = (C_1 + C_2 r)(2)^r$

Since f(r) = (r + 1)^2 is a polynomial of degree two, then we will assume the particular solution is

$a_r^{(p)} = A_0 + A_1 r + A_2 r^2$ …………………….. (2)

Putting the value of a_r in equation (1), we get

$$[A_0 + A_1 r + A_2 r^2] - 4[A_0 + A_1(r-1) + A_2(r-1)^2] + 4[A_0 + A_1(r-2) + A_2(r-2)^2] = (r+1)^2$$

$$\Rightarrow [A_0 + A_1 r + A_2 r^2] - 4[A_0 + A_1 r - A_1 + A_2 r^2 - 2A_2 r + A_2] + 4[A_0 + A_1 r - 2A_1 + A_2 r^2 - 4A_2 r + 4A_2] = (r+1)^2$$

$$\Rightarrow (A_0 - 4A_1 r + 12A_2) + (A_1 - 8A_2)r + A_2 r^2 = 1 + 2r + r^2$$

Equating the coefficient of r^0, r^1 and r^2 on both sides, we get

$A_0 - 4A_1 r + 12A_2 = 1$ ………………….. (4)

$A_1 - 8A_2 = 2$ ……………………………. (5)

and $A_2 = 1$ ……………………………. (6)

Solving the equation (4), (5) and (6) we get

$A_0 = 29$ and $A_1 = 10$

The total solution of equation (1) is

$a_r = a_r^{(h)} + a_r^{(p)}$

$a_r = (C_1 + C_2 r)(2)^r + r^2 + 10r + 29$

Example 34: Solve the recurrence relation $a_r + 6a_{r-1} + 9a_{r-2} = 3$ given that $a_0 = 0$, $a_1 = 1$

Solution: The recurrence relation is $a_r + 6a_{r-1} + 9a_{r-2} = 3$ …………………. (1)

The initial condition are $a_0 = 0$, $a_1 = 1$

The characteristic equation is $m^2 + 6m + 9 = 0$

$(m+3)^2 = 0 \quad \Rightarrow m = -2, -2$

The Homogeneous solution is $a_r^{(h)} = (C_1 + C_2 r)(-3)^r$

Since f(r) = 3 is a constant and not a part of Homogeneous solution, then we will assume the particular solution is

$$a_r^{(p)} = A_0 \text{ ……………………….. (2)}$$

Putting the value of a_r in equation (1), we get

$$A_0 + 6A_0 + 9A_0 = 3$$

$$A_0 = \frac{3}{16}$$

Putting the value in equation (2) we get $\quad a_r^{(p)} = \frac{3}{16}$

The total solution of the equation (1) is

$$a_r = a_r^{(h)} + a_r^{(p)}$$

$$a_r = (C_1 + C_2 r)(-3)^r + \frac{3}{16} \qquad \text{………………. (3)}$$

Putting $a_0 = 0$ i.e. $a_r = 0$ and r = 1 in equation (3), we get

$$(C_1 + C_2 1)(-3)^1 + \frac{3}{16} = 1$$

$$\Rightarrow 3C_1 + 3C_2 = -\frac{13}{16} \quad \text{………………………….. (4)}$$

Putting the value of C_1 in equation (4) we get

$$C_2 = -\frac{1}{12}$$

Putting the value in equation (3), we get

$$a_r = \left[-\frac{3}{16} - \frac{1}{12}r\right](-3)^r + \frac{3}{16}$$

Example 35: Solve the recurrence relation $a_r - 2a_{r-1} + a_{r-2} = 7$

Solution: The recurrence relation is $a_r - 2a_{r-1} + a_{r-2} = 7$ ……………… (1)

The characteristic equation is $m^2 - 2m + 1 = 0$

$$(m-1)^2 = 0 \qquad \Rightarrow m = 1{,}1$$

The Homogeneous solution is

$$a_r^{(h)} = (C_1 + C_2 r)(-1)^r = C_1 + C_2 r$$

Since f(r) = 7 = $7(1)^0$ is a constant and 1 is a double characteristic root of homogeneous solution. Then we will assume the particular solution is

$$a_r^{(p)} = A_0 r^2 \text{ …………………….. (2)}$$

Putting the value of a_r in equation (1) we get

$$A_0 r^2 - 2A_0(r-1)^2 + A_0(r-2)^2 = 7$$

$$\Rightarrow A_0 r^2 - 2A_0 r^2 + 4A_0 r - 2A_0 + A_0 r^2 - 4A_0 r + 4A_0 = 7$$

$$\Rightarrow \qquad 2A_0 = 7 \Rightarrow A_0 = \frac{7}{2}$$

Putting the value in equation (2), we get

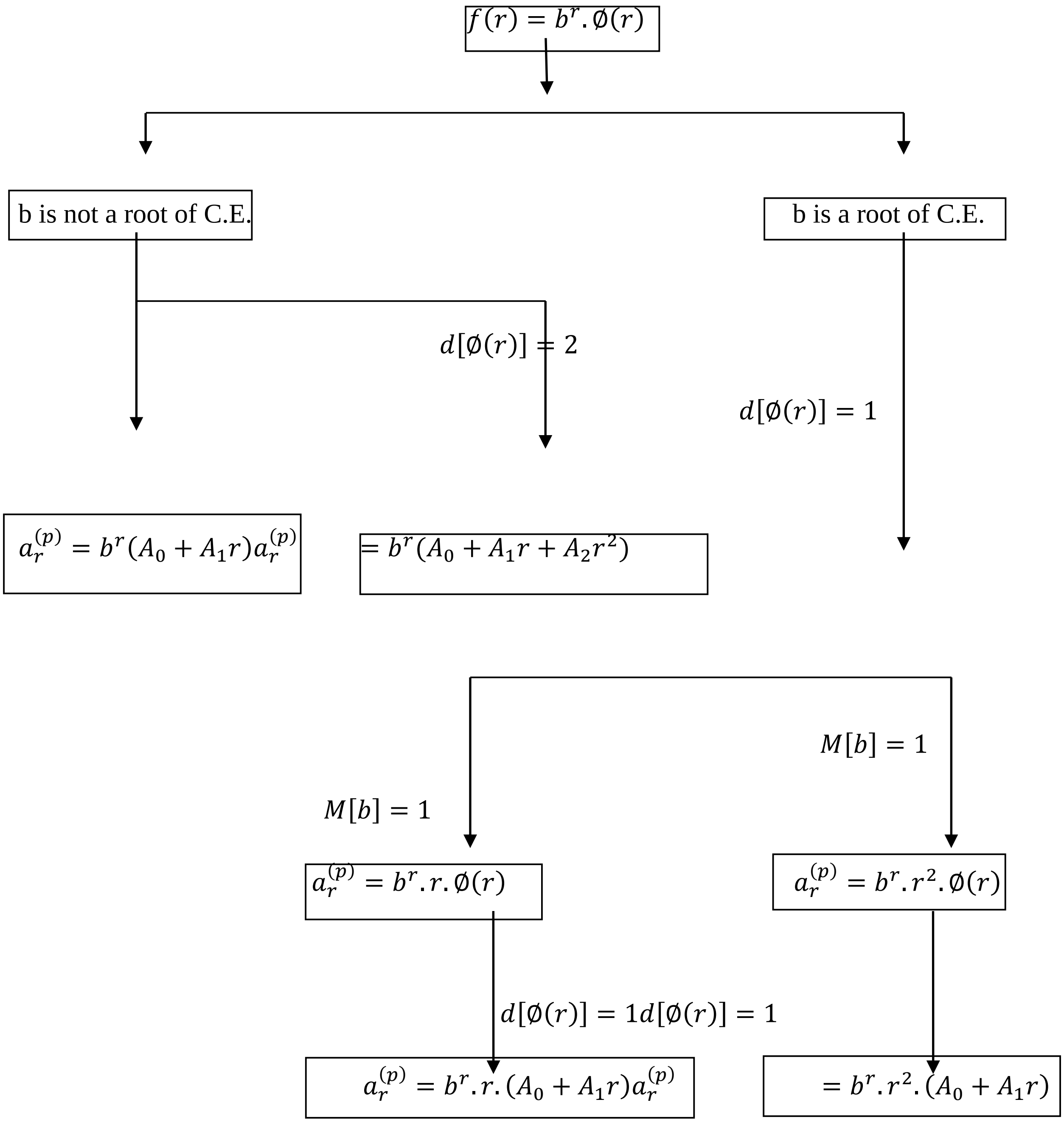
$$a_r^{(p)} = \frac{7}{2} r^2$$

The total solution of equation (1) is given by

$$a_r = a_r^{(h)} + a_r^{(p)}$$

$$a_r = C_1 + C_2 r + \frac{7}{2} r^2$$

Case IV:- When $f(r) = b^r.\emptyset(r)$, where Φ(r) is a polynomial in r and b is a constant.



Example 36: Solve the recurrence relation $a_r + a_{r-1} = 3r.2^r$

Solution: The recurrence relation is $a_r + a_{r-1} = 3r.2^r$ ………………. (1)

The characteristic equation is

$m + 1 = 0$

$\Rightarrow m = -1$

The homogenous solution is $a_r^{(h)} = C_1(-1)^r$

Since f(r) = 3r. 2^r, here 2 is not a root of characteristic equation and r is of degree one

$a_r^{(p)} = 2^r(A_0 + A_1 r)$ ……………….. (2)

Putting the value of a_r in equation (1), we get

$[2^r(A_0 + A_1 r)] + [2^{r-1}\{A_0 + A_1(r-1)\}] = 3r.2^r$

$\Rightarrow 2^r\left[A_0 + A_1 r + \frac{A_0}{2} + \frac{A_1}{2}r - \frac{A_1}{2}\right] = 3r.2^r$

$\Rightarrow \left(\frac{3A_0}{2} - \frac{A_1}{2}\right) + \frac{3}{2}A_1 r = 3r$

Equating both sides, we get

$\frac{3A_0}{2} - \frac{A_1}{2} = 0$ ……………………… (3)

and $\frac{3}{2}A_1 = 3 \Rightarrow A_1 = 2$

Putting the value of A_1 in equation (3), we get

$A_0 = \frac{2}{3}$

Putting the value in equation (2), we get

$a_r^{(p)} = 2^r\left(\frac{2}{3} + 2r\right)$

The total solution of the equation (1) is given by

$a_r = a_r^{(h)} + a_r^{(p)}$

$$a_r = C_1(-1)^r + 2^r\left(\frac{2}{3} + 2r\right)$$

Example 37: Solve the recurrence relation $a_r - 4a_{r-1} + 4a_{r-2} = (r+1).2^r$

Solution: The recurrence relation is $a_r - 4a_{r-1} + 4a_{r-2} = (r+1).2^r$ . .. (1)

The characteristic equation is

$$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0$$

$$\Rightarrow m = 2,2$$

The Homogeneous solution is $a_r^{(h)} = (C_1 + C_2 r)(2)^r$

Since f(r) = 2^r.(r + 1), here 2 is a double root of characteristic equation and (r + 1) is a polynomial of degree one, then we will assume the particular solution is

$$a_r^{(p)} = 2^r.r^2(A_0 + A_1 r) \text{ ……………… (2)}$$

Putting the value of a_r in equation (1) we get

$$2^r.r^2(A_0 + A_1 r) - 4.2^{r-1}.(r-1)^2\big(A_0 + A_1(r-1)\big) + 4.2^{r-2}.(r-2)^2\big(A_0 + A_1(r-2)\big) = (r+1).2^r$$

$$\Rightarrow 2^r[A_0 r^2 + A_1 r^3 - 2(r^2 - 2r + 1)(A_0 + A_1 r - A_1) + (r^2 - 4r + 4)(A_0 + A_1 r - 2A_1)] = (r+1).2^r$$

$$\Rightarrow A_0 r^2 + A_1 r^3 - 2A_0 r^2 - 2A_1 r^3 + 2A_1 r^2 + 4A_0 r + 4A_1 r^2 - 4A_1 r - 2A_0 - 2A_1 r + 2A_1 + A_0 r^2 + A_1 r^3 - 2A_1 r^2 - 4A_0 r - 4A_1 r^2 + 8A_1 r + 4A_0 + 4A_1 r - 8A_1 = r + 1$$

$$\Rightarrow 6A_1 r + 2A_0 - 6A_1 = r + 1$$

Equating the coefficient of r^0 and r^1 on both sides, we get

$6A_1 = 1 \Rightarrow A_1 = \frac{1}{6}$ and $2A_0 - 6A_1 = 1 \Rightarrow A_0 = 1$

Putting in equation (2), we get

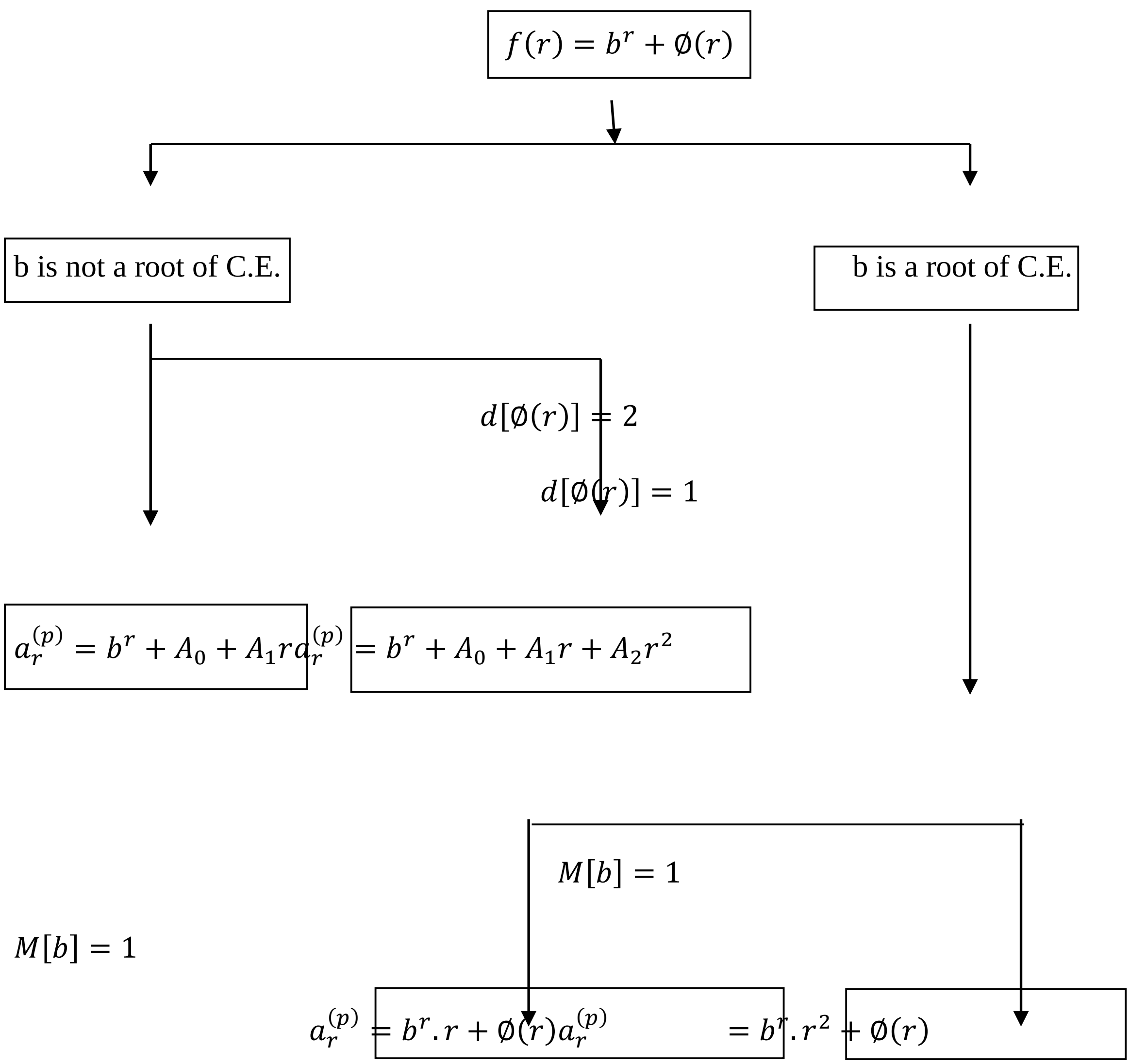
$$a_r^{(p)} = 2^r . r^2 \left(1 + \frac{r}{6}\right)$$

The total solution of equation (1) is given by

$$a_r = a_r^{(h)} + a_r^{(p)}$$

$$a_r = (C_1 + C_2 r)(2)^r + 2^r . r^2 \left(1 + \frac{r}{6}\right)$$

Case V:- When $f(r) = b^r + \emptyset(r)$, where $\emptyset(r)$ is a polynomial in r and b is constant



$d[\emptyset(r)] = 1$ $\qquad$ $d[\emptyset(r)] = 1$

$a_r^{(p)} = b^r.r + (A_0 + A_1 r)a_r^{(p)}$ $\qquad$ $= b^r.r^2 + (A_0 + A_1 r)$

Example 38: Solve the recurrence relation $a_r - 5a_{r-1} + 6a_{r-2} = 2^r + r$ given that $a_0 = 1$, $a_1 = 1$.

Solution: The recurrence relation is $a_r - 5a_{r-1} + 6a_{r-2} = 2^r + r$ ………….. (1)

The initial condition are $a_0 = 1$, $a_1 = 1$

The characteristic equation is

$m^2 - 5m + 6 = 0$

$\Rightarrow (m-2)(m-3) = 0$

$\Rightarrow m = 2,3$

The homogenous solution is $a_r^{(h)} = C_1(2)^r + C_2(3)^r$

Since f(r) = 3r. 2^r,+r here 2 is not a root of characteristic equation with multiplicity 1 and r is a polynomial of degree one, then we will assume the particular solution is

$a_r^{(p)} = A_0 2^r.r + A_1 + A_2 r$ ……………….. (2)

Putting the value of a_r in equation (1) we get

$$[A_0 2^r.r + A_1 + A_2 r] - 5[A_0 2^{r-1}.(r-1) + A_1 + A_2(r-1)] + 6[A_0 2^{r-2}.(r-2) + A_1 + A_2(r-2)] = 2^r + r$$

$$\Rightarrow A_0 2^r.r + A_1 + A_2 r - 5A_0 2^{r-1}.r + 5A_0 2^{r-1} - 5A_1 - 5A_2 r + 5A_2 + 5A_0 r 2^{r-2} - 12A_0 2^{r-2} + 6A_1 + 6A_2 r - 12A_2 = 2^r + r$$

$$\Rightarrow A_0 2^r.r\left[1-\frac{5}{2}+\frac{3}{2}\right]+A_0 2^r\left[\frac{5}{2}-3\right]+2A_2r+2A_1-7A_2=2^r+r$$

$$\Rightarrow -\frac{1}{2}A_0 2^r+2A_2r+2A_1-7A_2=2^r+r$$

Equating on both sides, we get

$-\frac{1}{2}A_0=1 \Rightarrow A_0=2$ and $2A_2=1 \Rightarrow A_2=\frac{1}{2}$ and

$$2A_1-7A_2=0 \Rightarrow 2A_1-7.\frac{1}{2} \Rightarrow A_1=\frac{7}{4}$$

Putting in equation (2), we get

$$a_r^{(p)}=-2.2^r.r+\frac{7}{4}+\frac{1}{2}r=-r.2^{r+1}+\frac{7}{4}+\frac{1}{2}r$$

The total solution of equation (1) is given by

$$a_r=a_r^{(h)}+a_r^{(p)}$$

$a_r=C_1(2)^r+C_2(3)^r-r.2^{r+1}+\frac{7}{4}+\frac{1}{2}r$ ……………….. (3)

Putting $a_0 = 1$ i.e. $a_r = 1$ and r = 0 in equation (3) we get

$$C_1(2)^0+C_2(3)^0-r.2^{0+1}+\frac{7}{4}+\frac{1}{2}.0=1$$

$\Rightarrow C_1+C_2=\frac{-3}{4}$ ……………….. (4)

Putting $a_1 = 1$ i.e. $a_r = 1$ and r = 1 in equation (3) we get

$$C_1(2)^1+C_2(3)^1-r.2^{1+1}+\frac{7}{4}+\frac{1}{2}.1=1$$

$\Rightarrow 2C_1+3C_2=\frac{11}{4}$ …………………………… (5)

solving equation (4) and (5), we get $C_1 = -5$ and $C_2 = \frac{17}{4}$

Putting in equation (3) we get $a_r = -5(2)^r + \frac{17}{4}(3)^r - r.2^{r+1} + \frac{7}{4} + \frac{1}{2}r$

Summary:

If { a_1, a_2, a_3, ….. a_r,……} is a sequence of real or complex numbers, then the power series given by $A(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_r z^r + \cdots$

$A(z) = \sum_{r=0}^{\infty} a_r z^r$ is called generating function for the given sequence.

Assume f is a function with the set of nonnegative integers as its domain We use two steps to define f.

Basis step- Specify the value of f(0).

Recursive step- Give a rule for f(x) using f(y) where y<0<x.

Recursively defined functions should be well defined. It means for every positive integer, the value of the function at this integer is determined in an unambiguous way.

In some recursive functions, the values of the function at the first k positive integers are specified. A rule is given to determine the value of the function at larger integer from its values at some of the preceding k integers.

If R is a set of real numbers. A function whose domain is the set {0, 1, 2, 3, …..} of non-negative integers and whose range is a subset of R, is called a discrete numeric function or numeric function and it is denoted by a_r or a.

a = {a_0, a_1, a_2, a_3, ……a_r….}

If a_r and b_r are two numeric functions, then the convolution of a_r and b_r is denoted by a_r*b_r and is a numeric function C_r defined as

$C_r = a_r * b_r = \sum_{k=0}^{r} a_k . b_{r-k}$ such that $C_0 = a_0 b_0$

Terminal Questions

Solve the following recurrence relation:

1. $a_r - 4a_{r-1} + 4a_{r-2} = 2^r r^2$

Ans. $a_r = (C_1 + C_2 r)(2)^r + \left(\frac{5}{11} + \frac{1}{3}r + \frac{1}{12}r^2\right)$

2. $a_r - 5a_{r-1} + 6a_{r-2} = 2^r + r^2 + r$

Ans. $a_r = -2^{r+4} + \frac{31}{4}3^r - r.2^{r+1} + \frac{1}{2}r^2 + 4r + \frac{37}{4}$

3. $a_r - 5a_{r-1} + 6a_{r-2} = r(r-1)$

Ans.$a_r = C_1(2)^r + C_2(3)^r + \frac{23}{2} + 3r + \frac{r^2}{2}$

4. $a_r - 2a_{r-1} = 7r^2$

Ans.$a_r = C_1(2)^r - 7r^2 - 28r - 42$

5. $a_r - a_{r-1} = 7$

Ans.$a_r = C_1(1)^r - 7r$

6. $a_{r+2} - 2a_{r+1} + a_r = 3r + 5$

Ans.$a_r = C_1 + C_2 r + \frac{1}{2}r(r-1)(r+3)$

7. $a_{r+2} + 5a_{r+1} + 6a_r = 3r^2$

Ans.$a_r = C_1(-2)^r + C_2(-3)^r + \frac{1}{4}r^2 + \frac{17}{24}r + \frac{115}{288}$.

Master of Science
PGMM -103N
Discrete Mathematics

U. P. Rajarshi Tandon
Open University

Block

3 Boolean Algebra

Block-3

Boolean Algebra

We study about Boolean algebra of different kinds.This is most basic unit of this block as it introduces the concept of Boolean algebra, Principle of Duality. We introduce the Subalgebra, Isomorphic Boolean Algebras. Boolean Algebra as Lattices, representation theorem for Finite Boolean Algebras, Boolean Functions and its applications in Logic Gates and Circuits, to stablish the formula for Minimization of Boolean Functions (Karnaugh Map) . It has got tremendous application in almost every field, social, economy, engineering, technology etc. In computer science concept of logic is a major tool to learn to understand it more clearly. This is most basic unit of this block as it introduces the concept of Lattice.

A non empty set A, together with a binary relation R is said to be a partially ordered set or a poset. If it satisfied the condition of reflexive, anti symmetry and transitive. Two elements a and b is a poset (S, ≤) are said to be comparable if either $a \leq b$ or $b \leq a$ Thus a and b are called incomparable if neither $a \leq b$ nor $b \leq a$.

A relation R on a set A is said to be total ordering relation if the relation R is reflexive, anti-symmetric, transitive and satisfies the following relation. For each $a, b \in A$, either a ≤ b or b ≤ a i.e. any two elements of A are comparable.

UNIT-9 : Boolean Algebra

Structure

9.1 Introduction

In this Unit, we shall study Boolean algebra as an abstract structure. The definition of a Boolean algebra which will be given now is one given by **Huntington** in 1904. In fact, Boolean algebra originated in the works of the English Mathematician **George Boole** (1813-1865). The original purpose of this algebra was to simplify logical statements and solve logic problems. Today it is the backbone of design and analysis of computer and other digital circuits.

This is most basic unit of this block as it introduces the concept of Boolean algebra, Principle of Duality. We introduce the Subalgebra, Isomorphic Boolean Algebras. Boolean Algebra as Lattices, representation theorem for Finite Boolean Algebras, Boolean Functions and its applications in Logic Gates and Circuits, to stablish the formula for Minimization of Boolean Functions (Karnaugh Map) . It has got tremendous application in almost every field, social, economy, engineering, technology etc. In computer science concept of logic is a major tool to learn to understand it more clearly.

9.2 Objectives

After reading this unit the learner should be able to understand about:

- the concept of Boolean Algbera, Principle of Duality
- Use of Boolean algebra, subalgebra, Isomorphic Boolean algebras
- Boolean algebra as Lattices, representation theorem for Finite Boolean algebras
- Use of Boolean Functions and its applications in Logic Gates and Circuits,
- to stablish the formula for Minimization of Boolean Functions (Karnaugh Map).

9.3 Boolean Algbera

Let B be a none-empty set with two binary operations + and *, a unary operation′, and two distinct elements 0 and 1. They B is called a Boolean algebra if the following axioms hold for any a, b, c $\in$ B.

[B1] Commutative laws: The operations + and * are commutative. In other words,

$$a+b = b + a \text{ and } a + b = b+a, \forall a, b \in B$$

[B2] Identity laws: For any a$\in$B a + 0 =a and a*1 = a

That is, both operations + and * have identity elements denoted by 0 and 1 respectively.

[B3] Distributive Laws: Each binary operation is distributive over the other. That is, for any $a, b, c \in B, a + (b * c) = (a+b) * (a+c)$ and $a * (b+c) = (a*b) + (a*c)$

[B3] Complements laws: For each a in B, there exists an element a in B such that a+a′ = 1 and a*a′=0

We sometimes denote a Boolean algebra by (B, +, *, ′ , 0, 1). The elements 0 and 1 are called zero element (identity for +) and unit element (identity for *) of B respectively while a′ is called complement of a in B.

We will usually drop the symbol * between a and b and write a * b simply as ab. Some authors use the symbols $\vee$ and $\wedge$ in place of the symbols + and * respectively and denote the complement of an element a by the symbol $\bar{a}$ instead of a′.

We mention here that there exist other sets of axioms which can equally well defined a Boolean algebra, though, of course, each set is derivable from the other. Moreover, we shall also give all alternative definition of a Boolean algebra in terms of an associated partial ordering. The following example shows that the algebra of sets is a Boolean algebra.

Example 1: Let S be an non-empty set and P(S) be the power set of S. then P(S) is a Boolean algebra with respect to union and intersection as two binary operations + and * respectively and complement of a set with respect to S as unary operation′, ϕ and S will act as 0 and 1 respectively.

Solution: We shall show that the power set P(S) of a non-empty set S forms a Boolean algebra with respect to union and intersection as two binary operations + and * respectively and complement of a subset A of S with respect to S, i.e, S – A as unary operation′ on a.

1. Commutative laws: We know from set theory that

$A \cup B = B \cap A$ and $A \cap B = B \cap A$, $\forall$ A, B $\in$P(S)

Thus commutative laws are satisfied.

2. Identity laws: We know that ϕ and S belong to P(S) such that

$A \cup \phi = A$ and $A \cap S = A$ for any A $\in$ P(S).

Thus ϕ and S act as 0 and 1 respectively.

3. Distributive laws: From set theory, we know that

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and

$A \cap (B \cap C) = (A \cap B) \cup (A \cap C)$ for all A, B, C $\in$ P(S).

Hence both operations ∪ and ∩ distribute over each other.

4. Complement laws: For any A$\in$P(S), S – A $\in$ P(S) such that

$A \cup (S - A) = S$ and $A \cap (S - A) = \phi$

Thus every element A in P(S) contains 2^n elements. The case when S contains three elements is considered in the following example.

Example 2: Let S = {a, b, c}. Then P(S) = (ϕ, {a}, {b}, {c}, {a}, {b, c}, {a, c}, S} is a boolean algebra in which +, * and ′ are taken as union ⋃, intersection ⋂and complement with respect to respectively with 0 = ϕ and 1 = S.

Solution: This is a particular case of example 5.1 above. Students are advised to reproduce the solution.

Example 3: Show that the set B = {0, 1} together with the operatons +, * and ′ defined by the following tables is a Boolean algebra.

+	0	1
0	0	1
1		1

*	0	1
0	0	0
1	0	1

′	0	1
	1	0

Solution: It is clear from the tables that + and * are binary operations on B and that ′ is a unary operation on B. We show that all axioms for a Boolean algebra are satisfied.

1. Commutative laws: Since the table for + and * are symmetrical about main diagonals, both operations are commutative.

2. Identity laws: It is clear from the tables that a+0 = a and a * 1 = a ∀a∈B.

Thus 0 is the identity for + and 1 is the identity for *

3. Distributive laws: It is easy to verify that both + and * are distributive over each other. That is, a+(b*c) = (a+b) * (a+c)and a* (b+c)=(a*b)+(a*c)∀a, b, c ∈B.

4. Complement laws: Given 0∈B, there exists 1 ∈B such that 0+1=1 and 0*1=0 and given 1∈B, we have 0∈B such that 1+0=1 and 1*0=0 Hence for every a in B, there exists a′=0 Thus (B, +, *, ′) is a Boolean algebra.

Example 4: let B = {1, 2, 5, 7, 10, 14, 35, 70}. For any a, b in B define +, * and ′ as follows: a+b = lcm (a, b), a*b = gcd (a, b) and $a = \frac{70}{a}$. Then it shows that B is a Boolean algebra with 1 as zero element and 70 as unit element.

Solution: We shall construct the composition tables for +, *, and ′,

+	1	2	5	7	10	14	35	70
1	1	2	5	7	10	14	35	70
2	2	2	10	14	10	14	70	70
5	5	10	5	35	10	70	35	70
7	7	14	35	7	70	14	35	70
10	10	10	10	70	10	70	70	70
14	14	14	70	14	70	14	70	70
35	37	70	25	35	70	70	35	70
70	70	70	70	70	70	70	70	70

*	1	2	5	7	10	14	35	70

1	1	1	1	1	1	1	1	1
2	1	2	1	1	2	2	1	2
5	1	1	5	1	5	1	5	5
7	1	1	1	7	1	7	7	7
10	1	2	5	1	10	2	5	10
14	1	2	1	7	2	14	7	14
35	1	1	5	7	5	7	35	35
70	1	1	5	7	10	14	35	70

′	1	2	5	7	10	14	35	70
	70	35	14	10	7	5	2	1

From the table we see that all the entries in the table are elements of the set B. Therefore both + and * are binary operations on B and ′ is a unary operation on B

1. Commutative laws: Since the composition tables for + and * are symmetrical with respect to main diagonals. Therefore operation + and * are commutative.

2. Identity laws: From the composition tables we see that a+1 =a and a*70= a $\forall a \in B$. Hence 1 and 70 are zero element and unit element of B, respectively.

3. Distributive laws: With the help of the composition tables for + and * it can be verified that a+(b*c)=(a+b)*(a+c) and a*(b+c)=(b+c)=(a*b)+(b*c)$\forall$a,b,c$\in$b.

Complement laws: For each a$\in$B, there exist $a' = \frac{70}{a}$ in b such that a+a′=70 and a*a′=1. Thus complement of every element in B exists in B. Hence B is a Boolean algebra.

9.4 Principle of Duality:

Observe the symmetry of the axioms $[B_1]$ to $[B_4]$ in the definition of a Boolan algebra b with respect to the two operations + an * and the two identities 0 and 1. For example, there are two complement laws and the second complement law can be obtained from the first complement law buy interchanging + and * and also interchange their identities 0 an 1. Because of this symmetry it follows that any statement deducible from the axioms of a Boolean algebra remains valid if the operations + and * are interchanged and also their identities 0 and 1 are interchanged throughout. The new statement so obtained (by interchanging + and * and also interchanging identies 0 and 1 in the given statement) is called dual of the given statement. Thus if a statement or algebraic identity holds in a Boolean algebra then its dual also holds. This result is known as Principle of Duality. We state this result as a theorem.

Theorem 1: (**Principle of Duality**) and theorem of Boolean algebra remains valid if + is interchanged with * and 0 is interchanged with 1 throughout in the theorem.

In the following theorem, each part contains two dual statements. In view of Principle of duality, it is sufficient to prove only one of them and the other will follow by the Principle of duality. However to illustrate the nature of duality, we shall give proofs of both statements in first part.

Theorem 2: The following holds in a Boolean algebra B.

1. **Idempotent laws:** a + a = a and a * a = a, ∀ a∈b
2. **Boundedness laws :** a + 1 = 1 and a * 0 = 0 , ∀a∈B
3. **Absorption laws:** a+ (a * b) = a and a*(a+b)=a, ∀a, b∈B
4. **Associative laws:** (a+b)+c= a+(b+c) and (a*b) * c = a * (b * c), ∀a, b, c∈B

Proof. (1): We first show that a + a = a, ∀ a∈B

we have a = a 0 by B_2

= a + a * a′ by B_4

=(a + a) * (a+a′) by B_3

= (a+a) * 1 by B_4

= a * a by B_2. Hence a = a + a

To show a * a = a, we write a = a * 1 by B_2

=a * (a+a′) by B_4

= a * a+a* a′ by B_3

=a * a +0 by B_4

= a * a by B_2. Thus a = a * a

Note that the step in the proof of a * a = a is dual to the steps in the proof of a+a = a and the justification for each steps is the same law in a*a = a as in a + a = a.

(2) We shall only prove a+1 = 1. The other statement will be obtained by the principle of duality. We have

1 = a + a′ by B_4

=a+a′*1 by B_2 (∵ a′*1 = a′)

= (a+a′)*(a+1) by B_3 =1* (a+1) = a+1 by B_2. Thus a+ 1 = 1

To prove a * 0 = 0, by principle of duality, since a + 1 =0 holds in a Boolean algebra, therefore its dual a * 0 = 0, also holds in the Boolean algebra by the principle of duality

(3) We first show that a + a * b = a ∀a, b, ∈B

we have a = a * 1 by B_2

= a * (1+b) by boundeness law, 1+b=1

=a*1+a*b by B_1 =a+a*b by B_2. Thus a+a*b = a

To prove a*(a+b) = a, we use principle of duality. Since a+a*b = a

Also holds in B by the principle of duality

Note: Students are advised to prove the result a * (a+b)=a without using the principle of duality.

(4) To prove (a * b)* c = a * (b * c), we first prove that a+(a*b)*c = a+a* (b *c) ∀a, b, c ∈b. By absorption law, we have a+a*(b*c) = a = a* (a+c)

by absorption law = (a+a*b)*(a+c)

Thus a+a*(b*c)=a+(a*b)*c by distributive laws …..(1)

Next, we will show that a′+a* (b*c) = a′+(b+c)*c

We have a′+a∗(b∗c) = (a′+a) ∗ (a′+b∗c) by distributive law

=1 ∗ (a′ +b ∗ c) by complement law

= a′ +b ∗ c by identity law = (a′+b)∗(a′+c) by distributive law

= [1∗(a′+b)] ∗(a′+c) by identity law

= [(a′+a∗b) ∗ (a′+b)]∗(′+c) ∵ a′+a = 1

=(a′+a∗b)∗(a′+c) by distributive law = a′+(a∗b)∗c by distributive law

Thus $a'+a*(b*c)=a'+(a*b)*c$ ……(2)

Now, $(a*b)*c = 0 + (a*b)*c = a * a' + (a* b) * cx$

$= [a+(a* (a*b) *c]*[a'+(a*b)*c]$ by distributive law

$= [a+a* (b*c)]*[a'+a*(b*c)$ by equations (1) and (2)

$= a * a' + a* (b*c)$ by distributive law

$= 0 +a* (b * c) = a * (b * c)$. This completes the proof.

Applying principle of duality on the result $(a* b) * c = a* (b*c)$

We get $(a+b)+c = a+(b+c)$

In view of this results, we shall write both $a * (b *c)$ and $(a * b)* c$ as $a * b *$ and similarly, we shall write both $(a+b) + c$ and $a+(b +c)$ as $a +b +c$.

Theorem 3: For each element a in a Boolean algebra B, a′ is unique. In other words, complement of an element a in Boolean algebra B is unique.

Proof: Let a by any element in a Boolean algebra B. If possible, suppose x and y be two complements o a in B. Then $a +x = 1$, $a* x = 0$ and $a + y = 1$, $a * y = 0$

Now $x = x *1$ by identity law

$= x * (a+y)$ by assumption

$= x * a + x*y$ by distributive l$= 0 + * y$ by assumption

$= x * y$ by identity law $= x * y + 0$ by identity law

$= x * y + a * y$ by assumption $= (x+a) *y$ by distributive law

$= 1 * y$ by assumption $= y$ by identity law

Thus complement of a is unique.

Theorem 4: For any element a in a Boolean algebra B, $(a')' = a$ (this result is known as involution law)

Proof: Since a′ is a complement of the element $a \in B$, therefore $a+a'=1$ and $a*a'=0$. But this is exactly the condition to be satisfied for a to be complement of a′. Now by uniqueness of the complement, we have $(a')'=a$

Theorem 5: In any Boolean algebra, $0' = 1$ and $1' = 0$.

Proof: By theorem 2, we have $1+0 = 1$ and $1 * 0 = 0$

$\Rightarrow 0' = 1$ and $1' = 0$,

Theorem: The following are equivalent in a Boolean algebra B

(1) $a + b = b$ (2) $a * b = a$ (3) $a' + b = 1$ (4) $a * b = 0$

Proof: (1) $\Rightarrow$(2) By absorption law, we have $a = a + a * b = (a + a) * (a + b)$

$= a * (a + b) = a + b$

(2) $\Rightarrow$ (1) Suppose that $a * b = a$. To show $a+b=b$.

We have $a + b = a*b+b$ by assumption $a = a*b$

$= b+a*b$ by commutative law $=b$ by absorption law

We now show (1) and (3) are equivalent.

(1) $\Rightarrow$ (3) Suppose (1) holds. Then

$a' + b = a' + (a + b)$ $\because$ by (1), $a + b = b$

$= (a'+a) + b$ by associative law $= 1 + b$ by complement law

$= 1$ by theorem (2)

(3) $\Rightarrow$ (1) Suppose that $a'+ b = 1$. To show $a+b = b$.

We have, $a + b = 1 * (a+b)$ by identity law

$= (a'+a) * (a+b)$ by assumption

$= a'*a+b$ by distributive law

$= 0 + b$ by complement law

$= b$ by identity law

Thus (1) and (3) are equivalent.

Finally we show that (3) and (4) are equivalent.

(3) $\Rightarrow$ (4). Suppose that $a' + b = 1$. To show $a*b' = 0$

We have $0 = 1' = (a'+ b)' = (a*)' = b'$ by De Morgain's law

$= a * b'$ by involution law

Thus (3) $\Rightarrow$ (4)

Suppose that $a*b' = 0$. To show $a' + b = 1$

We have $1 = 0' = (* b')'$ by assumption

$= a' + (b')$ by De Morgain's law

$= a' + b$

Thus (3) and (4) are equivalent. Consequently, all four are equivalent.

9.5 Sub-algebra:

Let (B, +, ∗, ′, 0, 1) be a Boolean algebra. A non-empty subset S of B is said to be a sub algebra (or a sub Boolean algebra) if S itself is a Boolean algebra with respect to the operation +, ∗ and ′ of B.

From the definition, it is clear that for any Boolean algebra B, the subsets {0, 1} containing identities of + and ∗ and the set B are both sub algebras of B. Observe that the identities of + and ∗ namely 0 and 1 must belong to every subalgebra. For if S is a subalgebra of a Boolean algebra B and $a \in S$ then by complement laws, $a' \in S$ and thus both $a+a' = 1$ and $a * a' = 0$ belong to S.

Theorem 1: A non-empty subset S of a Boolean algebra B is subalgebra of B if and only if S is closed under the three operations of B, i.e., +, ∗ and ′ .

Proof: Suppose that S is sub-algebra of a Boolean algebra B. Then S itself is a Boolean algebra under the three operations +, ∗ and ′ defined on B. Hence S is closed under the three operations. Thus $a, b \in S \Rightarrow a+b \in S$ and $a' \in S$

Conversely, suppose S is closed under the operations +, ∗ and ′ of B. That is,

$a, b \in S \Rightarrow a+b, a*b$ and $a' \in S$. To show S is a sub-algebra of B.

First of all, we show that both 0 and 1 are in S. Since S is non-empty suppose $a \in S$. We have $a \in S$ $\Rightarrow a' \in S$ by assumption S is closed under ′

Now $a \in S$ and $a' \Rightarrow a + a' \in S$ and $a*a' \in S$ because S is closed under + and ∗./

$\Rightarrow 1 \in S$ and $0 \in S$. Thus both identities 0 and 1 are in S.

Now we show that all the four axions $[B_1]$ to $[B_4]$ are satisfied for S.

1. **Commutative law :** Let $a, b \in S$ then $a, b \in B$ and therefore $a+b=b+a$ and $a*b=b*a$
2. **Identity laws :** For any $a \in S$, we have 0 and $1 \in S$, such that $a+0=a$ and $a* 1=a$, $\forall a \in S$

3. Distributive laws : Since operations + and ∗ are distributive over each other for elements of B, therefore they must also distributive over each other for all elements of S.

4. Complement laws : Let a∈S. Then by assumption, a′∈S such that a+a′ = 1 and a∗ a′ = 0. Hence S itself is a Boolean algebra under the operations of B. Thus S is a subalgebra of B.

Example 5: The subset S = (ϕ, {a}, {b,c}, {a,b,c}} of the Boolean algebra B = {ϕ, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b,c}} which respect to union intersection and complementation of sets is a sub-algebra of B.

Example 6: Consider the Boolean algebra B={1,2,5,7,10,14,35,70} Then S = {1, 7, 70} is a sub algebra of B.

Example 7: Let B be any Boolean algebra and a∈B such that a≠1. Then the subset S = {a, a′, 0, 1} is sub-algebra of B.

Solution: We have to show that S is closed with respect to the operations +, ∗ and ′ of B. We know that a+a=a, a∗a, a+1=1 and a∗0=0, ∀a∈B.

we construct composition tables for +, ∗ and ′ for the element of S.

+	a	a′	0	1
a	a	1	A	1
a′	1	a′	a′	1
0	a	a′	0	1

∗	a′	a′	0	1
a	a	0	0	a
a′	0	a′	0	a′
0	0	0	0	0

′	a	a′	0	1
	a′	a	1	0

1	1	1	1	1		1	a	a′	0	1

Since all entries in the tables are element of S, therefore S is closed with respect to operation +, ∗ and ′.

Theorem 2: A non-empty subset S of a Boolean algebra (B, +, ∗, ′ 0, 1) is a sub-algebra of B, if and only if S is closed with respect to operations + and′.

Proof: If S is sub-algebra of (B, +, ∗, ′, 0, 1) then S is closed with respect to operations + and ∗ and′ by theorem 1. Therefore S is closed with respect to operation + and ′.

Conversely, suppose S is closed with respect to operations + and′. To show that S is a sub-algebra, we need to show that S is also closed with respect to ∗. That is, we must show a, b∈S.

a,b∈S ⇒a′, b′∈ S ∵ S is closed w. r. to operation′.

⇒ a′ + b′∈ S ∵ S is closed w.r. to operation +.

= (a′+b′)′ = (a′)′∗ (b′)′ = a∗b. Thus a∗b ∈S.

Hence S is closed w.r. to ∗ also. Thus S is a subalgebra.

Theorem 3: If S_1 and S_2 are two subalgebras of a Boolean algebra B then $S1 \cap S_2$ is also a subalgebra of B.

Proof: Let S_1 and S_2 be any two subalgebra of a Boolean algebra B. We show that $S_1 \cap S_2$ is closed with respect to the operations +, ∗ and ′ of B (although in view of theorem 2, we need to show only for + and ′).

Clearly $S_1 \cap S_2$ is non-empty because 0, $1 \in S_1 \cap S_2$. Let a, $b \in S_1 \cap S_2$. We have a, $b \in S_1 \cap S_2 \Rightarrow$ a, b∈ and a, $b \in S_2$

Now, a, b$\in S_1$and S_2 is subalgebra $\Rightarrow$ a + b$\in S_1$, a $*$ b $\in S_1$ and a′$\in S_1$ similarly. a$*$b$\in S_1 \cap S_2$ and a + b $\in S_2 \Rightarrow$ a+b$\in S_1 \cap S_2$. Similarly, a$*$b$\in S_1 \cap S_2$ and a′$\in S_1 \cap S_2$.

Thus $S_1 \cap S_2$ is a subalgebra of B.

9.6 Isomorphic Boolean Algebras:

Two Boolean algebras B and B′and said to be isomorphic it there exists a bijective mapping f from B onto B′ such that

f(a+b) = f(a) + f(b), f(a$*$b) = f(a) $*$ f(b) and f(a′) = [f(a)]′ for all elements a, b of B.

In other words, two Boolean algebra's are said to be isomorphic if there exists a one-one, onto mapping f: B $\rightarrow$ B' which preserves that there operations in B and B′.

Note: In the above definition, we have used same symbols for operations in B and B′. If necessary, the students can use different symbols to denote the operations in B and B′.

If two Boolean algebras are isomorphic then they must have the same cardinality. If Boolean algebras B and B′ are isomorphic and one of them, say B, is finite then the Boolean algebra B′ must also be finite having the same number of elements as B.

Example 8: Let B = {0, 1} and operationis +, $*$ and ′ are defined on B as follows

+	0	1
0	0	1
1	1	1

$*$	0	1
0	0	0
1	0	1

′	0	1
	1	0

Then B is a Boolean algebra. Also, consider the set B′ = {a, b} together with the operations +$*$, $*$′ and – as follows

+′	a	b
a	a	b
b	a	b

a′	a	b
a	a	a
b	a	b

-	a	b
	b	a

Then B′ is also a Boolean algebra. The function f: B → B′ defined as f(0) = a and f(10 = b is a bijective mapping which preserves the three operations. Hence Boolean algebras B and B′ are isomorphic to each other.

Example 9: Consider Boolean algebra B of power set of {a, b, c} discussed in the Boolean algebra B′= {1, 2, 5, 7, 14, 35, 70}. B and B′ are isomorphic. In fact, the mapping f: B →B′ defined by

f(ϕ) = 1, f({a})= 2, f({b}) = 5, f({c})=7, f({a, b})=10, f({a, c}) =14, f({b, c})=35 and f({a, b, c})=70 is bijective mapping which preserves the three operations. Hence Boolean algebras B and B′ are isomorphic.

Theorem 4: Let Boolean algebras B and B′ be ismorphic and let f: B→B′ be the isomorphic mapping, then

(i) If 0 is the identity for + in B then f(0) is the identity for + in B′.

(ii) If 1 is the identity for ∗ in B then f(1) is the identity for ∗ in B′.

Proof: (i) Let 0 be the identity for + in B and 0∗ be the identity for + in g′. Then

$f(0) = f(a * a')$ $\quad\because\ a * a' = 0\ \forall\ a \in B.$

$= f(a) * f(a')$ $\quad\because$ f is isomorphic mapping

$= f(a) * [f(a)]'$ $\quad\because$ f is isomorphic mapping

= 0^* by complement law. Hence $0^* = f(0)$

(ii) Let 1 and $1*$ be the unit elements in B and B′. Then $f(1) = f(a+a')$ ∵ $a+a' = 1$

= $f(a) * f(a')$ ∵ f is isomorphic mapping

= $f(a) + [f(a)]'$ ∵ f is isomorphic mapping

= 1^* by complement law. Thus $1^* = f(1)$

Example 10: Prove that no Boolean algebra can have three distinct elements.

Solution: Let B be a Boolean algebra having three elements. Then B must have two distinct elements 0 and 1 as identities for the operations + and ∗ respectively. Let a be the third element of B. Since B in a Boolean algebra, there exists an element a′ in B such that $a + a' = 1$ and $a * a' = 0$

Now there are three cases : (i) $a' = a$ (ii) $a' = 0$ (iii) $a' = 1$

Case (i) If $a' = a$ then $a+a' = 1 \Rightarrow a + a = 1 \Rightarrow a = 1$

And $a * a' = 0 \Rightarrow a * a = 0 \Rightarrow a = 0$. But a is different from 0 and 1.

Therefore $a' = a$ is not possible.

Case (ii) if $a' = 0$. Then $a + a' = 1 \Rightarrow a + 0 = 1 \Rightarrow a = 1$.But a is not equal to 1. Hence $a' = 0$ is not possible.

Case (iii) if $a' = 1$. Then $a * a' = 0 \Rightarrow a * 1 = 0 \Rightarrow a = 0$

Thus $a' \neq 1$ because a is not equal to 0.

Therefore B either has only two elements 0 and 1, or B has four elements because if there is an element a in B different from 0 and 1, then B must have another element f different from 0, 1 and a. Hence no Booleam algebra can have exactly three elements.

Example 11: Prove that for any, a, b and c in a Boolean algebra the following are equal.

(**a**) $(a+b)(a'+c)(b+c)$ (**b**) $ac + a'b + bc$ (**c**) $(a + b)(a'+c)$ (**d**) $ac + a'b$

Solution: We have $(a+b)(a'+c)(b+c) = (a+b)(a'b+c)$ by distributive law

$=a(a'b+c)+b(a'b+c)$ by distributive law

$=aa'b+ac+ba'b+bc$ by distributive law

$=(aa')b+ac+a'bb+bc$ by commutative and associative law

$=0b + ac + a'b+bc$ $\because aa' = 0$ and $bb=b$ $=ac+a'b+bc$ $\because 0b=0$

Thus (a) and (b) are equal

Now we show that (c) is equal to (b).

$(a+b)(a'+c) = a(a'+c) + b(a'+c)$ by distributive law

$= aa' +ac+ba'+bc$ by distributive law

$= 0 + ac + a'b + bc$ by commutative and complement law

Thus (b) is equal to (c)

Finally, we show that (b) is equal to (d)

$ac+a'b+bc = ac + a'b + (a+a')bc$ $\because a+a'=1$

$= ac+a'b+abc+a'bc$ by distributive law

$= (ac+abc)+(a'b+a'bc)$ by associative and commutative law

$= ac(1+b)+a'b(1+c)$ $\because 1+b =1$ and $1+c =1$

$= ac + a'b$ by absorption law. Hence (b) and (d) are equal

Thus all the four are equal.

Example 12: In any Boolean algebra, show that

(1) $(a+b)(b+c)(c+a) = ab + bc + ca$

(2) $(a+b')(b+c')(c+a') = (a'+b)(b'+c)(c'+a)$

Solution (1) L.H.S $= (a+b)(b+c)(c+a)$

$= (a+b)[(b+c)(c+a)]$ by associative law

$= (a+b)[(c+b)(c+a)]$ by commutative law

$= (a+b)[c+ba]$ by distributive law

$= a(c+ba) + b(c+ba)$ by distributive law

$= ac + aba + bc + bba$ by distributive and associative laws

$= ac + (aa)b+bc+cbb)a$ by commutative and associative law $\because aa = a$

$= ac + ab + bc+ba$ by associative and commutative laws

$= ab+bc+ac$

R.H.S. (2) $(a+b')(b+c')(c+a') = [(a+b')(b+c')](c+a')$

$=[(a+b')b+(a+b')c'](c+a') = (ab + b'b+ac'+b'c')(c+a')$

$= (ab+0+ac'+b'c')(c+a') = (ab + ac'+b'c')(c+a')$

$=(ab + ac'+b'c')c+(ab+ac'+b'c')a'= abc+ac'c+b'c'c+aba'+ac'a'+b'c'a'$

$= abc+0+0+0+0+b'c'a' = abc + a'b'c'$

Similarly, we can show that $(a'+b)(b'+c)(c'+a) = abc + a'b'c'$

Hence we have $(a+b')(b+c')(c+a') = (a'+b)(b'+c)(c'+a)$

Example 13: in a Boolean algebra, if $a + x= b + x$ and $a + x' = b+x'$ then prove that $a = b$

Solution: We are given that $a + x = b + x$ … (1)

and $a + x' = b + x'$ … (2)

now $a = a + 0$ by identity law

$= a + xx'$ $\because xx' = 0$

$= (a+x)(a+x')$ by distributive law

$= (b+x)(b+x')$ by (1) and (2)

$= b+xx'$ by distributive law

$= b + 0,$

$\because xx' = 0$ $= b$ by identity law

Example 14: In anyBoolean algebra, prove that $b = c$ if and only of both $a+b = a+c$ and $ab= ac$ holds.

Proof: If $b = c$ then we have $a + b = a + c$ and $ab = ac$ both hold

Now we show that $a+b = a+c$ and $ab = ac \Rightarrow b = c$

$b = b(b+a)$ by absorption law

$=b(c+a)$ $\because a+b=a+c \Rightarrow b+a=c+a$

$= bc+ba$ by distributive law

$=bc+ab$ by commutative law

$=bc+ac$ by given conditioin

= (b+a)c by distributive law

= (c+a)c = c by absorption law

Check your progress

1. Write the dual of each of the following

$(a)\ (a * 1) * (0 + a') = 0,\ (b)\ a + a'b = a + b$

Ans: (1) $(a)\ (a + 0) + (1 * a') = 1 (b)\ a(a' + b) = ab$

2. Show that the set B = {0, a, b, 1} together with the operation ∨,∧ and ′ defined by

∨	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

∧	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

′	
0	1
a	b
b	a
1	0

is a Boolean algebra.

3. Show that algebra of sets is a Boolean algebra with respect to suitable operations.

4. Show that a non-empty subset S of a Boolean algebra is a sub algebra if it is closed under ∗ and ′.

5. Show that a mapping f from a Boolean algebra B to another Boolean algebra B′ which preserves the operations + and ′ also preserves the operation ∗.

6. If a and b are element of a Boolean algebra B then show that a=b if and only if ab′ + a′b = 0

7. prove that in any Boolean algebra (a) a+a′b = a+b

(b) if ax = bx and ax′ bx′ then a=b, (c) ab+ab′+a′b+a′b′ = 1

8. Let B_n be the set of n-tuples of the form $(a_1, a_2, \ldots a_n)$ where each a_1 is either 0 or 1. Define suitable operations on B_n so that it becomes a Boolean algebra.

[Hint ; Define $(a_1,a_2, \ldots,a_n)+ (b_1,b_2,\ldots b_n) = (a_1,b_2, a_2,+b_2,\ldots, a_n,b_n)$, $(a_1, a_2,\ldots a_n) * (a_1,b_2,\ldots b_n) = (a_1, b_2, a_2 ,b_2\ldots, a_n,b_n)$ and $(a_1,a_2\ldots,a_n)'=(a'_1, a'_2,\ldots a'_n)$].

9.7 Boolean Algebra as Lattices:

Theorem.1: Let B a Boolean algebra and a, b∈ B. Then a ≤ b if and only if a+b = b.

Proof: We know that ab′=0 is equivalent to a+b=b. Thus a ≤ b if and only if a+b = b

Theorem 2: Let B be a Boolean algebra. Then the relation ≤ defined as a ≤ b if and only if ab′ = 0 is a partial order on B.

Proof: ≤ is reflexive. Since aa′=0 therefore a ≤ a for all a∈ B

≤ is anti symmetric. Let a, b∈b such that

a ≤ b and b ≤ a. Then ab′=0 and ba′=0

Now a = a.1 = a(b+b′) = ab + ab′ = ab+0 = ab + ba′ = ba + ba′ = b(a+a′) = b.1 = b

Thus ≤ is anti symmetric

$\leq$ is transitive. Suppose that $a \leq b$ and $b \leq c$. Then $ab'=0$ and $bc'=0$.

Now $ac' = a.1.c' = a(b+b')c' =(ab + ab')c' =a(bc') + (ab')c'$

$= a.0+0.c' = 0+0=0$. Hence $a\leq c$. Thus $\leq$ is a partial order onB.

Theorem 3: Let B be a Boolean algebra. Then (B, $\leq$) where $\leq$ is defined as $a \leq b$ if and only if $ab' = 0$, is a lattice. Moreover the identities 0 and 1 are the least and the greastest elements of this lattice.

Proof: We have already shown that (B, $\leq$) is a partial ordered set. To show that (B, $\leq$) is a lattice, we will show that for any elements $a, b \in B$, join of a and b is $a+b$ and meet of a and b is ab. That is, we will show that

Sup $\{a, b\} = a\vee b = a + b$ and inf $\{a, b\} = a\wedge b = ab$, $\forall a, b\in B$

Since $a (a+b)' = a(a'b') = (aa')b'=0b'=0$, $a \leq a + b$

Similarly, $b \leq a + b$

$a \leq a + b$ and $b \leq a + b \Rightarrow a + b$ is an upper bound of the set $\{a, b\}$.

Let c be any other upper bound of $\{a, b\}$. Then $a \leq c$ and $b \leq c$

$\Rightarrow ac' = 0$ and $bc' = 0 \Rightarrow ac' + bc' = 0$

$\Rightarrow (a + b) c' = 0 \Rightarrow a + b \leq c$

Thus $a + b$ is the least upper bound of the set $\{a, b\}$, which by definition, in the joint of a abd be denoted by $a \vee b$. Similarly we can show (or by duality) that ab is the infimum of $\{a, b\}$. Thus inf $\{a, b\} = a \wedge b = ab$.

Hence B is a lattice where + and $'$ are join and meet operations.

Finally, if $a \in B$ then $0\ a' = 0$ and hence $0 \leq a$. This shows that 0 is the least element of B. Similarly $a.\ 1' = a0 = 0$ for all $a \in B$ implies that $a \leq 1$ for all $a \in B$. Thus 1 is the greatest element of B. Thus B is bounded lattice.

Theorem 4: Let $(B, +, ., ')$ be a Boolean algebra. Then the lattice $(B \vee, \wedge)$, where $a \vee b = a + b$ and $a \wedge b = ab$ is bounded, complemented and distributive. Conversely, if $(B, \vee, \wedge)$ is a bounded, complemented and distributive lattice then $(B, +, ., ')$ is a Boolean algebra, where $a+b = a \vee b$, $ab = a \wedge b$ and a' is a complement of a in $(B, \vee, \wedge)$.

Proof: Let $(B, +, ., ')$ be a Boolean algebra. Then $(B, \vee, \wedge)$ is a bounded lattice. Since $\vee$ and $\wedge$ are precisely + and respectively, axioms B_2 and B_4 in definition of a Boolean algebra show that $(B, \vee, \wedge)$ is also distributive and complemented.

Conversely suppose $(B, \vee, \wedge)$ is bounded, complemented and distributive lattice with 0 and 1 as the least and the greatest elements. For $a, b \in B$, define

$a + b = a \vee b$ and $a . b = a \wedge b$

Then the binary operation + and . are commutative with 0 and 1 as their identities. The distributive laws follow from the definition of a distributive lattice. Thus the axioms B_1 to B_3 in the definitions for a Boolean algebra are verified.

Since $(b, \vee, \wedge)$ is a complemented lattice, we can find a complement of each $a \in B$. We denote this complement of a by a'. Now we have $a+a' = 1$ and $aa' = 0$

Thus axiom B_4 is also satisfied. Thus $(B, +, . , ')$ is a Boolean algebra.

Remark: Many authors define a Boolean algebra as a bounded, complemented and distributive lattice. The preceding theorem shows that the definition is equivalent to ours.

9.8 Representation Theorem for Finite Boolean Algebras:

The partial order structure induced on the set B of a Boolean algebra (B, +, . , ′) also enables us to prove the representation theorem for finite Boolean algebras. By a finite Boolean algebra we mean a Boolean algebra with a finite number of elements. We shall show that a finite Boolean algebra has exactly 2^n elements for some n>0. Moreover, there is a unique Boolean algebra of 2^n elements for every n>0.

Let (B, + , . , ′) be a Boolean algebra. (Then (B, ≤) is lattice, where a ≤ b if and only if ab′ = 0. We recall that an element a in B is an atom if it covers 0. In other words, an elements a in B is called an atom if 0 < a and there is no element b in B such that 0 < b and b < a. For example, atoms of a power set Boolean algebra P(S), are precisely singleton subsets of S. As another example, consider the Boolean algebra B = {1, 2, 5, 7, 10, 14, 35, 70} of factors of 70 under the operations l.c.m (for +) and g.c.d (for.) The atoms of this Boolean algebra are precisely 2, 5 and 7.

In the following lemma, we give some simple properties about atoms in a finite Boolean algebra.

Lemma: Let (B, +, . , ′) be a finite Boolean algebra. Then

(i) For every noon-zero element b, there exists atleast one atom a such that a≤b.

(ii) If a and b are distinct aotms then ab = 0

(iii) If b is any non-zero element in B and a_1, a_2, a_3…., a_4 be all atoms of B such that a_i≤ b, i=1….k, then b = $a_1+a_2+\ldots+a_k$ and this representation in unique.

Proof: (i) Let (B, +, . , ′) be a finite Boolean algebra with 0 as the least element. Let b be any non-zero element of B. We shall show that there exists atleast one atoms a in B such that a≤b. If b itself is an atom then we have nothing to do. If b is not an atom then there exists b_1 in B such that 0 < b_1 < b. If b_1 is an atom, we are done. Otherwise there exists b_2 in B such that 0 < b_2< b_1< b. Continuing in this manner, since B is finite, there exists and atom b_i for some i such that 0 < b_i< …..b_2<b_1<b. It follows that for every non-zero element b, there exists atleast one atom a such that a ≤ b.

(ii) Let a and b be two distinct atoms of B. If $ab \neq 0$, there exists an atom c such that $c \leq ab$. Note $ab = a \wedge b = \inf \{a, b\}$ Therefore $c \leq ab \leq a$. Since a itself is an atom it follows that a = c. Similarly b = c. Hence a = b. In other words, if a and b are distinct atoms then ab = 0 .

(iii) Let $a_1, a_2, \ldots, a_k$ be distinct atoms of B such that $a_1 \leq b, a_2 \leq b, \ldots, a_k \leq b$

we claim that $b = a_1+a_2+\ldots+a_k$

since $a_i \leq b$ for each i = 1, 2, …, k $\Rightarrow \sup \{a_1, a_2, \ldots A_k\} \leq b$

$\Rightarrow a_1+ a_1 +\ldots + a_k \leq b$. We now show that $b \leq a_1+ a_2 + \ldots + a_k$

for notational convenience , let $c = a_1+ a_2+\ldots a_k$.

To show $b \leq c$, we shal show $bc'=0$. $b \leq c$ will follow by (1). If possible, suppose $bc' \neq 0$. Then by (i), there exists an atom a such that $a \leq bc'$

Since $bc' \leq b$ and $bc'c'$ by transitive property of ≤, we have $a \leq b$ and $a \leq c'$

since a is an atom and $a \leq b$, therefore a must be equal t one of the atoms $a_1, a_2, \ldots, a_4$. Thus $a \leq a_1+ a_2+\ldots+a_k=c$

Now $a \leq c'$ and $a \leq c \Rightarrow a \leq c \wedge c' = cc' = 0 \Rightarrow a = 0$, which is impossible because a is an atom. Thus $bc' = 0$ which implies $b \leq c$ by (i)

Now $b \leq c$, $c \leq b$ and ≤ is anti-symmetric give $b = c = a_1+ a_2 + \ldots a_k$

Uniqueness: suppose $b = b_1 + b_2 + \ldots b_r$, where each b_i is an atom and $b_i \neq b_j$ be another representation of b.

$\Rightarrow b_i \leq b$ for each i, because b is the supremum of $b_1, b_2, \ldots b_r$

Now consider an atom b_i, $i \leq i \leq r$. Since $b_i \leq b$, we have $\inf\{b_i, b\} = b_i$

$\Rightarrow b_i\, b = b_i,\ 1 \leq i \leq r$

$\Rightarrow b_i (a_1 + a_2 + \ldots + a_k) = b_i$ $\quad\because a_1 + a_2 + \ldots a_k = b$

$\Rightarrow b_i a_1 + b_i a_2 + \ldots.. b_i a_k = b_i$ by distributive law

$\Rightarrow$ for some a_j, $1 \le j \le k$, $b_i a_j \neq 0 \Rightarrow b_i = a_j$ by (ii)

This shows that every b_i is equal to some a_j and hence representation of b as sum of atoms is unique (except for order).

Corollary: (B, + , …., ′ , 0, 1) be a Boolean algebra, Then sum of all atoms in B equal 1. Now we state and prove representation theorem for finite boolan algebras.

Theorem: Let (B, +, . , ′) be a finite Boolean algebra. Let S be the set of atoms of b. Then show that (B, +, . , ′) is isomorphic to the Boolean algebra (P(S), ∪, ∩-) of power set of S.

Proof: Let (B, +, . , ′) be a finite Boolean algebra and let $S = \{a_1, a_2, \ldots, a_n\}$ be the set of all distinct atoms of B. By the last lemma, every element $x \neq 0$ has a unique representation as a sum of atoms. That is,

$x = a_1 + a_2 + \ldots + a_k$, where a_i, s are atoms $f(0) = \phi$ and $f(x) = \{ a_1, a_2, \ldots.. a_k)$ where

$x = a_1 + a_2 + \ldots. + a_k$ is the unique representation of x as a sum of atoms.

Suppose x, y are any two elements in B. suppose

$x = a_1 + a_2 + \ldots. + a_r + b_1 + b_2 + \ldots. + b_S$

and $y = b_1 + b_2 + \ldots. + b_s + c_1 + c_2 + + \ldots. c_t$

where each a_i, $1 \le i \le r$, b_j, $1 \le j \le s$ and c_k, $1 \le k \le t$ are atoms of B.

Then $x+y = a_1 + a_2 + \ldots. + a_r + b_1 + b_2 + \ldots b_s + c_1 + c_2 + \ldots + c_1$

And $xy = b_1 + b_2 + \ldots + b_s$

Because if a_i and a_j are distinct atoms then $a_i a_j = 0$ and

$a + a = a$ and $a.a = a$ for any $a \in B$.

Hence $f(x+y) = \{a_1, a_2, \ldots, a_r, b_1, b_2 \ldots, b_s, c_1, c_2 \ldots c_t\}$

$= \{ a_1, a_2, .. a_r, b_1, b_2, \ldots b_s\} \cup \{ b_1, b_2 \ldots b_s, c_1, c_2 \ldots c_1\}$ and $f(xy) = \{b_1, b_2 \ldots b_s\}$

$= \{a_1, a_2, .. a_r, b_1, b_2, \ldots b_s\} \cap \{ b_1, b_2 \ldots b_s, c_1, c_2 \ldots c_1\} = f(x) \cap f(y)$

Let $S = \{a_1, a_2, .. a_r, b_1, b_2, \ldots b_s\} = \{d_1, d_2, .. d_p\}$

We claim $z = d_1, d_2, .. d_p$ is the complement of x. For this, we shall show $x + z = 1$ and $xz = 0$. Clearly, $x + z = a_1 + a_2 + \ldots a_r + b_1 + b_2 + \ldots b_s + d_1 + d_2 + \ldots d_p$ is the sum of all atoms of B. Therefore, $x + z = 1$. Also $xz = 0$ because $a_i a_j = 0$ for distinct atoms a_i and a_j. Hence $z = x'$

Now $f(x') = \{d_1, d_2, \ldots, d_p\} = S - \{a_1, a_2, \ldots a_r, b_1, b_2, \ldots, b_s\} = S - f(x)$ complement of $f(x)$ Further, by uniqueness of the representation, we see that f is one and onto.

(To show f in one-one, consider $x, y \in B$ such that $x \neq y$, we can write

$x = a_1 + a_2 + \ldots + a_m$, and $y = b_1, b_2, \ldots, b_k$ as sums of atoms

$x \neq y \Rightarrow a_1 + a_2 + \ldots a_m \neq b_1, b_2, \ldots, b_k \Rightarrow \{ a_1 + a_2 + \ldots a_m \} \neq \{b_1, b_2, \ldots, b_k \}$

$\Rightarrow f(x) \neq f(y)$. Thus f in one-one

To show that f is onto, let $\{a_1 + a_2 + \ldots a_n\}$ be any subset of S. then $x = a_1 + a_2 + \ldots a_n$

is a unique element in B and $f(x) = \{ a_1 + a_2 + \ldots a_n\}$. Hence f is onto.

Hence f is an isomorphism. Thus Boolean algebra $(B, +, . , ')$ and $(P(S), \cup, \cap, -)$ are isomorphic to each other.

Corollary: Every finite Boolean algebra has 2^n elements for some positive integer n.

Proof: Since, by above theorem, every finite Boolean algebra B is isomorphic to power set Boolean algebra P(S) and power set P(S) has 2^n elements, where n is the number of elements in S, the set of atoms of B. Hence B has 2^n elements for some n>0.

Example 15: Consider the Boolean algebra $B = \{1, 2, 5, 7, 10, 14, 35, 70\}$ with +, . and $'$ defined as follows: $a + b = lcm\,(a, b)$, $ab = gcd\,(a, b)$ and $a' = 70/a$.

In this Boolean algebra, atoms are 2, 5 and 7 and B is isomorphic to the Boolean algebra (P(S), ∪, ∩, -), where S = {2,5,7}.

Example 16: in any Boolean algebra B, show that $a \leq b \Rightarrow a + bc == b\,(a + c\,)$, where $a, b, c \in B$.

Proof: We have $a + b = a\,(b + b') + b$

$= ab + ab' + b$ [distributive law]

$= ab + 0 + b \because a \leq b \Leftrightarrow ab' = 0$

$= ab + b$ [identity law]

$= b$ [absorption]. Therefore, $a + bc = (a + b)(a + c)$by distributive law

$= b(a + c)$ in view of the result just proved.

Example 18: Consider the lattice {1, 2, 4, 5, 10, 20} of all positive divisors of 20 under the divisibility relation. Then this lattice cannot be a Boolean algebra because it has six elements and $6 \neq 2^n$ for any integer n > 0. Thus we conclude that divisors of 20 is not a Boolean algebra.

9.9 Boolean Functions:

Let (B, +, ·, ′) be a Boolean algebra. By a constant, we shall mean any symbol, such as 0 and 1, which represents a specified element of B. By a variable, we mean a symbol, which represents an arbitrary element of B.

A Boolean function or a Boolean polynomial is an expression derived from a finite number of applications of the operations +, ·, and ′ to the elements of a Boolean algebra. Expression such as ab, (a′ + b)′ + ab′x + ab, and a′+b′ are Boolean functions. In any Boolean algebra, we know that $2a = a + a = a, 3a = a + a + a = a$ and in general na = a where n is any positive integer.

Also a^2= a. a = a, a^3=a . a. a = a and in general a^k = a, where k is any positive integer. Thus no multiples or powers appear in the Boolean polynomials.

A Boolean expression of n variables $x_1, x_2, \ldots x_n$ is said to be a minterm or a minimal polynomial if it is of the form $f_1(x_1) \cdot f_2(x_2) \cdot f_3(x_3) \ldots f_n(x_n)$

where $f_i(x_i) = x_i$ or x_i' for all i = 1, 2, ….n

for example $x_1.x_2.x_1'. x_2, x_1. x_2'$ are min-terms in two variables x_1 and x_2. Similarly $x_1x_2'x_3$ and x_1' x_2' is an example of minterms in three variables x_1, x_2 and x_3.

Theorem 1: There are exactly 2^n minterms in variables $x_1, x_2 \ldots X_n$ is an expression of the form $f_1(x_1) \cdot f_2(x_2) \ldots, f(x_n)$, where each $f_i(x_i) = x_i$ or x_i' for all i = 1, ……,n

clearly, there are two ways of selecting $f_i(x_i)$ namely x_i or x_i' for each i = 1, ….,n. Thus there are 2^n different minterms in n variables.

9.10 Minimization of Boolean Functions (Karnaugh Map)

In this section, we shall concern with the problem of obtaining a minimal Boolean function equivalent to a given Boolean function. These problems arise in the design of switching circuits because the cost of the circuit, to some extent, depends on the number of switches in the circuit. The goal of minimization is to reduce to minimum number of switches or gate required by a circuit. A general method of simplifying a Boolean function obtain a minimal form is to use basic laws and identities such as a+ab = a of a Boolean algebra.

Example 19: Simplifying the Boolean function

$f = ab'cd + cb + cd' + ac' + a'bc' + b'c'd'$

Solution: Here $f = ab'cd + cb + cd' + ac' + a'b\,c' + b'c'd'$

$$= (ab'd + b + d')c + (a + a'b + b'd')c'$$

$= c(b + d' + ab'd) + c'\,[(a + a')\,(a + b) + b'd']$

using $a + a'b = (a + a')\,(a + b)$

$= c[(b + d' + b')\,(b + d' + ad)] + c'[a + b + b'd']$ using distributive law and $a + a' = 1$

$= c\,(b + d' + a)\,(b + d' + d) + c'(a + b + b'(a + b + d')$

$= c\,(a + b + d') + c'(a + b + d')$

$\because\ d + d'1, b + b' = 1 \quad = a + b + d'$

Form the above, it is clear that the choice of which Boolean laws to use in any particular simplification operation is primarily determined by the skill of the person performing Bolean manipulations and this skill is partly a matter of experience. Because of this difficulty in reducing a Boolean functions to its simplest (minimal) form, a method base on Karnugh maps has been developed.

Karnaugh Maps: The Karnaugh map is a pictorial representation of truth table of the Boolean functions. This method is easy to use when Boolean function has six or fewer variables. Since function of one variable can be simplified easily, there is no need to illustrate it. We illustrate the method when number of variables in a function is 2, 3 and 4.

Case of two variables

We consider the case when the Boolean function f is of two variables, say x and y, in the first figure below, we have constructed a 2×2 matrix of squares with each square containing one possible input combination of variable x and y. The Karnaugh map of the function is the 2×2 matrix obtained by placing 0s and 1s in the square according to whether the functional value is 0 or 1 or the input combination associated with that square. For example, the Boolean function $f(x,y) = xy + x'y$ is represented by the Karnaugh map as shown I second figure below

	x	x'
y	xy	$x'y$
y'	xy'	$x'y'$

	x	x'
y	1	1
y'	0	0

Karnaugh map of $f(x,y) = xy\ +\ x'y$

We now consider the method to obtain minimal form of the function by using Karnaugh map. The application of the Boolean law $xy + x'y = y$, when seen in the context of a Karnaugh map, becomes the replacement of two adjacent squares (squares having one side in common) containing

1s by a rectangle containing two squares. The absorption law $x + xy = x$ has its counterpart on a Karnaugh map as well. It is simply the grouping of adjacent squares into the largest possible rectangle of such squares and we still use the largest rectangle instead of individual squares. Of find minimal form of the function, we first consider all largest rectangles composed of the adjacent squares with 1s in them. From the set of these largest rectangles, the minimum number of rectangles are taken such that every square with 1 is part of atleast one such rectangle.

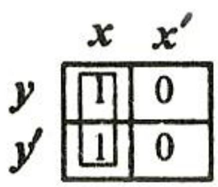
Example 20: use the karnaugh map method to find a minimal DN form (sum of products form) of the following functions

$(\boldsymbol{a})\ f(x, y) = xy + xy'$

$(\boldsymbol{b})\ f(x, y) = xy + x'y + x'y'$

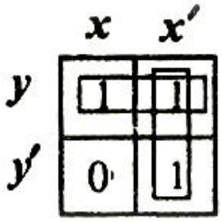
$(\boldsymbol{c})\ f(x, y) = xy + x'y'$

Solution: (a) We first represent $f(x, y)$ by a Karnaugh map. The Karnaugh map representation of $f(x, y) = xy = xy'$is the following



We have represented two adjacent squares with 1s in them by a rectangle. This rectangle represents x, Hence $f(x, y) = x$.

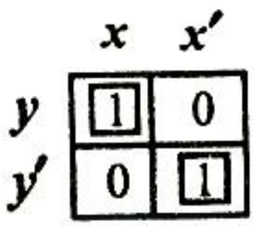
(b) The representation of $f(x, y) = xy + x'y + x'y'$ by Karnaugh map is as follows:



The function f(x,y) contains two pairs of adjacent squares with 1 (indicated by two rectangles) which includes all the squares of f(x,y) which contain 1. The horizontal pair (rectangle) represents y and vertical pair (rectangle) represents x′, Hence.

$f(x, y) = y + x' = x' + y$ is its minimal form.

(c) The Karaugh map representation of $f(x, y) = xy + x'y'$ is given below



Observe that f(x,y) consider of two rectangles as shown in the figure. Thus $f(x, y) = xy + x'y'$ is the minimal form.

Case of three Variables:

We now turn to the case of a function of three variables, say x, y and z. The Kanaugh map corresponding to Boolean functions f(x, y, z0 is shown in figure 3(a)

	xy	xy′	x′y′	x′y
z				
z′				

Fig. 3(a)

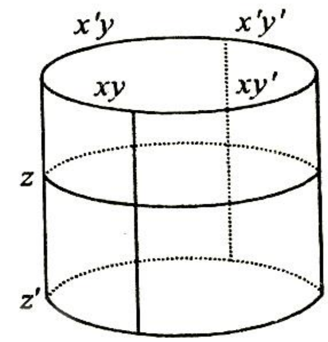


Fig. 3(b)

In figure 3(a), each square represents the minterm corresponding to the column and row intersecting in that square. In order that every pair of adjacent product in figure 3(a) are geometrically adjacent, the right and left edge of the must be identified. This is equivalent to cutting out, bending and gluing the map along the identified edge to obtain the cylindrical figure as shown in figure 3(b), by a basic rectangle in Karnaugh map with three variables, we mean a square, two adjacent squares or four squares which form 1×4 or a 2×2 rectangle.

Suppose that the Boolean function f(x,y,z) has been represented in the Karnaugh map by placing 0s and 1s in the appropriate squares. A minimal form of f(x, y, z0 will consist of the least number of maximal basic rectangles (a basic rectangle which is not contained in any larger basic rectangle) of f which together include all the squares with 1 (in them) of f.

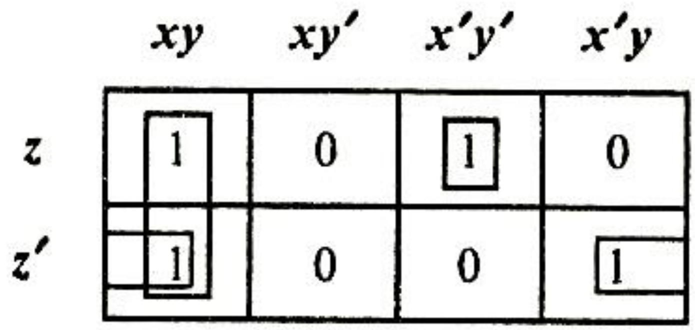
Example 21: Find using Karaugh maps a minimal form for each of the following Boolean functions :

$(\boldsymbol{a})\ f(x,y,z) = xyz + xyz' + x'yz' + x'y'z$

$(\boldsymbol{b})\ f(x,y,z) = xyz + xyz' + xy'z + x'yz + x'y'z$

$(\boldsymbol{c}) f(x,y,z) = xyz + xyz' + x'yz' + x'y'z' + x'y'z$

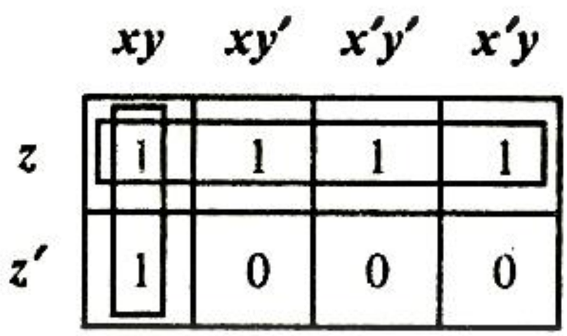
Solution: (a) The Karnaugh map corresponding to the given function is given below



From the Karnaugh map, we see that f(x, y, z) has three maximal basic rectangles containing squares with 1 which are shown by rectangles. Observe that squars corresponding to xyz′ and x′yz′ are adjacent. Thus the symbols are left open ended to signify that they join in one rectangle. The resulting minimal Boolean function is $xy + yz' + x'y'z$

(b) The Karnaugh map corresponding to the function f = xyz+xyz′+xy′z+x′yz+x′y′z

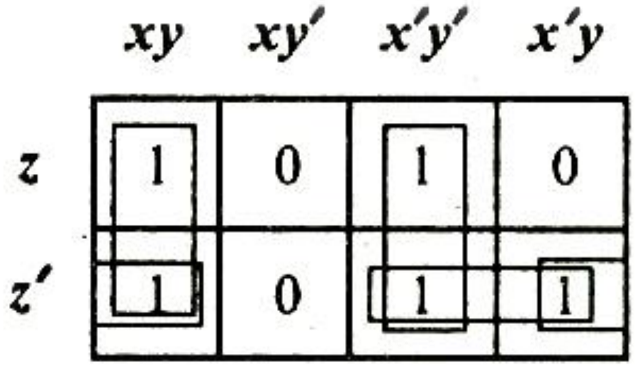
is given below which has five squares with 1s in them corresponding to the five miniterms of f.



From the Karnaugh map, we see tht $f(x, y, z)$ has two maximal basic rectangles containing all the squares with 1, which are shown by rectangles. One of the maximal basic rectangle is the two adjacent squares which represents xy and the other is the 1×4 square which represent z. Both are needed to cover all the squares with 1. So, the minimal form of $f(x, y, z)$ is given by $f(x, y, z) = xy + z$

(c) The Karnugh map corresponding to the function

$f(x,y,z) = xyz + xyz' + x'yz' + x'y'z' + x'y'z$ is given below which has five squares with 1s in the corresponding to the five minterms of f.

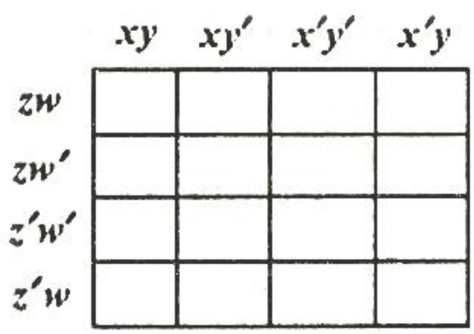


As shown by the rectangles, $f(x,y,z)$ has four maximal basic rectangles. To cover all squares with 1s in them, it is necessary here to include basic rectangles which represent xy and x′y′ and only one of two rectangles which correspond to $x'z'$and yz'. Thus $f(x,y,z)$ has two minimal forms: $f(x,y,z) = xy + x'y' + x'z'$

and $f(x,y,z) = xy + x'y' + yz'$

Case of four variables:

The Karnaugh map corresponding to Boolean function f(x, y, z, w) with four variables x, y, z and w is shown below. Each of the 16 squares corresponds to one of the 16 min-terms with



four variables $xyzw, xyzw', \ldots, x'yz'w$

Here again, we consider the first and last column to be adjacent and the first and last rows to be adjacent, both by wrap around.

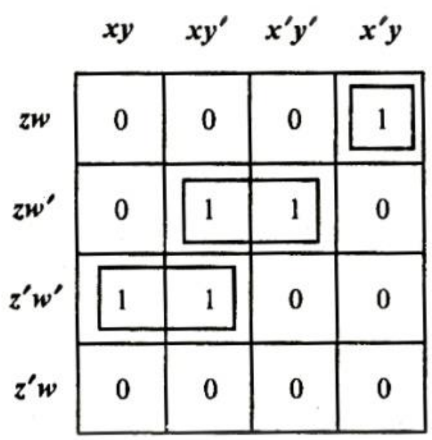
A basic rectangle in a four variable Karnaugh ma is a square, two adjacent squares, four squares which form a 1×4 or 2×2 rectangle or eight squares which form a 2× 4 rectangle. The minimization technique for a Boolean function f(x, y, z, w) is the same as for three variables function.

Example 22: Use Karnaugh maps to find a minimal form for the following Boolean functions

$(\boldsymbol{a})\ f(x, y, z, w) = x'yzw + xy'zw' + x'y'zw' + xyz'w' + xy'z'w'$

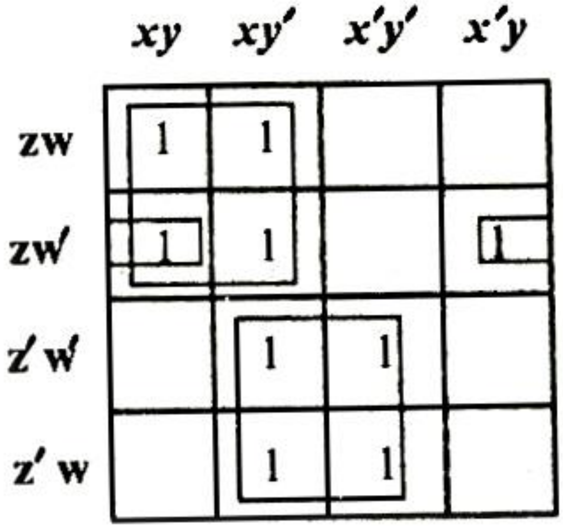
$(\boldsymbol{b})\ f(x, y, z, w) = xy' + xyz + x'y'z' + x'yzw'$

Solution: (a) The Karnaugh map representation of the given function is shown below which has five squares with 1s in the corresponding to the five minterms of f



in the corresponding to the five min-terms of f. A minimal cover of all 1s of the map consists of the three maximal basic rectangles as shown in the figure. Thus the minimal form is $f(x, y, z, w) = y'zw' + xz'w' + x'yzw$

(b) The Karanugh map representation of the given function is shown below. Observe that there are four squares with 1s in them representing xy′. Similarly, there are two squares with 1 representing xyz and so on



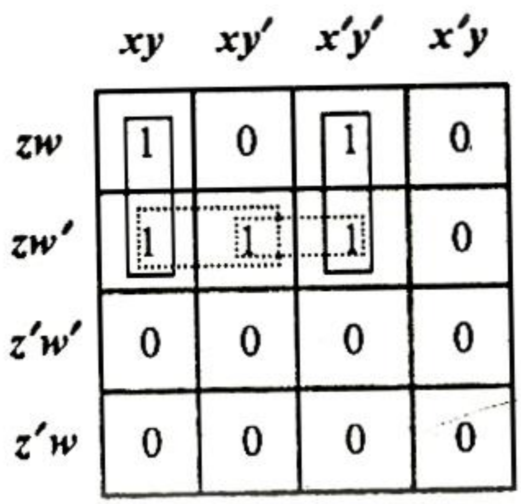
The minimum number of maximal basic rectangles to cover all 1s of the map is 3 as shown in the figure. Thus the minimal form is $f\ (x, y, z)\ =\ xz\ +\ y'z'+\ yzw'$

Observe that the upper left 2×2 rectangle represent xz while the other 2×2 rectangles represents $y'z'$.

Example 23: Use a Karnaugh map to find a minimal form of the function

$$f(x, y, z, w)\ =\ xyzw\ +\ xyzw' +\ xy'zw' + x'y'zw + x'y'zw'$$

Solution: The Karnaugh map of the given function is shown below



As shown by the rectangles, $f(x, y, z, w)$ has four maximal basic rectangles of 1×2 size. To cover all 1s, it is necessary to include basic rectangles which represents xz and $x'y'z$ and only one of the two dotted rectangles which cover 1 at the square corresponding to $xy'zw'$. Hence we obtain two minimal forms, namely

$f\ (x, y, z, w) = xyz + x'y'z + xzw'$ and $f(x, y, z, w) = xyz + x'y'z + y\,zw$.

9.11 Summary:

Let B be a none-empty set with two binary operations + and *, a unary operation′, and two distinct elements 0 and 1. They B is called a Boolean algebra if the following axioms hold for any a, b, c ∈ B.

[B1] Commutative laws: The operations + and * are commutative. In other words,

$a + b = b + a$ and $a + b = b + a, \forall a, b \in B$

[B2] Identity laws: For any a∈B a + 0 =a and a*1 = a

That is, both operations + and * have identity elements denoted by 0 and 1 respectively.

[B3] Distributive Laws: Each binary operation is distributive over the other. That is, for any $a, b, c \in B, a + (b * c) = (a + b) * (a + c)$ and $a * (b + c) = (a * b) + (a * c)$

[B3] Complements laws: For each a in B, there exists an element a in B such that a+a′ = 1 and a*a′=0.

We sometimes denote a Boolean algebra by (B, +, *, ′ , 0, 1). The elements 0 and 1 are called zero element (identity for +) and unit element (identity for *) of B respectively while a′ is called complement of a in B.

Let (B, +, ∗, ′, 0, 1) be a Boolean algebra. A non-empty subset S of B is said to be a sub algebra (or a sub Boolean algebra) if S itself is a Boolean algebra with respect to the operation +, ∗ and ′ of B.

Two Boolean algebra's are said to be isomorphic if there exists a one-one, onto mapping f: B → B' which preserves that there operations in B and B′.

Let B a Boolean algebra and a, b∈ B. Then a ≤ b if and only if a+b = b.

Let (B, + , . , ′) be a Boolean algebra. (Then (B, ≤) is lattice, where a ≤ b if and only if ab′ = 0. We recall that an element a in B is an atom if it covers 0. In other words, an elements a in B is called an atom if 0 < a and there is no element b in B such that 0 < b and b < a.

Let (B, +, ·, ′) be a Boolean algebra. By a constant, we shall mean any symbol, such as 0 and 1, which represents a specified element of B. By a variable, we mean a symbol, which represents an arbitrary element of B.

9.12 Terminal Questions:

1. Describe the Karnaugh maps for three and four variables.

2. Use the Karnaugh map representation to find a minimal form of each of the following functions:

(a) $f(x, y) = x'y + xy$

(b) $f(x,y,z) = xyz + xy'z + x'yz + x'y'z$

(c) $f(x,y,z) = xyz' + xy'z + x'y'z' + x'y'z' + x'yz + x'yz'$

3. Use the Karnaugh map to find a minimal form of each of the following functions:

a. $f = xyz'w' + xyz'w + xyz'w + xy'zw' + x'y'zw + x'y'zw' + x'yw'z'$

b. $f = xyzw' + xy'zw' + xy'z'w' + xy'z'w + x'y'zw + x'y'zw' + x'y'z'w' + x'yz'w'$

4. Find the minimal form of the Boolean function of four variables represented by the Karnaugh map given below:

	xy	xy′	x′y′	x′y
zw	1	0	0	1
zw′	0	0	0	0
z′w′	0	0	0	0
z′w	1	0	0	1

Answer

2. $(a)\ f = y\ (b)\ f = z\ (c)\ f = z' + x'z$

3. $(a)\ f = y'z + xyz' + yz'w'\ (b)\ f = xzw' + xy'z' + x'y'z + x'z'w$ (4). $f = yw$.

UNIT-10: Lattices

Structure

10.1 Introduction

This is most basic unit of this block as it introduces the concept of Lattice. A non empty set A, together with a binary relation R is said to be a partially ordered set or a poset. If it satisfied the condition of reflexive, anti-symmetry and transitive. Two elements a and b is a poset (S, ≤) are said to be comparable if either $a \leq b$ or $b \leq a$ Thus a and b are called incomparable if neither $a \leq b$ nor $b \leq a$. A relation R on a set A is said to be total ordering relation if the relation R is reflexive, anti-symmetric, transitive and satisfies the following relation.For each$a, b \in A$, either a ≤ b or b ≤ a i.e. any two elements of A are comparable.

A graphical representation of a partial ordering relation in which all arrow heads are understood to be pointing upward is known as the Hasse Diagram of the relation.

In Hasse diagram, we represent each element of A as a node or vertex of the graph and we draw a rising line from a to b if b covers$a$. These lines intersected only at the vertices of the graph.

10.2 Objectives

After reading this unit we should be able to

- understand the concept of Partially ordered sets
- use the concept of Hasse Diagram
- understand he Maximal and Minimal element of a partially ordered set
- understand the concept of Lattice, Bounded Lattices
- Distributive Lattices and Complete Lattice

10.3 Partial Order Sets

Partial Order Set (Poset)

A non empty set A, together with a binary relation R is said to be a partially ordered set or a poset. If following condition are satisfied.

(P_1) Reflexivity: $aRa \forall a \in A$

(P_2) Anti-symmetry: If $a, b \in A$, then aRb and $bRa \Rightarrow a = b$

(P_3) Transitivity: If $a, b, c \in A$ then aRb and $bRc \Rightarrow aRc$

The relation R on set A is called partial order relation. The poset is denoted by (A, R) or (A, ≤).

Remarks: We generally use the symbol ≤ in place of R. We read ≤ as less than or equal to (although it may have nothing to do with the usual less than or equal to relation.)

Comparable: Two elements a and b is a poset (S, ≤) are said to be comparable if either a ≤ b or b ≤ Thus a and b are called incomparable if neither a ≤ b nor b ≤ a

Illustration: In poset (Z^+, '|'), the integer 3 and 9 are comparable. Since 3 '|'9. But the integers 5 and 7 are incomparable because neither 5 '|'7 nor 7 '|'5.

Total Ordering Relation or Linear Order or Chain:

A relation R on a set A is said to be total ordering relation if the relation R is reflexive, anti-symmetric, and transitive and satisfies the following relation.

Law of Dichotomy: For each $a, b \in A$, either a ≤ b or b ≤ a i.e. any two elements of A are comparable.

Illustration: The set (Z^+, '|') is not linearly ordered. Since 4 and 7 are incomparable as neither 4 '|'7 nor 7 '|'4. But a = (2, 6, 12, 36) is linearly ordered subset of Z^+. Since 2 '|'6 6 '|'12 and 12 '|'36.

Illustration: The set N of positive integers along with usual order ≤ (less that or equal to) is totally or linearly ordered.

10.4 Hasse Diagram or Partially Ordered Set or Representation of Posets

A graphical representation of a partial ordering relation in which all arrow heads are understood to be pointing upward is known as the Hasse Diagram of the relation.

In Hasse diagram, we represent each element of A as a node or vertex of the graph and we draw a rising line from a to b if b covers a. These lines intersected only at the vertices of the graph. Subgraphs of the types shown in figure 1 do not appear in Hasse diagrams of posets because in each case the left branch says that cover b, whereas the right branch contradicts this on the other hand figure 2 represent distinct posets.

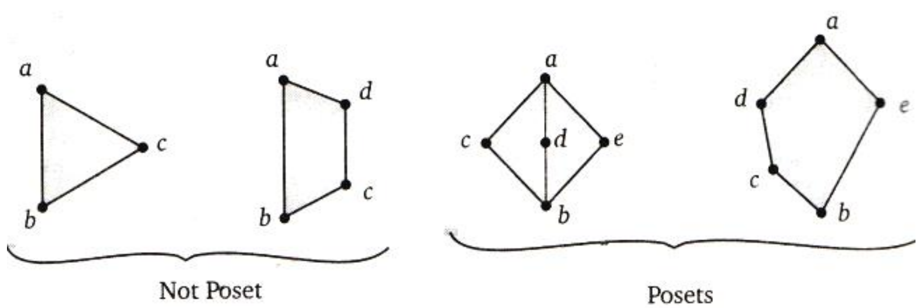


Fig. 1

Procedure for Drawing Hasse Diagram:

We use the following steps for drawing Hasse diagram:

1. Draw the diagraph of given relation.
2. Delete all cycles for digraph.
3. Eliminate all edges that are implied by the transitive relation.
4. Draw the digraph of a partial order with edges pointing upward so that arrows may be omitted from edges.

5. Replace the circles representing the vertices by dots.

Example 1: (i) Draw the Hasse diagram of (A, ≤), Where A = (3, 4, 12, 24, 48, 72) and relation ≤ be such that a ≤ b if a divides b.

(ii) Draw the Hasse diagram of the relation S defined as "divides" on set B where B = (2, 3, 4, 6, 12, 36, 48)

Solution (i) The Hasse diagram is given below (ii) The Hasse diagram is given below

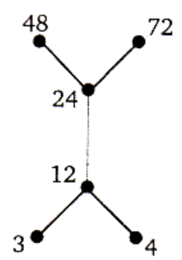


Fig. 2

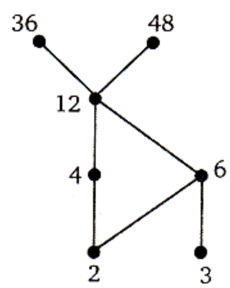


Fig 3

Example 2: Let A be the set of factors of a particular positive integer m and Let ≤ be the relation "divides" i.e. S={ $(x, y): x \in A, y \in A$ and x '│'y} draw Hasse diagrams for

$(a)\, m = 12 (b)\, m = 30 (c)\, m = 45$

Solution: $(a)\, A = \{1, 2, 3, 4, 6, 12\}$ $(b)\, A = \{1, 2, 3, 5, 6, 10, 15\}$

$(c)\, A = \{1, 3, 5, 9, 15\}$

The Hasse diagrams is given below

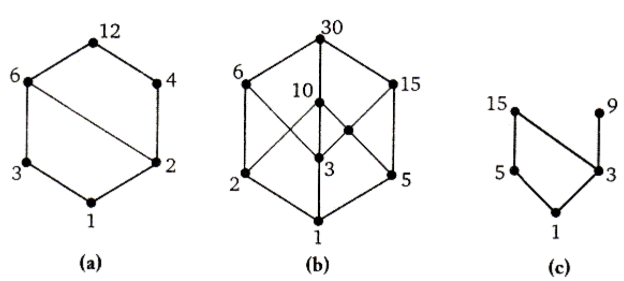


Fig. 4

Example 3: Draw the Hasse diagram for the partial ordering {(A, B): A⊆B} on the power set P(S) where S = {a, b, c}.

Solution: Let $S = \{a, b, c\}, then$

$$P(S) = \{\Phi, (a), (b), (c), (a, b), (a, c), (b, c), (a, b, c)\}$$

The Hasse diagram of the poset {P(S), ⊆} is

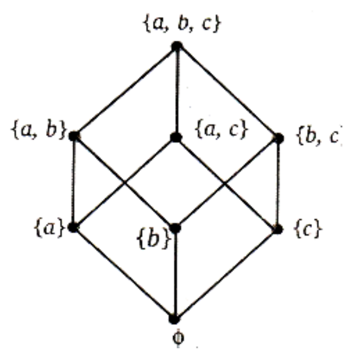


Fig. 5

Example 4: Let N be the set of partial integer. Prove that the relation ≤ where ≤ has its usual meaning, is a positive order relation on N.

Solution: (P$_1$) Reflexivity: For each $a \in N$, we have a is equal to a itself

⇒ a is less than or equal to a itself ⇒ $a \leq a$

Hence, the relation ≤ is reflexive.

(P$_2$) Antisymmetry: Let a, b ∈ N and a ≤ b, b ≤ a then a ≤ b, b ≤ a ⇒a = b

Hence, the relation ≤ is antisymmetry.

(P$_3$) Transitivity: Let $a, b, c \in N$ and $a \leq b, b \leq c$ then $a \leq b, b \leq c \Rightarrow a = c.$

Hence, the relation ≤ is transitive.

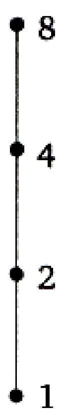
Thus, the relation “less than or equal” denoted by ≤ is a partial order relation on the set N.

Example 5: Draw the Hasse diagrams of $(i)\ (D8, '|')$ $(ii)\ (D6, '|')$

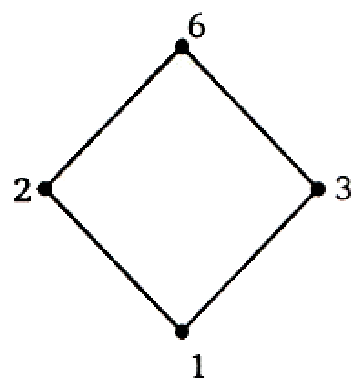
$(iii)\ A = \{2, 3, 5, 30, 60, 120, 180, 360, '|'\}$ $(iv)\ h = \{1, 2, 3, 4, 6, 9, '|'\}$

Solution: The Hasse diagrams of all these posets are given as

(i) We have $D8 = (1, 2, 4, 8)$, Relation ≤ ‘ | ’ = division



$(D_8, '|')$



$D_6, '|'$

Fig. 6

(ii) We have $D_6 = (1, 2, 3, 6)$, Relation ≤ = ‘ | ’ = division

(iii) We have $A = \{2, 3, 5, 30, 60, 120, 180, 360, '|'\}$ relation is divisor i.e. a | b

(iv) We have $H = \{1, 2, 3, 4, 6, 9, '|'\}$ relation is divisor i.e. a | b.

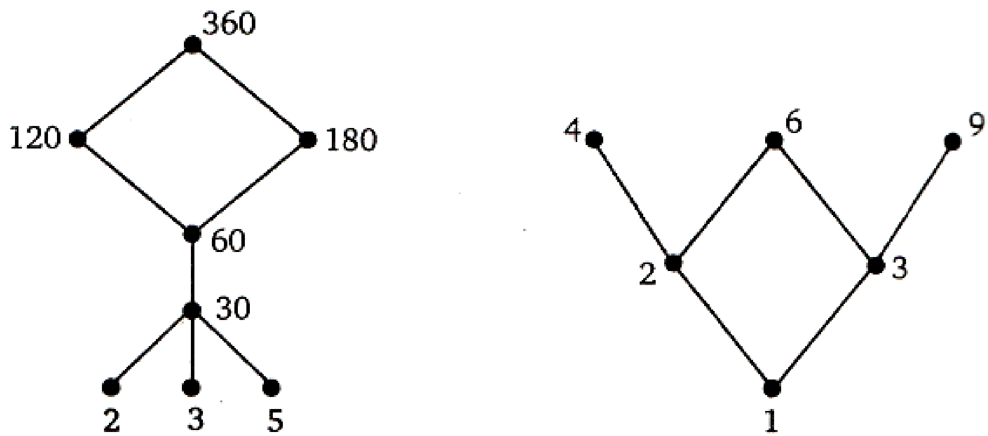


Fig. 7

Example 6: Let $A = \{a, b, c\}$. Show that $(P(A), \subseteq)$ is a poset and draw its Hasse diagram.

Solution: We have $A = \{a, b, c\}$. Then

$$P(A) = \{\Phi, (a), (b), (c), (a, b), (a, c), (b, c), (a, b, c)\}$$

Then $(P(A), \subseteq)$ will be posets if

(P_1) Reflexivity: For each set $B \subseteq P(A)$. We have

$B \subseteq B$ i.e. $B\ R\ B$. So $\subseteq$ is reflexive

(P_2) Antisymmetry: For any $B, C \subseteq P(A)$. we have

$B \subseteq C, \quad C \subseteq B \Rightarrow B = C$

i.e. $B\ R\ C, C\ R\ B \Rightarrow B = C$

So $\subseteq$ is antisymmetric

(P_3) Transitivity: For any $B, C, D \in P(A)$, we have

$B \subseteq C, \quad C \subseteq D \Rightarrow B \subseteq D$

i.e. $B\ R\ C, C\ R\ D \Rightarrow BRD$

So ⊆ is transitive on P(A)

Thus ⊆ is a partial order relation on $P(A)$

The relation R on $P(A)$ is as

R = {(Φ, Φ), (Φ, {a}), (Φ, {b}), (Φ, {c}), (Φ, {a, b}), (Φ, {a, c}), (Φ, {b, c}), (Φ, {a, b, c}), ({a}, {b}), ({a}, {c}), ({a}, {a, b}), ({a}, {a, c}), ({a}, {b, c}), ({a}, {a, b, c}), ({b}, {c}), ({b}, {a, b}), ({b}, {a, c}), ({b}, {b, c}), ({b}, {a, b, c}), ({c}, {a, b}), ({c}, {a, c}), ({c}, {b, c}), ({c}, {a, b, c}), }), ({a, b}, {a, c}), ({a, b}, {b, c}), ({a, b}, {a, b, c}), ({a, c}, {b, c}), ({a, c}, {a, b, c}), ({b, c}, {a, b, c})}

The matrix of above relation R is as follows:

$$M_R = \begin{array}{c} \\ \phi \\ \{a\} \\ \{b\} \\ \{c\} \\ \{a, b\} \\ \{a, c\} \\ \{b, c\} \\ \{a, b, c\} \end{array} \begin{array}{cccccccc} \phi\{a\} & \{b\} & \{c\} & \{a, b\} & \{a, c\} & \{b, c\} & \{b, c\} & \{a, b, c\} \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

Digraph of this matrix M_R is

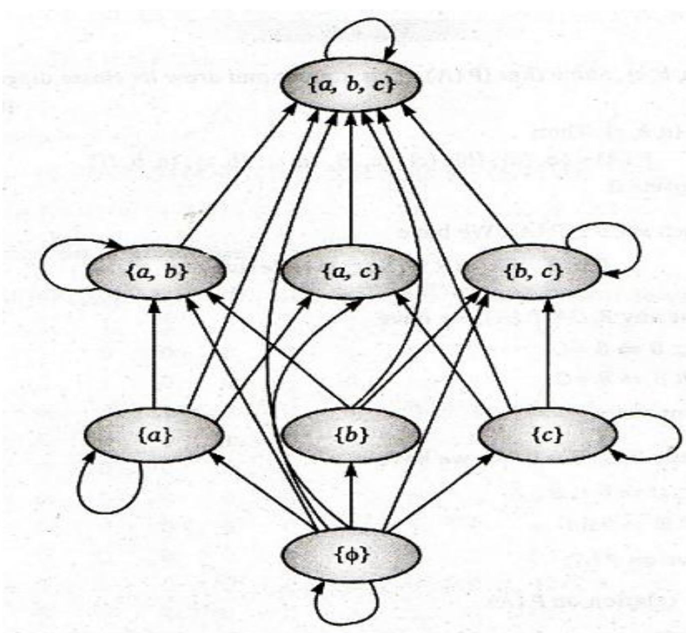


Fig. 8

To convert this digraph into Hasse diagram.

Step 1: Remove Cycles

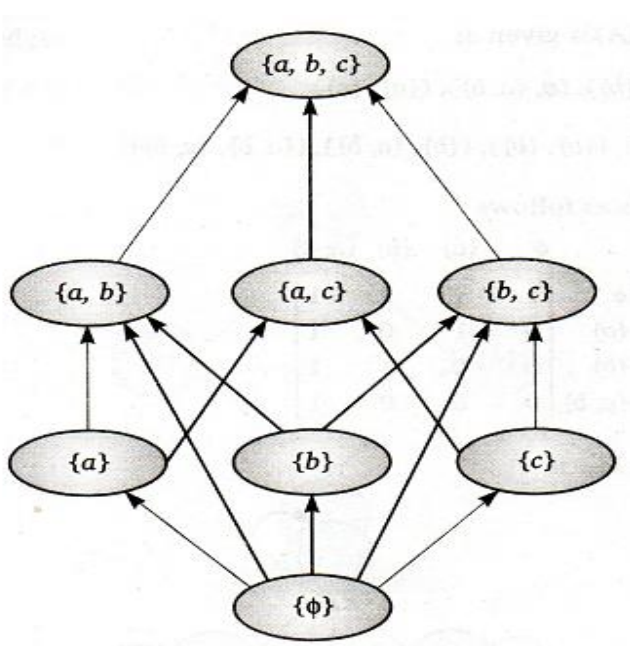


Fig. 9

Step 2: Remove transitive edges

$(\Phi, \{a, b\}), (\Phi, \{a, c\}), (\Phi, \{b, c\}), (\Phi, \{a, b, c\}), (\{a\}, \{a, b, c\}), (\{b\}, \{a, b, c\}), (\{c\}, \{a, b, c\}),$

Step 3: All edges are pointing upwards. Now replace circles by dots and remove arrow from edges.

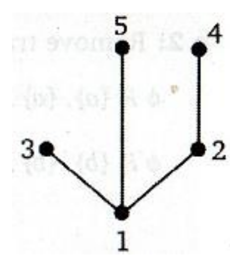
Hence, this is the required Hasse diagram.

10.5 Maximal and Minimal Element:

An element belonging to a point (a ≤) is said to be Maximal element of A if there is no element c in A such that a ≤ c. An element b∈A is said to be minimal element of A if there is no element c in A such that c ≤ b.

Example 8: Let (P, ≤) be a partially ordered set. Where $P = \{1, 2, 3, 4, 5\}$ and ≤ is the relation of division which partially ordered the set P. Draw the Hasse diagram of P.

Solution: The Hasse diagram is shown in figure



The maximal elements of P is 3, 4, 5

The minimal element of P is 1.

Example 9: Find all the maximal and minimal elements of posets whose Hasse diagrams are given in the figure below.

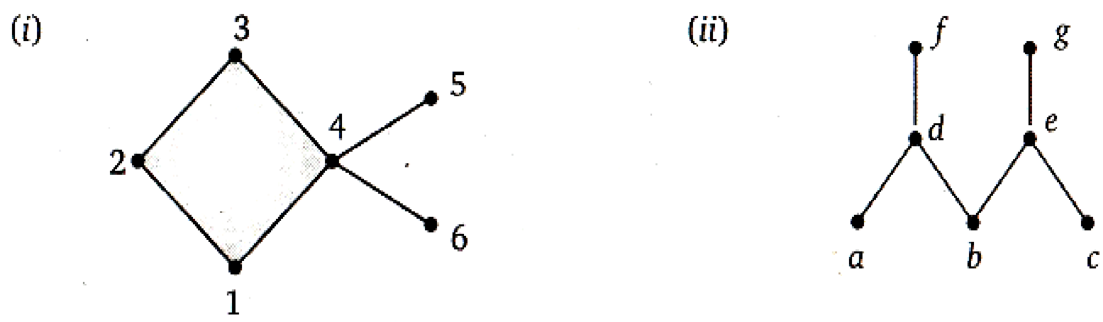


Fig. 10

Solution: (i) $\begin{cases} Maximal\ elements: 3,5 \\ Minimal\ elements: 1,6 \end{cases}$

(ii) $\begin{cases} Maximal\ elements: f, g \\ Minimal\ elements: a, b, c \end{cases}$

Example 10: Find all the maximal and minimal elements of the posets whose Hasse diagrams are given in the figure.

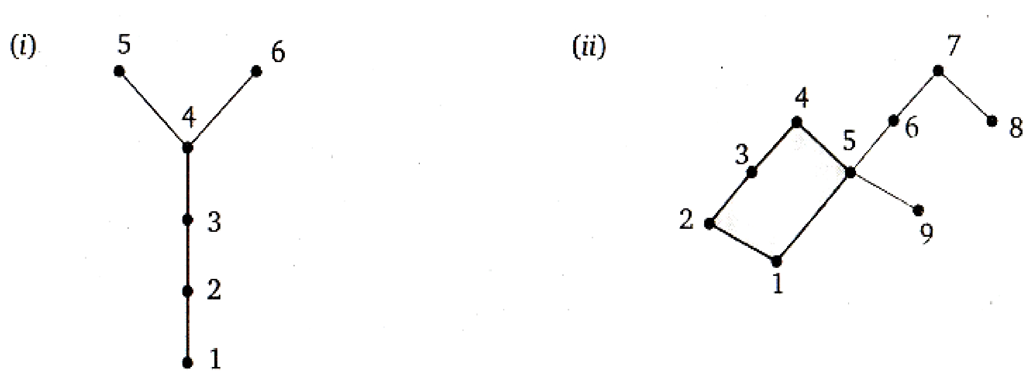


Fig. 11

Solution: (i) $\begin{cases} Maximal\ elements: 5,6 \\ Minimal\ elements: 1 \end{cases}$

(ii) $\begin{cases} Maximal\ elements: 4,7 \\ Minimal\ elements: 1,9,8 \end{cases}$

10.6 Greatest and Least Element:

An element a ∈ A is said to be a greatest element of A if x ≤ a for all x∈A. An element a ∈ A is called a least element of A if a ≤ x for all x ∈ A and $a \in A$ is called generator if $a \geq x, \forall x \in A$.

The least element is also called first element or zero element of A. The least element if exists is unique. It may happen that the least element does not exist. the least element is generally denoted by 0.

The greatest element is also called last element or unit element of A. the greatest element if exists is unique. It may happen that the greatest element does not exist. The greatest element is generally denoted by 1.

Note: Greatest element is also called as universal upper bound. Least element is also called as universal upper bound.

Example 11: Find the greatest and least elements of following Hasse diagrams.

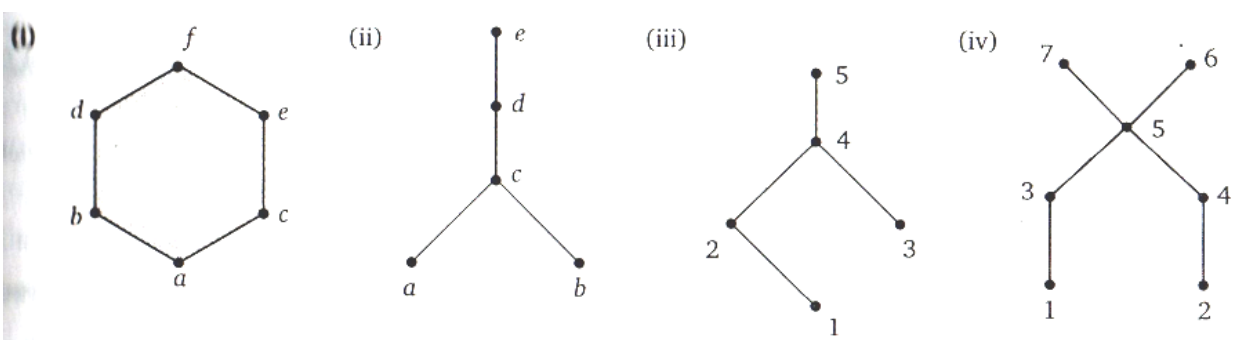


Fig. 12

Solution: (i) $\begin{cases} Greatest\ element = f \\ Least\ element \quad = a \end{cases}$ (ii) $\begin{cases} Greatest\ element = e \\ Least\ eleme \quad = none \end{cases}$

(iii) $\begin{cases} Greatest\ element = 5 \\ Least\ eleme \quad = none \end{cases}$ (iv) $\begin{cases} Greatest\ element = none \\ Least\ eleme \quad = none \end{cases}$

Upper Bounds and Least Upper Bound: Let (P, ≤) be a poset and let A be a subset of P. An element x ∈ P is called an upper bound of A if $a \leq x \; \forall a \in A$. Let (P, ≤) be poset and A be a subset of P. An element x ∈ P is called an lower bound of A if x ≤ $a \; \forall a \in A$.

Least upper bound or supremum: Let (P, ≤) be a poset and let A ≤ P. An element x ∈ P is said to be a least upper bound or supremum of A if x is an upper bound of A and x ≤ y for all upper bounds y of A. Least upper bound, if it exists, is unique and will be denoted by "lub" or "sup".

Greatest Lower Bound or Infimum: Let (P, ≤) be a poset and A ⊆ P. An element x ∈ P is said to be a greatest lower bound or infimum of A written as glb (A) or inf(A), if x is a lower bound and y ≤ x for all lower bounds y of A. A greatest lower bound, if it exists is unique.

Illustration: Consider the poset whose diagram is given by

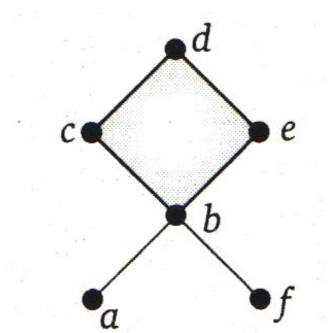


Fig. 13

(i) Upper bound for (c, e) is d lub (c, e) or sup (c, e) = d, lower bound for (c, e) are the elements b, a and f glb (c, e) = inf (c, e) = b

(ii) Lower bound for (a, f) are the elements b, c, e, d lub (a, f) = b = sup (a, f) lower bounds of a, f do not exist.

Example 12: Show that a poset has atmost one greatest and atmost one least element.

Solution: Let a and b are the greatest element of a poset A. Then, since b is the greatest element, we have a ≤ b

Similarly, since a is the greatest element, we have b ≤ a

Then from (1) and (2) b = 1 by antisymmetric property of poset.

Similarly it can be proved for least element. Hence a poset has atmost one greatest and one least element.

Example 13: Consider the set A = {1, 2, 3, 4, 5}. Define the relation < on A such that x < y if (x mod 3) < (y mod 3).

(i) Prove that (A, ≤) is a poset (ii) Draw the Hasse diagram for (A, ≤)

(iii) What are the maximal elements (iv) What are the minimal elements?

Solution: (i) (A, ≤) will be poset if

(a) Reflexivity: Since every element of A is related to itself.

i.e. 1 ≤ 1→ (1 mod 3) ≤ (1 mod 3). Hence it is reflexive

(b) Antisymmetric: if a ≤ b → (a mod 3) ≤ (b mod 3) and b ≤ a → (b mod 3) ≤ (a mod 3). Then, we get a = b. Hence, it is antisymmetric

(c) Transitive: if a ≤ b and b ≤ a. Then $(a \, mod \, 3) \leq (b \, mod \, 3) \; and \; (b \, mod \, 3) \leq (c \, mod \, 3)$

$$\therefore \quad (a \, mod \, 3) \leq (c \, mod \, 3) \; or, \quad a \leq c$$

Hence, it is transitive. therefore (A, ≤) is poset.

(ii) The Hasse diagram for A = {1, 2, 3, 4, 5}

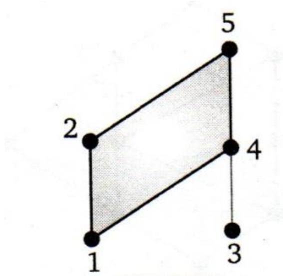


Fig. 14

(iii) $\begin{cases} Maximal\ elements: 5 \\ Minimal\ elements: 1,3 \end{cases}$

Example 14: Consider the partially order set A = {2, 4, 6, 8} where 2 | 4 means 2 divide 4 show with reason whether the following statements are true or false.

(i) Every pair of elements in the poset has a greatest lower bound.

(ii) Every pair of elements in the poset has a least upper bound.

(iii) This poset is a lattice.

Solution: Now first we draw the Hasse diagram under relation of divisor.

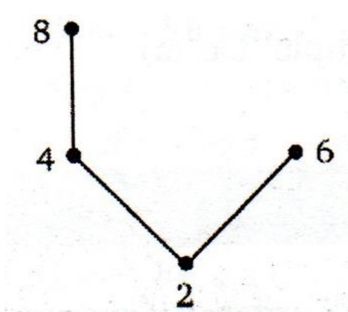


Fig. 15

(i) True its g.l.b. table is

^	2	4	5	8
2	2	2	2	2
4	2	4	2	4
6	2	2	6	2
8	2	4	2	8

Here a^b = H.C.F. of (a, b). Hence every pair of elements has g.l.b.

(ii) False, since there exists no l.u.b. of 6 and 8.

(iii) False, since 6 and 8 has no l.u.b.. this is not a lattice.

Example 15: Consider the poset A = {1, 2, 3, 4, 6, 9, 12, 18, 36, ‘ | ’}. Find the greatest lower bound and least upper bound of the set (6, 18) and (4, 6, 9).

Solution: Hasse diagram under relation of divisibility

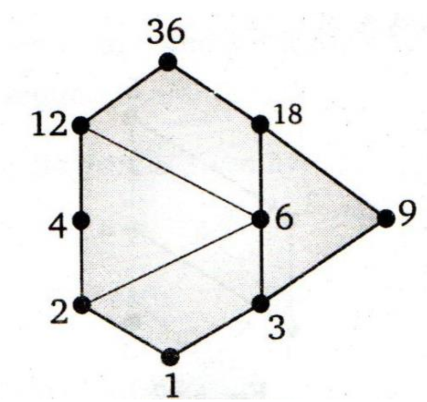


Fig. 16

An integer is a lower bound of (6, 18) if 6 and 18 are divisible by this integer. Only such integers are 1 and 6. since 1 | 6, 6 is the greatest lower bound of (6, 18). Hence glb (6, 18) = 6.

An integer is an upper bound of (6, 18) if and only if its is divisible by 6 and 18 which is 18. Hence lub (6, 18) = 18

The only lower bound of (4, 6, 9) = 1, Hence glb (4, 6, 9) = 1

The only upper bound of (4, 6, 9) = 36, Hence lub (4, 6, 9) = 36

Remark: $\begin{cases} \text{glp (a, b)} = \text{greatest common divisor(g. c. d)} \\ \text{lub (ab)} = \text{lowest common multiple (l. c. m)} \end{cases}$

Well Ordered Set: A partially ordered set (A, ≤) is said to be will ordered if every non-empty subset of S has a least element.

Illustration: The set of real numbers R and the set of all integer I (with the usual ordering ≤) are not well ordered.

Illustration: The set of all positive integer I+ or N is well ordered.

Theorem 1: Every well ordered set is totally ordered.

Proof: Let x, y ∈ X. (X, ≤) being a well ordered set. Then (x, y) contains a first element.

Hence, x ≤ y or y ≤ x. This means that (X, ≤) is totally ordered.

Complete Order: A linear order ≤ on a set X is called complete if every non-empty subset of X which is bounded above has a supremum in X.

Illustration: Every well order is complete.

Dual if a Poset: Let R be relation on a set X. Then converse of R denoted by $\bar{R}$ is a relation on X defined by $a\bar{R}b \Leftrightarrow bRa \forall a, b \in X$. If R is a partial order relation denoted by ≤ on X, then $\bar{R}$ the converse of $\bar{R}$ is denoted by ≥.

Product of Two Posets: Let A and B be two posets.

Then product of two poset is define as A x B = {(a, b):a∈A, b∈B} Under the relation

$(a_1.b_2) \leq (a_2.b_2)$ if $a_1 \leq a_2$ on A and $b_1 \leq b_2$ on B.

10.7 Isomorphic Posets:

Two posets A and B are said to be isomorphic if there is a function $f: A \rightarrow B$ such that f is one-one and onto function. And.

$x \leq y \Leftrightarrow f(x) \leq f(y)$ Or. $xRy \Leftrightarrow f(x)R'f(y)$

Example 16: Let A = {1, 2, 4, 8} and let ≤ be the partial order of divisibility on A, let A' = (0, 1, 2, 3) and let ≤ be the usual relation "less than or equal to" on integers. Show that (A, ≤) and (A', ≤) are isomorphic posets.

Solution: The Hasse diagram of (A, '|') is

Fig. 17

And the Hasse diagram of (A', '|') is

Fig. 18

The mapping $f: A \rightarrow A'$

Since both have same Hasse diagram, every element of A maps to only a single element in A and both have same number of elements. Hence it is one-one onto.

If we define f(1) = 0, f(2) = 1, f(4) = 2, f(8) = 3

Let a $|$ b iff $f(a) \leq f(b)$

1 $| 2 \Rightarrow f(1) < f(2) \Rightarrow 0 < 1$

And 2 $| 4 \Rightarrow f(2) < f(4) \Rightarrow 1 < 2$

Hence $f: A \rightarrow A'$ is an isomorphism

Example 17: Let A = {1, 2, 3, 6} and let ≤ the divisibility relation on A. Let B = {Φ, (a), (b), (a, b)} and let ≤ be the relation ⊆ then (A, ≤) and (B, ⊆) are isomorphic.

Solution: Let A → B is defined as

f(1) = Φ, f(2) = (a), f(3) = (b), f(6) = (a, b)

Then f is bijective isomorphic from (A, ≤) to (B, ⊆). Hence (A, ≤) and (B, ⊆) are isomorphic.

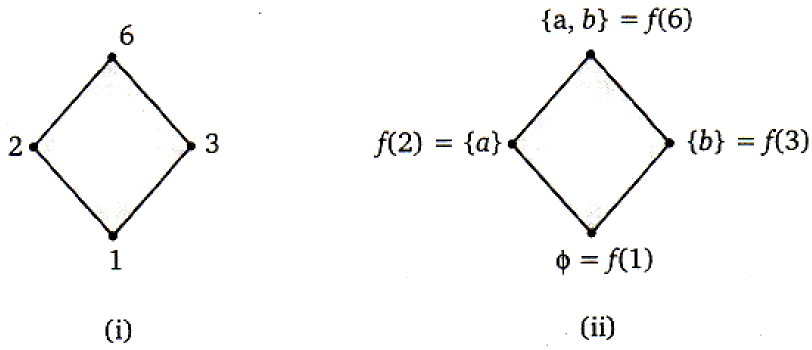


Fig. 19

(i) and (ii) are the Hasse diagram of (A, ≤) and (B, ⊆).

10.8 Lattice:

A partially ordered set (L, ≤) is said to be a lattice if every two elements in the set L has unique least upper bound (sup) and a unique greatest lower bound (inf). i.e.

The poset (I, ≤) is a lattice if for every a, b∈ L sup {a, b} and inf {a, b} exists in L. i.e.

sup (a, b) = $a \vee b = a$ joint b and inf {a, b} = $a \wedge b = a$ meet b.

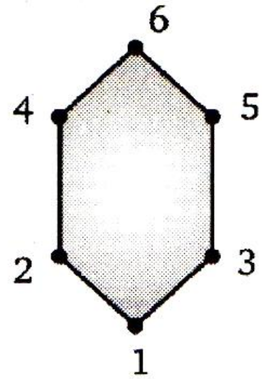
Illustration: The set N of natural numbers under divisibility relation '$|$' formed a lattice in which

$$a \vee b = lcm(a, b) \in N$$

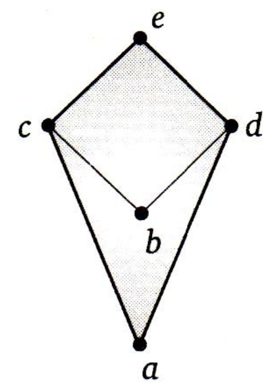
$$a \wedge b = gcd(a, b) \in N$$

Example 18: Determine whether the following Hasse diagrams represent lattice or not.

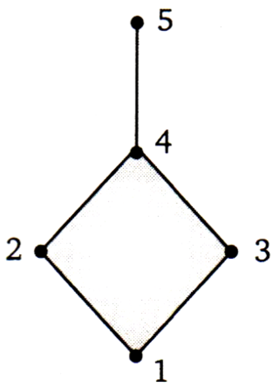
(i)



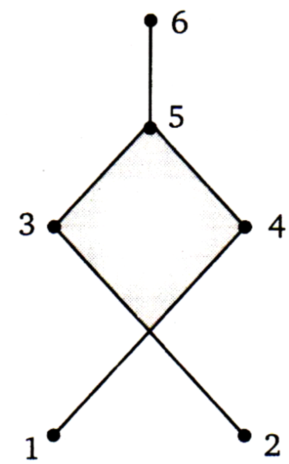
(ii)



(iii)



(iv)



(i) Construct the closure tables for lub(∨) and gib (∧)

lub:

∨	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	2	6	4	6	6
3	3	6	3	6	6	6
4	4	4	6	6	6	6
5	5	6	5	6	5	6
6	6	6	6	6	6	6

glb:

∧	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	1	2	1	1
3	1	1	3	1	3	3
4	1	2	1	4	1	4
5	1	1	3	1	5	5
6	1	2	3	4	5	6

Since each subset of two elements has least upper bound and a greatest lower bound. So this is the lattice.

(ii) Construct the closure tables for lub(∨) and gib (∧)

lub:

∨	a	b	c	d	e
a	a	b	c	d	e
b	b	b	c	d	e
c	c	c	c	e	e
d	d	d	e	d	e
e	e	e	e	e	e

glb:

∧	a	b	c	d	e
a	a	a	a	a	a
b	a	b	b	b	b
c	a	b	c	b	c
d	a	b	b	d	d
e	a	b	c	d	e

Since each subset of two element has least upper bound and a greatest lower bound. so this is the lattice.

(iii) Construct the closure tables for lub(∨) and gib (∧)

lub:

∨	1	2	3	4	5
1	1	2	3	4	5
2	2	2	4	4	5
3	3	4	3	4	5
4	4	4	4	4	5
5	5	5	5	5	5

glb:

∧	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	2
3	1	1	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

Since each subset of two elements has least upper bound and greatest lower bound. so this the lattice.

(iv) Since 1∧3 does not exist so it is not lattice.

Example 19: For any positive integer m, D_m denote the set of divisors of m ordered of divisibility, then (D_m, ‘ | ’) is lattice, where

$$sup(a, b) = lcm(a, b)$$

$$inf(a, b) = gcm(a, b)$$

for any pair a, b in Dm. i.e. D_{36} = {1, 2, 3, 4, 6, 9, 12, 18, 36} is lattice.

Solution: The Hasse diagram of (D_{36}, '|') is

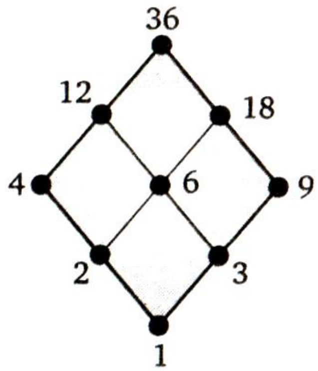


Fig. 20

Since each subset of two elements least upper bound and a greatest lower bound. So this is the lattice.

Dual Lattice: Let (L, ≤) be a lattice, for any a, b ∈ L, the converse of relation ≤, denoted by ≥ is defined as $a \geq b \leftrightarrow b \leq a$. Thus (L, ≥) is also a lattice called dual lattice of (L, ≤).

Properties of Lattices:

Theorem: Let (L, ≤) be a lattice. Then the following results hold.

1. Idempotent Law: for each a ∈ L

(i) $a \wedge b = a$ (ii) $a \vee b = a$

2. Commutative Law: For each a, b ∈ L.

(i) $a \wedge b = b \wedge a$ (ii) $a \vee b = b \vee a$

3. Associative Law: For any a, b, c ∈ L.

(i) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ (ii) $(a \vee b) \vee c = a \vee (b \vee c)$

4. Absorption Law: For a, b ∈ L.

(i) $a \wedge (a \vee b) = a$ (ii) $a \vee (a \wedge b) = a$

Proof: (1) Let a ∈ L. Then

(i) $(a \wedge a) = inf\{a, a\} = a$. Then $a \wedge a = a$

(ii) $(a \vee a) = sup\{a, a\} = a$. Then $a \vee a = a$

(2) Let a, b ∈ L. Then

(i) $(a \wedge b) = inf\{a, b\} = inf\{b, a\} = b \wedge a$

(ii) $(a \vee b) = sup\{a, b\} = sup\{b, a\} = b \vee a$

(3) Let a, b, c ∈ L. Then show

(i) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ (ii) $(a \vee b) \vee c = a \vee (b \vee c)$

Let $x = a \wedge (b \wedge c)$ and$y = (a \wedge b) \wedge c$

Now, $x = a \wedge (b \wedge c) \Rightarrow x \leq a, x \leq b \wedge c$

$\Rightarrow x \leq a, x \leq b, x \leq c$

$\Rightarrow x \leq a \wedge b, x \leq c$

$\Rightarrow x \leq (a \wedge b) \wedge c \leq y$

$\therefore (a \wedge b) \wedge c = a \wedge (b \wedge c)$

(ii) By principle of duality $(a \vee b) \vee c = a \vee (b \vee c)$

(4) For a, b ∈ L. Then show

(i) $a \wedge (a \vee b) = a$ (ii) $a \vee (a \wedge b) = a$

by definition for any a ∈ L, we have

$a \leq a, a \leq a \vee b \Rightarrow a \leq a \wedge (a \vee b)$ ……………… (1)

$a \wedge (a \vee b) \leq a$ …………….. (2)

Then from (1) and (2)

$a \wedge (a \vee b) = a$

By principle of duality $a \vee (a \wedge b) = a$

Theorem: Show that dual of a lattice is a lattice

Solution: Let (L, ≤) be a lattice and let (L, ≥) be its dual, where the relation ≥ is defined as x ≥ y if and only if y ≤ x.

We now show that ≥ is reflexive, anti-symmetric and transitive.

(P_1) ≥ is reflexive: Let a ∈ L. Since ≤ is reflexive, we have

$a \leq a \forall a \in L \Rightarrow a \geq a \forall a \in L \Rightarrow \geq$ is reflexive.

(P_2) ≥ is anti-symmetric: Let a, b ∈ L. be such that a ≥ b and b ≥ a.

Then $a \geq b \text{ and } b \geq a \Rightarrow b \leq a \text{ and } a \leq b$

⇒ a = b [≤ is anti-symmetric]

Thus, a ≥ b and b ≥ a ⇒ a = b. Hence, ≥ is anti-symmetric

(P_3) ≥ is transitive: a, b, c ∈ L be such that a ≥ b and b ≥ c. Then

$$a \geq b \text{ and } b \geq c \Rightarrow b \leq a \text{ and } c \leq b$$

⇒ c ≤ b and b ≤ a ⇒ c ≤ a ⇒ a ≥ c

Thus, a ≥ b and b ≥ c ⇒a ≥ c Hence, ≥ is transitive.

Therefore, ≥ is a partial order relation and L, and so (L, ≥) is a poset.

Let a, b ∈ L. Then since (L, ≤) is a lattice, sup {a, b} exists in (L, ≤)

Let a ∨ b = sup {a, b} in (L, ≤). Then $a \leq a \vee b$ *and* $b \leq a \vee b$

Now $a \vee b \geq a$ *and* $a \vee b \geq b$

⇒a ∨ b is a lower bound of {a, b} in (L, ≥)

We shall show that a ∨ b is the greatest lower bound of {a, b} in (L, ≥)

Let l be any lower bound of {a, b} in (L, ≥), Then $l \geq a$ and

$l \geq b \Rightarrow a \leq l$ *and* $b \leq l$

⇒ l is an upper bound of {a, b} in (l, ≤)

$\Rightarrow lub\,\{a, b\} \leq l\ in\ (L, \leq)$

$\Rightarrow a \vee b \leq l\ in\ (L, \leq)$

$\Rightarrow l \geq a \vee b$

⇒ a ∨ b is greatest lower bound of {a, b} in (l, ≤)

Similarly, we can show that $a \wedge b$ is the least upper bound of {a, b} in (l, ≤). Hence (l, ≥) is a lattice.

Sub Lattice: A non empty subset M of lattice (L, ≤) is said to be a sub lattice of L if M is closed with respect to meet (A) and joint (V) i.e. $x, y \in M \Rightarrow x \vee y \in M$ and $x \wedge y \in M$. i.e. A non empty subset of M of lattice (L, ≤) is said to be sub-lattice of L if M itself formed lattice with respect to $\vee$ *and* $\wedge$ operation.

Isomorphic Lattice: Two lattice L1 and L2 are isomorphic if there exists a none-to-one correspondence $f: L_1 \rightarrow L_2$ such that $f(a \wedge b) = f(a) \wedge f(b)$

and $f(a \vee b) = f(a) \vee f(b) \forall a, b \in L_1$ and $f(a), f(b) \in L_2$

Example 20: Show that the lattice L and L' given below are not isomorphic?

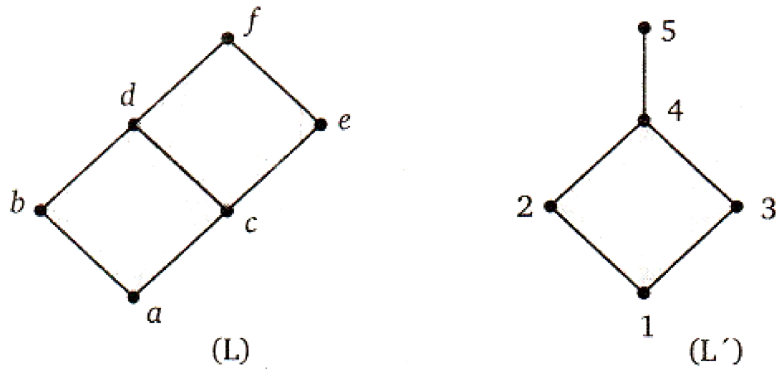


Fig. 21

Solution: Consider the mapping

f = (a, 1), (b, 2), (c, 3), (d, 4), (e, not defined)

Since there is no one-one corresponding between L and L' so L and L' are not isomorphic.

10.9 Bounded and Distributive Lattices

Let (L, ≤) be a lattice. Then L is said to be bounded lattice if it has a least element 0 and a greatest element 1, 0 is called the identity of joint and 1 is called the identity of meet in a bounded lattice (L, ∧, ∨).

A lattice L is called distributive lattice if for any element a, b and c of L, it satisfies the following properties.

(i) $a \vee (a \wedge b) = (a \vee b) \wedge (a \vee c)$

(ii) $a \wedge (a \vee b) = (a \wedge b) \vee (a \wedge c)$

Theorem: Let $a, b, c \in L$ where (L, ≤) is a distributive lattice. Then $a \vee b = a \vee c$ and $a \wedge b = a \wedge c \Rightarrow b = c$

Proof: We know that

$b = b \vee (b \wedge a)$ [absorption]

$= b \vee (a \wedge b)$ [commutative]

$= b \vee (a \wedge c)$ [$a \wedge b = a \wedge c$]

$= (b \vee a) \wedge (b \vee c)$ [distributive]

$= (a \vee b) \wedge (c \vee b)$ [commutative]

$= (a \vee c) \wedge (c \vee b)$ [$a \vee b = a \vee c$]

$= (c \vee a) \wedge (c \vee b)$

$= c \vee (a \wedge c)$ [$a \wedge b = a \wedge c$]

$= c \vee (c \wedge a) = c.$

10.10 Complete Lattice

Let (L, ≤) be lattice. Then L is said to be complete if every subset A of L ∧A and ∨A exist in L. Thus in every complete lattice (L, ≤) there exist a greatest element g and a least element L.

.**Theorem:** For any a and b in a Boolean algebra B.

(i) $(a')' = a$ (ii) $(a \vee b)' = a' \wedge b'$ (iii) $(a \wedge b)' = a' \vee b'$

(iv) $\begin{cases} a \vee (a' \wedge b) = a \vee b \\ a \wedge (a' \vee b) = a \wedge b \end{cases}$

Proof: (i) To show $(a')' = a$

Let complement of $a \in B$ be $a' \in B$ then by definition of compliment

$a \vee a' = 1$ and $a \wedge a' = 0$

Since the operation $\vee$ and $\wedge$ are commutative then. From (1)

$a' \vee a = 1$ and $a' \wedge a = 0$

This show $a \in B$ be the complement of a'

i.e. $(a')' = a$

(ii) To show $(a \vee b)' = a' \wedge b'$

if $a' \wedge b'$ be complement of $a \vee b$ then show

$(a \vee b) \vee (a' \wedge b') = 1$ and $(a \vee b) \wedge (a' \wedge b') = 0$

Now, $(a \vee b) \vee (a' \wedge b') = [(a \vee b) \vee a'] \wedge [(a \vee b) \vee b']$ [by distributive]

$= [a \vee (b \vee a')] \wedge [a \vee (b \vee b')]$ [by associativity]

$= [a \vee (a' \vee b)] \wedge [a \vee 1]$

$= [(a \vee a') \vee b] \wedge 1 = [1 \vee b] \wedge 1$

$= 1 \wedge 1 = 1$

And $(a \vee b) \wedge (a' \wedge b') = [a \wedge (a' \wedge b')] \vee [b \wedge (a' \wedge b')]$

$= [(a \wedge a') \wedge b'] \vee [(b \wedge a') \wedge b'] = (0 \wedge b') \vee (a' \wedge b) \wedge b'$

$= 0 \vee [a' \wedge (b \wedge b')] = 0 \vee [a' \wedge 0] = 0 \vee 0 = 0$

Thus $a' \wedge b'$ is the complement of $a \vee b$ i.e.

$$(a \vee b)' = a' \wedge b'$$

(iii) Applying principle of duality on $(a \vee b)' = a' \wedge b'$

We get $$(a \wedge b)' = a' \vee b'$$

(iv) We have $a \vee (a' \wedge b) = (a \vee a') \wedge (a \vee b)$

$= 1 \wedge (a \vee b) = a \vee b \qquad [a \vee a' = 1]$

Again, $a \wedge (a' \vee b) = (a \wedge a') \vee (a \wedge b)$

$= 0 \vee (a \wedge b) = a \wedge b$

Theorem: Prove that in a distributive lattice, if an element has a complement then this complement is unique.

Proof: Let (L, ≤) be a bounded distributive lattice, Let $a \in L$ having two complement b and c then show b = c. Since b and c be complement of a then

$a \vee b = 1 a \wedge b = 0$ and $a \vee c = 1 a \wedge c = 0$

Now, $b = b \wedge 1 = b \wedge (a \vee c)$

$= (b \wedge a) \vee (b \wedge c)$

$= (a \wedge b) \vee (b \wedge c) \qquad [a \wedge b = b \wedge a]$

$= 0 \vee (b \wedge c) \qquad [a \wedge b = 0]$

$= (a \wedge c) \vee (b \wedge c) \qquad [0 = a \wedge c]$

$= (a \vee b) \wedge c \qquad [a \vee b = 1]$

$= 1 \wedge c = c$

Hence, complement of a is unique.

Example 21: In the lattice defined by the Hasse diagram given by the following figure.

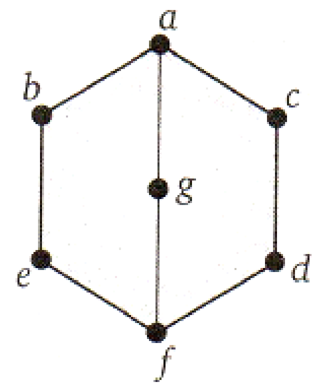


Fig. 22

How many complements does the elements ‘e’ have? Given all.

Solution: Since $e \wedge g = f, e \vee g = a$ and $e \wedge d = f, e \vee d = a$

Where a is universal upper bound and f be universal lower bound. Hence, d, g be two complements of e.

10.11 Summary:

A non empty set A, together with a binary relation R is said to be a partially ordered set or a poset. If following condition are satisfied.

(P_1) Reflexivity: $aRa \forall a \in A$

(P_2) Anti-symmetry: If $a, b \in A$, then aRb and $bRa \Rightarrow a = b$

(P_3) Transitivity: If $a, b, c \in A$ then aRb and $bRc \Rightarrow aRc$

The relation R on set A is called partial order relation. The poset is denoted by (A, R) or (A, ≤).

A graphical representation of a partial ordering relation in which all arrow heads are understood to be pointing upward is known as the Hasse Diagram of the relation.

An element belonging to a point (a ≤) is said to be Maximal element of A if there is no element c in A such that a ≤ c. An element b∈A is said to be minimal element of A if there is no element c in A such that c ≤ b.

An element a ∈ A is said to be a greatest element of A if x ≤ a for all x∈A. An element a ∈ A is called a least element of A if a ≤ x for all x ∈ A and $a \in A$ is called generator if $a \geq x, \forall x \in A$.

The least element is also called first element or zero element of A. The least element if exists is unique. It may happen that the least element does not exist. the least element is generally denoted by 0.

The greatest element is also called last element or unit element of A. the greatest element if exists is unique. It may happen that the greatest element does not exist. The greatest element is generally denoted by 1.

Two posets A and B are said to be isomorphic if there is a function $f: A \rightarrow B$ such that f is one-one and onto function. And.

$x \leq y \Leftrightarrow f(x) \leq f(y)$ Or. $xRy \Leftrightarrow f(x)R'f(y)$

A partially ordered set (L, ≤) is said to be a lattice if every two elements in the set L has unique least upper bound (sup) and a unique greatest lower bound (inf). i.e.

The poset (I, ≤) is a lattice if for every a, b∈ L sup {a, b} and inf {a, b} exists in L. i.e.

sup (a, b) = $a \vee b = a$ joint b and inf {a, b} = $a \wedge b = a$ meet b.

Let (L, ≤) be a lattice, for any a, b ∈ L, the converse of relation ≤, denoted by ≥ is defined as a ≥ b ↔ b ≤ a. Thus (L, ≥) is also a lattice called dual lattice of (L, ≤).

A non empty subset M of lattice (L, ≤) is said to be a sub lattice of L if M is closed with respect to meet (A) and joint (∨) i.e. $x, y \in M \Rightarrow x \vee y \in M$ and $x \wedge y \in M$. i.e. A non empty subset of M of lattice (L, ≤) is said to be sub-lattice of L if M itself formed lattice with respect to $\vee$ *and* $\wedge$ operation.

Let (L, ≤) be a lattice. Then L is said to be bounded lattice if it has a least element 0 and a greatest element 1, 0 is called the identity of joint and 1 is called the identity of meet in a bounded lattice (L, ∧, ∨).

A lattice L is called distributive lattice if for any element a, b and c of L, it satisfies the following properties.

(i) $a \vee (a \wedge b) = (a \vee b) \wedge (a \vee c)$

(ii) $a \wedge (a \vee b) = (a \wedge b) \vee (a \wedge c)$

Let (L, ≤) be lattice. Then L is said to be complete if every subset A of L ∧A and ∨A exist in L. Thus in every complete lattice (L, ≤) there exist a greatest element g and a least element L.

10.12 Terminal Questions

1. Let (P, ≤) be a partially ordered set. Where $P = \{1, 2, 3, 4, 5\}$ and ≤ is the relation of division which partially ordered the set P. Draw the Hasse diagram of P.

2. For any positive integer m, D_m denote the set of divisors of m ordered of divisibility, then (D_m, ‘ | ’) is lattice, where

(f) $sup(a, b) = lcm(a, b)$

(g) $inf(a, b) = gcm(a, b)$

for any pair a, b in Dm. i.e. D_{36} = {1, 2, 3, 4, 6, 9, 12, 18, 36} is lattice.

3. Show that a poset has atmost one greatest and atmost one least element.

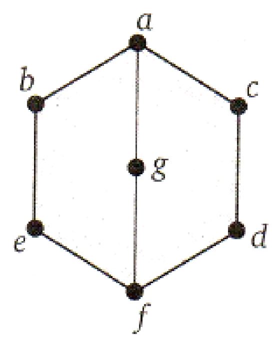
4. Draw the Hasse diagram of the relation S defined as “divides” on set B where B = (2, 3, 4, 6, 12, 36, 48).

5. Draw the Hasse diagrams of $(i)\ (D8, ‘|’)$ $(ii)\ (D6, ‘|’)$

$(iii)\ A = \{2, 3, 5, 30, 60, 120, 180, 360, `|'\}$ $(iv)\ h = \{1, 2, 3, 4, 6, 9, `|'\}$.

6. Consider the poset A = {1, 2, 3, 4, 6, 9, 12, 18, 36, ‘ | ’}. Find the greatest lower bound and least upper bound of the set (6, 18) and (4, 6, 9).

7. In the lattice defined by the Hasse diagram given by the following figure.



How many complements does the elements ‘e’ have? Given all.

Master of Science
PGMM -103N
Discrete Mathematics

U. P. Rajarshi Tandon
Open University

Block

4 Graph Theory

Block-4

Graph Theory

We study about Graph theory of different kinds, Graph theory can be described as a study of the graph. A graph is a type of mathematical structure which is used to show a particular function with the help of connecting a set of points. We can use graphs to create a pair wise relationship between objects. The graph is created with the help of vertices and edges. The vertices are also known as the nodes, and edges are also known as the lines. In any graph, the edges are used to connect the vertices. We can use the application of linear graphs not only in discrete mathematics but we can also use it in the field of Biology, Computer science, Linguistics, Physics, Chemistry, etc. GPS (Global positioning system) is the best real-life example of graph structure because GPS has used to track the path or to know about the road's direction. We will also introduce a pictorial way of describing sets. Knowledge of the material covered in this unit is necessary for studying any mathematics course, so please study this unit carefully.

A tree is a type of graph which has undirected networks. The tree can have only one path to connect any two vertices. British mathematician Arthur ***Cayley*** was introduced the concept of a tree in 1857. The tree cannot have loops and cycles In any graph, the degree can be calculated by the number of edges which are connected to a vertex. The symbol deg (v) is used to indicate the degree where v is used to show the vertex of a graph. So basically, the degree can be described as the measure of a vertex. In any graph, a cycle can be described as a closed path that forms a loop. A cycle will be formed in a graph if there is the same starting and end vertex of the graph, which contains a set of vertices. A cycle will be known as a simple cycle if it does not have any repetition of a vertex in a closed circuit. With the help of symbolC_n, we can indicate the cycle graph. The cycle graph can be of two types, i.e., Even cycle and Odd cycle. A simple graph will be known as the bipartite graph if there are two independent sets which contain the set of vertices. The vertices of this graph will be connected in such a way that each edge in this graph can have a connection from the first set to the second set. That means the vertices of a first set can only connect with the vertices of a second set. Similarly, the vertices of a second set can only connect with the vertices of a first set. But this graph does not contain any edge which can connect the vertices of same set.

UNIT-11: Introduction to Graph

Structure

11.1 Introduction

Graph theory is intimately related to many branches of mathematic. It is widely applied in subjects like, Computer Technology, Communication Science, Electrical Engineering, Physics, Architecture, Operation Research, Economics, Sociology, Genetics, etc. in the earlier stages it was called slum Topology. Euler, Cayley, Sir William Hamilton, Lewin and Kirchhoff, laid foundations to the graph theory.

Graph theory was born in 1736 with Euler's paper on Konigsberg bridge problem. The Konigsberg bridge problem is the best known example in graph Theory. It was a long pending problem. Euler solved this problem by means of a graph. Euler became father of graph theory. Kirchhoff, Cayley, Mobius, Hamilton and De Morgan have laid strong foundation and contributed much to the development of the subject. In this unit basic concepts and terms of graph theory have been introduced.

11.2 Objectives

After reading this unit the learner should be able to understand about:

- to solve problems using basic graph theory.
- to write precise and accurate mathematical definitions of objects in graph theory.
- to use definitions in graph theory to identify and construct examples
- the self-loop, multigraph, simple graph, null graph.
- Weighted graph, finite graph.
- Incidence and Degree and Degree of a Vertex
- Labeled Graph, K-regular Graph and Complete Graph
- Order of a Graph, Size of a Graph and Degree Sequence of a Graph

11.3 Graph

A Graph G is a pair of sets (V, E), where V is non-empty set. The set V is called the set of vertices and the set E is called the set of edges (or lines).

A Graph may be represented by a diagram in which each vertex is represented by a point in the plane and each edge is represented by a straight line (or curve) joining the points. The objects shown below are graphs

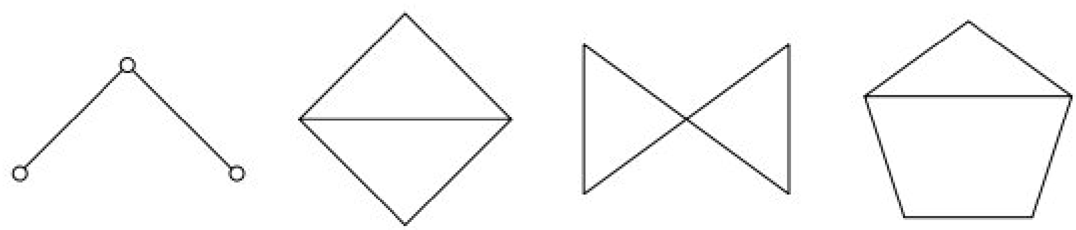


Fig.1

Note:

1. In a graph G = (V, E), the sets V and E are assumed to be finite sets.

2. A vertex of a graph is called a node a point, a junction or 0-cell. An edge of a graph is called a line, a branch a 1-cell or an arc.

3. If G = (V, E); {or (V, E), E(G)} is a graph, then the number of vertices in G is denoted by $|V|(|V(G)|)$, and the number of edges in a is denote$|E|(or\ |E(G)|)$.

4. If (u, v) is an edge in a graph G, then the vertices and v are said to be adjacent.

11.4 Non-Directed Graph:

Let G = (V, E) be a graph. If the elements of E are unordered pairs of vertices of G then G is called a non-directed graph.

The graph in Fig. 2 are non-directed graphs

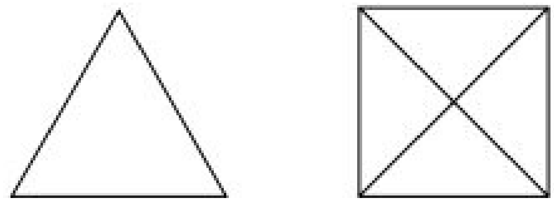


Fig. 2

Note: If e is an edge of a non-directed graph G, connecting the vertices u and v of G, then it is denoted by e = {u, v}.

The point's u and v are called the end points of the edge e.

11.5 Directed Graph (or Digraph)

Let G = (V, E) be a graph. If the elements of E are ordered pairs of vertices, then the graph G is called a directed graph.

Note: If e is an edge of a directed graph G, denoted by e = (u, v), then e is a directed edge in G. The edge e begins at the point u and ends at u. The vertex u is called the origin or initial point of the directed edge e and v is called the destination or terminal point of e. The graphs in Fig. 3 are directed graphs.

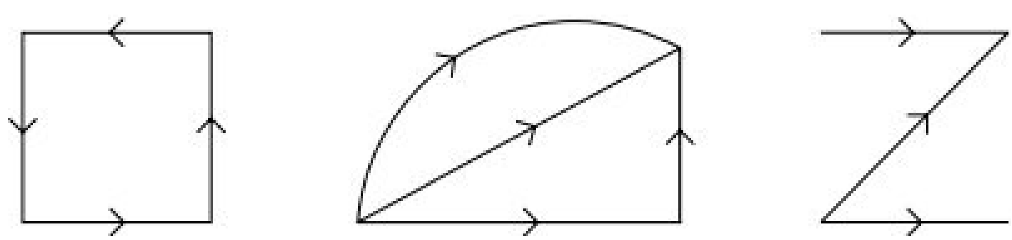


Fig.3 Digraph

11.6 Self-Loop

An edge associated with the unordered pair (v_i, v_i} where $v_i \in V$ of a graph G = (V, E) is called a self-loop in a graph G is an edge joining a vertex to itself.

In Fig.4, there is a loop incident on the vertex v.

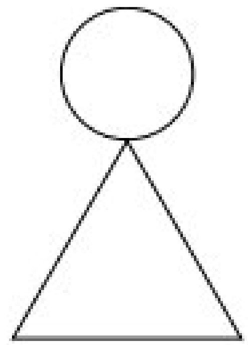


Fig. 4

11.7 Multigraph

A graph which allows more than one edge to join a pair of vertices is called a Multigraph.

Fig. 5(a) is a multigraph in which, we have

G = {{ a, b}, {a, d}, 2 (a, c), (c, d)}

Fig. 5(b) is a multigraph in which, we have

G = {{a, b}, {a, c}, 3 {b, d}, {c, d}, {a, d}}

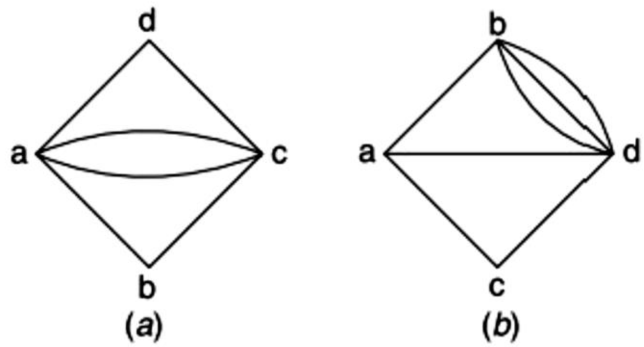


Fig.5 Multigraph

11.8 Simple Graph

A graph G with no self-loops is called a simple graph.

The graphs in Fig.6 are simple graphs.

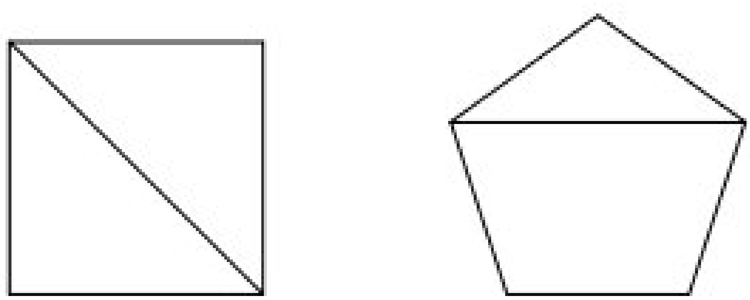


Fig. 6 Simple graphs

Non-Simple Graph:

If graph G is not a simple graph, then it is called a non-simple graph (see Fig.7)

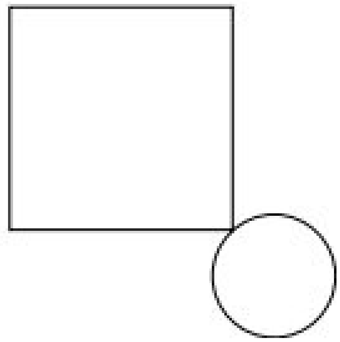


Fig.7 Non Simple graph

11.9 Null Graph

A graph G = (V, E) in which the set of edges E is empty is called a Null graph (see Fig.8)

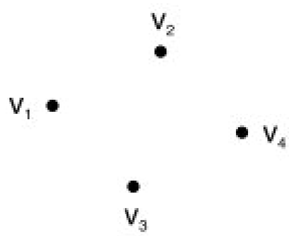


Fig.8 Null Graph

Note: A finite graph with one vertex and no edges is called a trivial graph.

11.10 Weighted Graph

A graph G is in which weight are assigned to every edge is called a weighted graph (Fig.9)

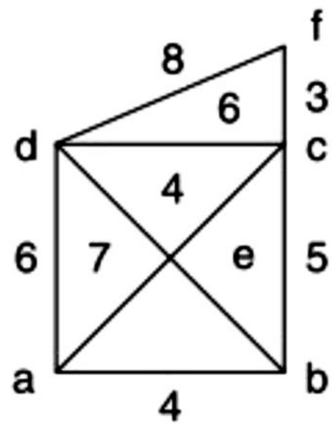


Fig.9 Weighted graph

11.11 Finite Graph

A graph G = (V,E) in which both V (G) and E (G) are finite sets is called a finite graph.

Pseudo graph:

A graph having loops but no multiple edges is called a Pseudo graph (see Fig. 10)

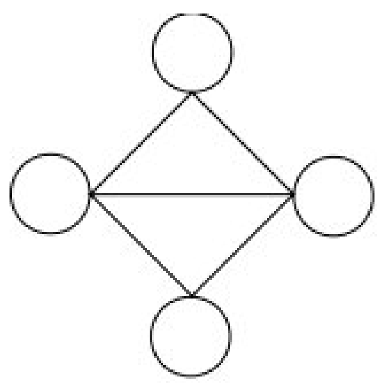


Fig.10 Pseudo graph

11.12 Incidence and Degree

Let G be a non-directed graph. An unordered pair {u, v} is an edge incident on u and v. If G is a directed graph. An edge {u, v} is said to be incident from u and to be incident to v.

lndegree:

In a directed graph G, the number of edges ending at vertex v of G is called the indegree of v.

The indegree of a vertex v of G is denoted by deg $^{+}_{G}(v)$ (or by indeg (v)).

Example: In the graph given below (Fig.11) the indegree of the vertex V_1 is 3:

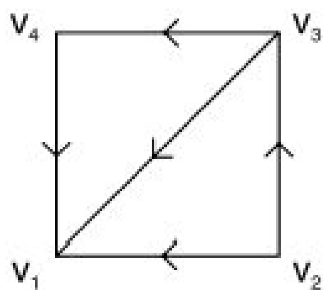


Fig.11

Out degree

Let G be a directed graph and v be a vertex of G. The outdegree of v is the number of edges beginning at v.

The outdegree of a vertex v in G is denoted by deg (; (v) (or by outdegree (v)).

In the Fig.12, the outdegree of the vertex v_1 is 2.

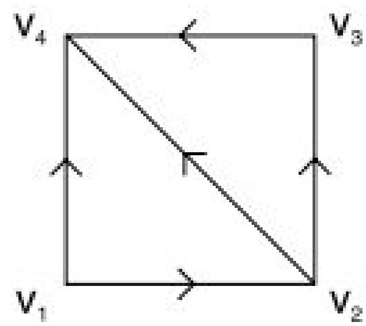


Fig.12

11.13 Degree of a Vertex

In a non-directed graph G, the degree of a vertex v is determined by counting each loop incident on v twice and each other edge once. The degree of the vertex v in G is denoted by d (v) or by deg (v) and is defined as follows:

The number of edges incident with a vertex v of a graph, with self-loops counted twice is called the degree of the vertex v.

Note: A vertex of odd degree is an odd vertex of G and that of even degree is called even vertex.

Example 1: In the graph of Fig.13

deg (v_1) = 1, deg (v_2) = 3, deg (v_3) = 1, deg (v_4) = 1,

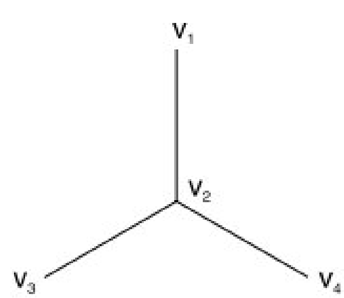


Fig.13

Example.2: In the graph of Fig. 14, we have deg (v_1) = 3, deg (v_2) = 4, deg (v_3) = 2, deg (v_4) = 3 or d_1, = 3, d_2 = 4, d_3 = 2, d_4 = 3

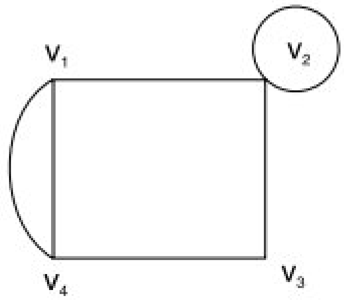


Fig.14

Minimum degree and maximum degree:

For a graph G = (V, E), we introduce the following symbols:

δ (G) = Minimum of all the degrees of the vertices of a graph G.

Δ (G) = Maximum of all the degrees of the vertices of a graph G.

Thus

$\delta (G) = \min \{\deg (v_i): v_i \epsilon V\}$

$\Delta (G) = \max \{\deg (v_i): v_i \epsilon V\}$.

11.14 Labeled Graph

A graph G in which each vertex is assigned a unique label is called a labeled graph. The graph G in Fig.15 is a labeled graph.

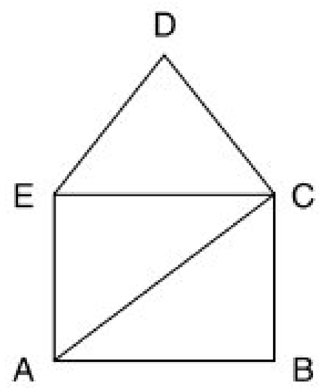


Fig.15 Labeled graph

Isolated Vertex:

A vertex of degree zero in a graph is called an isolated vertex.

The vertex v_4 in Fig.16 is an isolated vertex.

Note: An Isolated vertex in a graph G has no edges incident with it. Every vertex in a null graph is an isolated vertex.

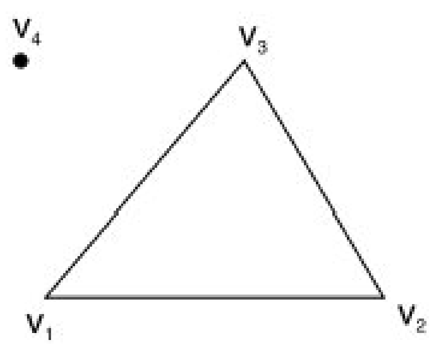


Fig.16

Pendant Vertex:

A vertex of a graph with degree one is called a pendant vertex. (or an end vertex). In the graph shown in Fig.17, the vertices v_1 and v_4 are pendant vertices.

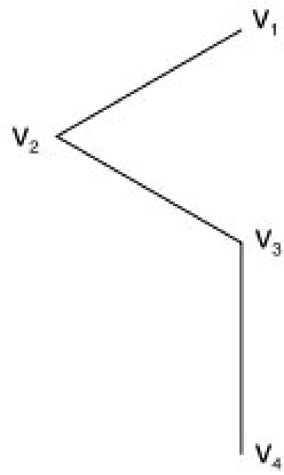


Fig.17

11.15 K-regular Graph

A graph G is said to be k-regular, if every vertex of G has degree k.

Note:

(i) For a k-regular graph

δ (G) = .Δ (G) = K

i.e., all the vertices (points) of G have the same degree K.

(ii) A regular graph of degree zero has no lines.

(iii) In a regular graph of degree I , every component contains exactly one line.

(iii) If G is a 2-regular graph, then every component has a cycle.

(v) If G is a regular graph of degree 3, it is called a cubic graph .(see Fig.18). Every cubic graph has an even number of points.

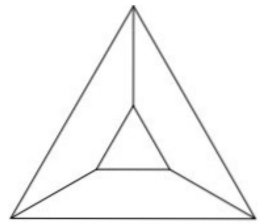


Fig.18 Cubic Graph

11.16 Complete Graph

A simple graph G, in which every pair of distinct vertices are adjacent is called a complete graph. lf G is a complete graph of n vertices then it is denoted by k_n.

Fig.19 shows K_3 K_4 and K_5.

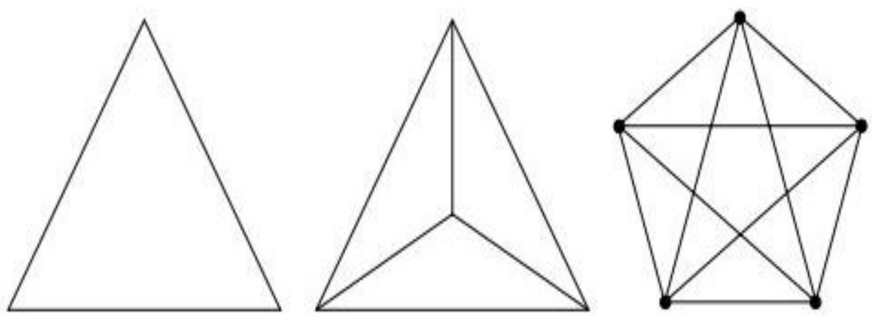


Fig.19 Complete graphs K_3. K_4 and K_5.

Note:

(i) In a complete graph, there is an edge between every pair of vertices.

(ii) K_n.is called (n - I)-regular graph.

(iii) K_n has exactly $\frac{n(n-1)}{2}$ edges.

11.17 Order of a Graph

lf G = (V, E) is a finite then the number of vertices denoted by $|V|$ is called the order of G.

Thus, the cardinality of the vertex set V of G the order of G.

Example: The graph shown in Fig.20 is of order 6.

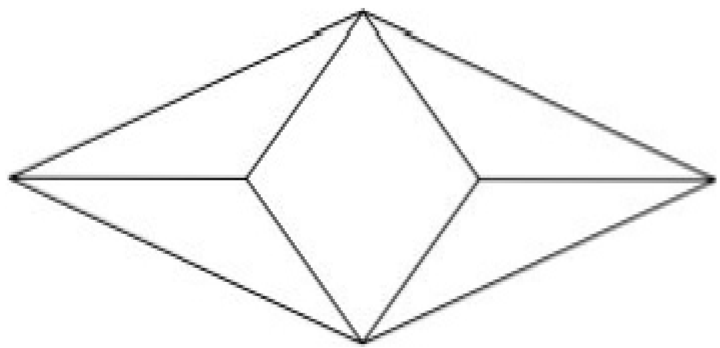


Fig.20

11.18 Size of a Graph

lf G = (V. E) is a finite graph, then the number of edges in G is called the size of G. It is denoted by $|E|$ (cardinality of E).

Example 1: The size of the graph shown in Fig. is 21.

We shall often refer to a graph of order n and size m an (n, m)-graph.

If G is a (p, q) graph, then G has p vertices (points) and q edges (lines).

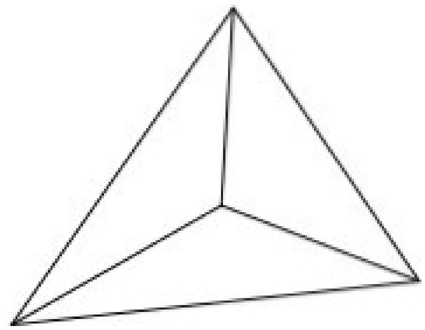


Fig.21

Example 2: Let $V = \{v_1, v_2, v_3, v_4\}$, and

$$E = \{ (v_1, v_2), (v_1, v_3), (v_1, v_4) \}$$

G = (V, E) is a (4, 3) graph G can be represented by the Fig.22.

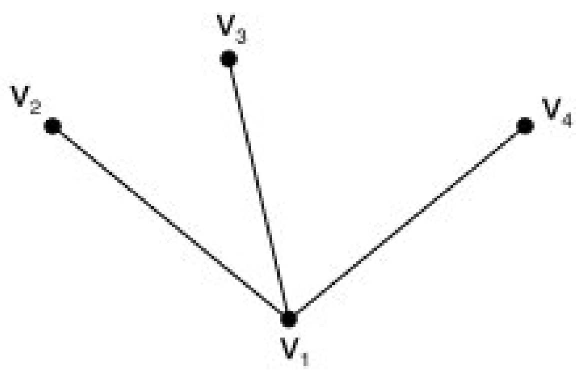


Fig.22

11.19 Degree Sequence of a Graph

Let G be graph with $v = \{v_1, v_2, v_3, v_4, \ldots..v_n \}$ as the vertex set. Also let $d_i = \deg (v_i)$, then the sequence $(d_1 , d_2, \ldots d_n)$ in any order is called the degree sequence of G.

Note:

(i) The vertices of a graph G, are ordered so that the degree sequence is monotonically increasing so that $\delta(G) = d_1 \leq d_2 \leq d_3 \leq \ldots \leq d_n = \Delta(G)$

(ii) The set of distinct non-negative integers occurring in a degree sequence of a graph G is called its degree set.

(iii) Two graphs with the same degree sequence are said to be degree equivalent.

(iv) It is customary to denote the degree sequence in power notation.

If (2, 2, 2. 3, 3, 4. 5, 5, 6) is the degree sequence of a graph G, then it is represented in power notation as $2^3\ 3^2\ 4^1\ 5^2\ 6^1$ the degree set being (1, 2, 3).

Example: Consider the graph shown in Fig.23

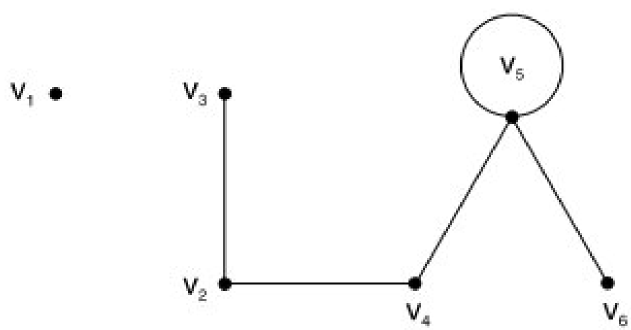


Fig.23

We have d (v_1) = 0, d (v_2) = 2, d (v_3) = 1, d (v_4) = 2, d (v_5) = 4, d (v_6) = 1

v_1 is not adjacent with any other vertex of G, hence v_1 is an isolated vertex of G.

The degree sequence of G is (0, 1, 1 , 2, 2, 4). There are two vertices of odd degree in G (The vertices v_3 and v_6 are odd).

Theorem 11.1: (i) The sum of degrees of the vertices of a non-directed graph G is twice the number of edges in G i.e.,

$$\sum_{i=1}^{n} \deg(v_i) = 2|E|$$

(ii) If G is directed graph

$$\sum_{i=1}^{n} deg_G^-(v_i) = \sum_{i=1}^{n} deg_G^+(v_i)$$

where $|V|$ = number of vertices in G = n

Proof:(i) Let G be a non-directed graph. Each edge of G is incident with two vertices and hence contributes 2 to the sum of degree of all the vertices of the non-directed graph G.

Then the sum of degrees of all the ve1tices in G is t\vice the number of edges in G.

i.e.

$$\sum_{i=1}^{n} \deg_G(v_i) = 2|E|$$

(ii) Let G be digraph and e be an edge associated with a vertex pair (v_p, v_q). The edge e contributes one to the outdegree of v_P and one to the indegree of v_q. This is true for all the edges in G.

Hence

$$\sum_{i=1}^{n} deg_G^+(v_i) = \sum_{i=1}^{n} deg_G^-(v_i) = |E|$$

Corollary 1: In a non-directed graph, the number of vertices of odd degree vertices is even.

Proof: Let G = (V, E) be a non-directed graph. Let W denote the set of odd degree vertices and U denote the set of even degree vertices in G.

Then

$$\sum_{v_i \in V} \deg(v_i) = \sum_{v_i \in W} \deg(v_i) = \sum_{v_i \in U} \deg(v_i)$$

Or

$\sum_{v_i \in V} \deg(v_i) - \sum_{v_i \in U} \deg(v_i) = \sum_{v_i \in W} \deg(v_i)$ …..(1)

$\sum_{v_i \in V} \deg(v_i)$ is even and$\sum_{v_i \in U} \deg(v_i)$ is also even, therefore

$\sum_{v_i \in V} \deg(v_i) - \sum_{v_i \in U} \deg(v_i)$ is even L.H.S. of (1) is even

Thus each deg (v_i) on R.H.S. is odd the number of summands must be even.

∴ The number of odd degree vertices in G is even.

Corollary 2: If K = 8 (G), is the minimum degree of the vertices of a non-directed graph G = (V, E) then

$$K|V| \leq 2|E|$$

in particular , if G is a k-regular graph then

$$K|V| = 2|E|$$

If G is a simple graph, then G is without parallel edges or self-loops. Let G be a simple graph with one vertex. The number of edges in G is zero i.e., (1-1). The maximum degree of a vertex in a simple graph G with one vertex is zero. If G is a simple graph with 2 vertices then the maximum degree of any vertex in a is 1= (2-1). In general it can be shown that the maximum degree of any vertex in a simple graph with n vertices is (n-1). This can be stated in the form a theorem as follows:

Note: The maximum degree of any vertex in a simple with n vertices is n - 1.

Example 1: Draw a simple graph with 3 vertices.

Solution: The graph shown in Fig.24 a simple graph with 3 vertices.

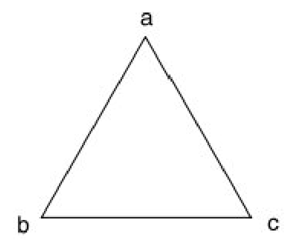


Fig.24

Example 2: Draw a graph representing the problem of three houses and three utilities say water, gas and electricity.

Solution: Let H_1, H_2 and H_3 denote the houses. The utilities, water, gas and electricity be denoted by W, G and E respectively. The houses can be connected by the utilities as shown

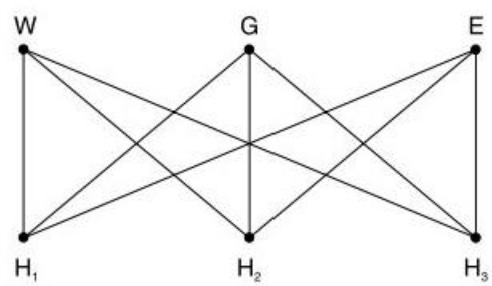


Fig.25

Example 3: Draw the graphs of the chemical compounds.

(a) C_2H_6 (b) C_4H_{10}

Solution: (a) The graph of C_2H_4 is

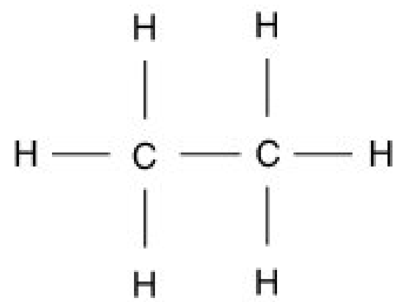


Fig. 26(a)

(b) The graph of C_4H_{10} is

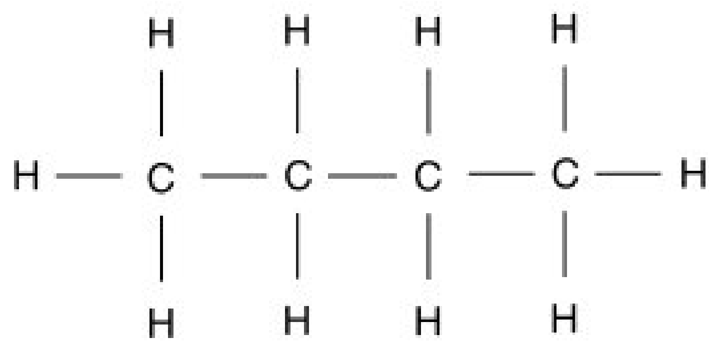


Fig. 26(b)

Example 4: Represent the graph

G = {(l, 2, 3, 4), (x, 4): $|x - 4| \leq$l)}

Solution: We have

V = {l, 2, 3, 4}

E = {(l, 1), (1, 2), (2, 2), (2, 3), (3, 3), (3, 4)}

and G = (V, E) is a non-directed graph. It can be represented as shown in Fig.27.

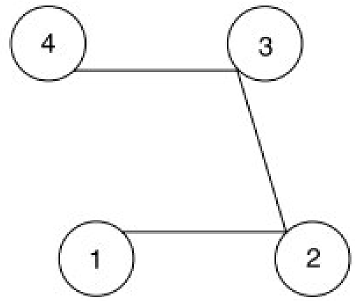


Fig.27

Example 5: For the graph shown in Fig.28. Verify $\sum \deg(v_i) = 2|E|$

Solution: We have V = {v_1, v_2, v_3, v_4, v_5}

E = {e_1, e_2, e_3, e_4, e_5, e_6, e_7)

d_1 = deg (V_1) = 3, d_2 = deg (V_2) = 4, d_3 = deg (V_3) = 2,

d_4 = deg (V_4) = 3, d_5 = deg (V_5) = 1 and $|E|$= 7

$\sum \deg(v_i)$ = $d_1 + d_2 + d_3 + d_4 + d_5$

= 3 + 4 + 3 + 3 + 1= 14

$\therefore \sum \deg(v_i)$ = 14 = 2 x 7 = 2 $|E|$

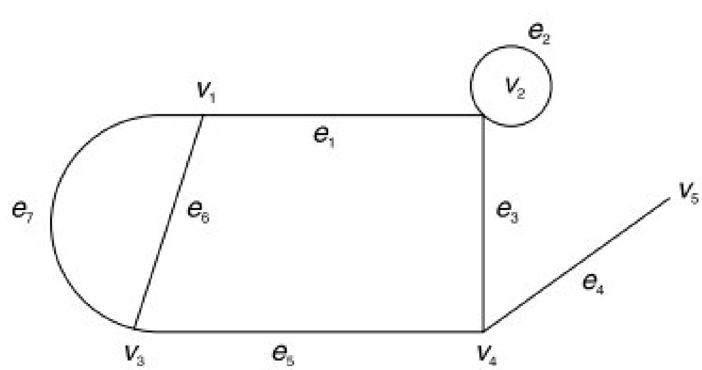


Fig.28

Example 6: A sequence d = (d_1, d_2, ... d_n) is graphic, if there is a simple non-directed graph with the degree sequence d. Show that the following sequences are not graphic:

(i) (2, 3, 4, 5, 6, 7)   (ii) (2, 2, 4)

Solution: (i) The number of odd vertices in the degree sequence is 3 i.e., odd and the number of vertices in G = 6.

Maximum degree of any vertex in a simple graph is = n - 1= 6 - 1 = 5. But the maximum degree in the given degree sequence is 7, therefore the given degree sequence is not graphic.

(ii) The graph contains 3 vertices. The maximum degree in the graph must be (3 - 1) = 2, but the maximum degree in the sequence is 4. Hence, the given degree sequences is not graphic.

Example 7: Draw a picture of the following graph and state whether it is directed or non-directed and whether it is simple:

G = (V, E) where

V = {a, b, c, d, e} and E = {(a, b), (a, c), (a, d), (a, e), (e, e), (c, d), (a, a), (b, .c), (c, c)}.

Solution: The given graph G is a directed graph and is not simple:

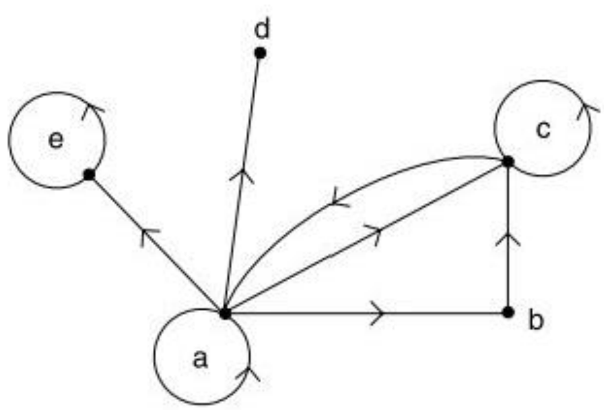


Fig.29

Example 8: Find the order and size of the graph G shown in Fig.30.

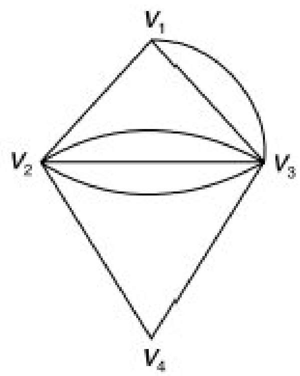


Fig.30

Solution:

$|V|$ = order of G = 4

$|E|$ = size of G = 8

Example 9: Give an example of:

(i) A Simple graph,

(ii) A Pseudo graph, and

(iii) A Multigraph.

Solution:

(i) Figure 8.31 (a) is a Simple graph.

(ii) Figure 8.31 (b) is a Pseudo graph.

(iii) Figure shown in Fig.31 (c) is a Multigraph.

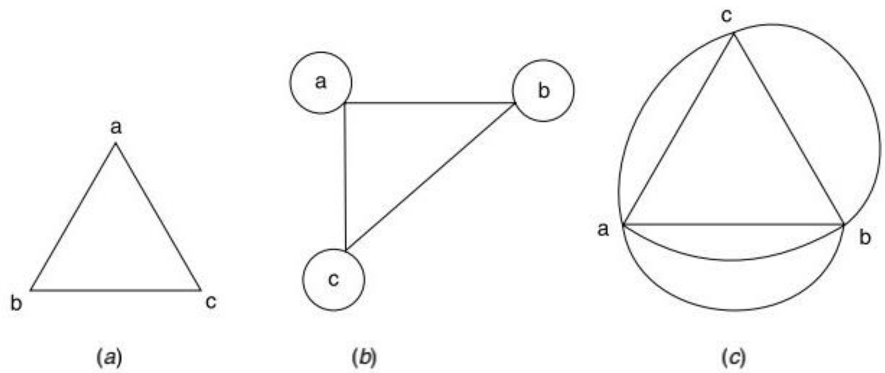


Fig. 31(a) Simple Graph (b) Pseudo Graph (c) Multigraph

Example 10: Draw a non-simple graphs G with degree sequence (1, I, 3, 3, 3, 4, 6, 7).

Solution: G is non-directed, therefore G permits self-loops in it. It be drawn as shown in Fig.32.

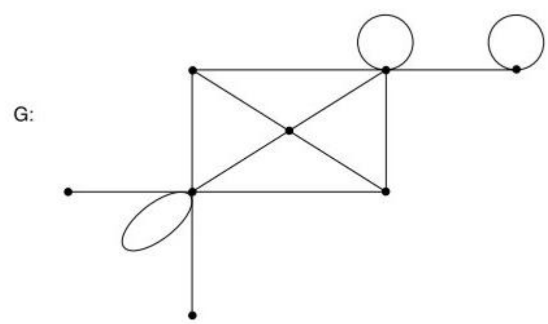


Fig.32

Example 11: Show that every cubic graph bas even number of vertices.

Solution: Let G be a cubic graph with p vertices.

The $\sum \deg(v_i) = 3p$ ... (i)

L.H.S. of (i) is even, therefore R.H.S. i.e., 3p is even hence p is even.

Example 12: If G = (V, E) is a (p, q) graph then show that $\delta \leq \frac{2q}{p} \leq \Delta$

Solution: Let V = {v_1, v_2, ..., v_p} then we have $\delta \le \deg(v_i) \le \Delta$

Or $p\delta \le \sum_{i=1}^{p} \deg(v_i) \le p\Delta$

Or $p\delta \le 2q \le p\Delta$

Hence $\delta \le \frac{2q}{p} \le \Delta$.

Example 13: Suppose is a non-directed graph with 12 edges. If G bas 6 vertices each of degree 3 and the rest have degree less than 3, what is the minimum number of vertices G can have?

Solution: Number of edges in G = 12

Hence $\sum \deg(v_i) = 2|E| = 2 \times 12 = 24$

we have 6 vertices of degree 3. Let n denote the number of vertices each of whose degree is less than 3.

Then $\sum \deg(v_i) < 6.3 + 3x$

Or $24 < 18 + 3x$

Or $3x > 6$

Or $x > 2$

The least positive integer for which the inequality x > 2 bolds is x = 3. Hence, the minimum number of vertices G can have is 3 + 6 = 9.

Example 14: A non-directed graph G bas 8 edges. Find the number of vertices, if the degree of each vertex is 2.

Solution: Given $|E|$= 8

We have

$$\sum_{i=1} \deg(v_i) = 2|E|$$

i.e. $$2|V| = 2 \times 8$$

or $$|V| = 8$$

Thus number of vertices in G = 8

Example 15: Show that a simple graph of order 4 and size 7 does not exist.

Solution: Let G be a group with order 4 and size 7, we have IVI= n = 4 and $|E|$= number of edges = 7 Maximum number of edges in G

$$= \frac{1}{2} n(n-1)$$

$$= \frac{1}{2} 4(4-1)$$

$$= 6$$

Maximum number of edges, G can have is 6

It is given that number of edges in G is 7

$$|E| = 7 > 6$$

which is contradiction Hence, G cannot exist

i.e., there cannot be a Simple graph with order 4 and size 7.

Circuit and Cycle

A closed walk in which no vertex (except its terminal vertices) appear more than once is called a circuit.

A circuit in a graph is a closed non-intersecting walk in which every vertex is of degree two.

A circuit with no other repeated vertices except its end points is called a cycle. The terms a circuit and cycle or synonymous.

In the Fig.33 (a) v_3 - v_4 - v_5 - v_3 is a cycle and in the Fig. 34 (b), v_1 - v_2 - v_4 - v_5 - v_1 is a cycle.

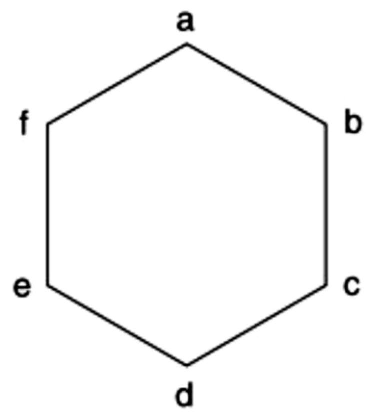


Fig. 33

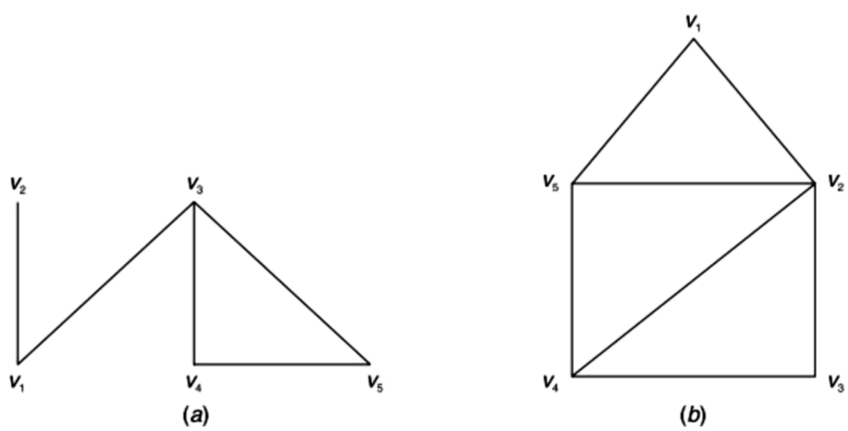


Fig.34

Theorem 11.3: In a graph G, any v_0 - v_n walk contains a path.

Proof: We prove the theorem by induction on the length of the walk.

If the length of the v_0- v_n path 0 or 1, then the walk is obviously a path.

Now, let us assume that the result bolds for all walks of length less than n.

Let v_0, v_1,v_2, ... v_n be a walk of length n. If all the vertices v_i; $1 \leq i \leq n$ are distinct then the walk is a path, if not there exists i and j such that $v_i = v_j$ for some i, j such that ; $1 \leq i \leq j \leq n$.

Now the walk v_0 - v_1 - v_2 - ... v_i, v_{j+r}, .. v_n is a v_0 - v_n walk, whose length is less then n, which by induction hypothesis contains a v_0 - v_n path.

Theorem 11.4: If $\delta(G) \geq K$; then graph G has a path of length k.

Proof: Let v_1 bean arbitrary vertex in G, choose a vertex say v_2 which is adjacent to v 1 Since $\delta(G) \geq K$, there exist atleast k - 1 vertices other them v_1, which are adjacent to v_2. Choose another vertex $v_3 \neq v_1$ such that v_3 is adjacent to v_2• In this may we can find vertices v_4, v_5 , v_6, ..vi where $1 \leq i \leq \delta(G)$. Having chosen the vertices v_1 , v_2, ... v_i, where $1 \leq i \leq \delta(G)$. We can find another vertex v_{i+1} which is different from the vertices v_1 , v_2, ...v_i such that v_{i+1} i is adjacent to v_i. Proceeding in this way. We can find a path of length k in G.

We now state the following theorem without proof:

Note: A closed walk of odd length in a graph G contains a cycle.

11.20 Summary

A Graph G is a pair of sets (V, E), where V is non-empty set. The set V is called the set of vertices and the set E is called the set of edges (or lines). Let G = (V, E) be a graph. If the elements of E are unordered pairs of vertices of G then G is called a non-directed graph.

Let G = (V, E) be a graph. If the elements of E are ordered pairs of vertices, then the graph G is called a directed graph. An edge associated with the unordered pair (v_i, v_i} where $v_i \in V$ of a graph G = (V, E) is called a self-loop in a graph G is an edge joining a vertex to itself.

A graph which allows more than one edge to join a pair of vertices is called a Multigraph. A graph G with no self-loops is called a simple graph. If graph G is not a simple graph, then it is called a non-simple graph.

A graph G = (V, E) in which the set of edges E is empty is called a Null graph. A graph G is in which weight are assigned to every edge is called a weighted graph. A graph G = (V,E) in which both V (G) and E (G) are finite sets is called a finite graph.

Let G be a non-directed graph. An unordered pair {u, v} is an edge incident on u and v. If G is a directed graph. An edge {u, v} is said to be incident from u and to be incident to v.

In a non-directed graph G, the degree of a vertex v is determined by counting each loop incident on v twice and each other edge once. The degree of the vertex v in G is denoted by d (v) or by deg (v) and is defined as follows: the number of edges incident with a vertex v of a graph, with self-loops counted twice is called the degree of the vertex v.

A vertex of odd degree is an odd vertex of G and that of even degree is called even vertex. A graph G in which each vertex is assigned a unique label is called a labeled graph. A vertex of degree zero in a graph is called an isolated vertex. A vertex of a graph with degree one is called a pendant vertex. A graph G is said to be k-regular, if every vertex of G has degree k. A simple graph G, in which every pair of distinct vertices are adjacent is called a complete graph. lf G is a complete graph of n vertices then it is denoted by k_n.

lf G = (V, E) is a finite then the number of vertices denoted by $|V|$ is called the order of G.

Thus, the cardinality of the vertex set V of G the order of G. lf G = (V. E) is a finite graph, then the number of edges in G is called the size of G. It is denoted by $|E|$ (cardinality of E).

Let G be graph with v = {v_1, v_2, v_3, v_4, …..v_n } as the vertex set. Also let d_i = deg (v_i), then the sequence (d_1 , d_2, …d_n) in any order is called the degree sequence of G.

11.21 Terminal Questions:

1. Define the following with examples:

a) Graph

b) Self-loop

c) Digraph

d) Multi-graph

e) Pseudo graph

f) Order of a graph

g) Size of a graph

2. Draw a diagram for each of the following graphs:

(a) V = {(a, d), (a, f), (b, c), (b, f) , (c, e)}

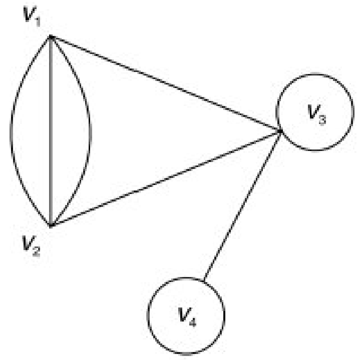
(b) V = {v_1,v_2, v_3, v_4, v_5), E = {(v_1, v_1), (v_2, v_3), (v_2, $_4$v), (v_4, v_5)}

(c) V = {a, b, c, d, e}, E = {(a, a), ... (a, b), (b, c), (c, d), (c, e), (d, e)}

(d) V = {a, b, c, d}, E = {(a, a), (a, b), (b, c), (c, c), (c, d), (d, a)}

3. Describe the graph G, given below:

G:



4. (a) Give two example for a regular graph of degree 1.

(b) Give two examples for a regular graph of degree 2.

5. Draw a simple graph of

(i) Two vertices (ii) Four vertices.

6. Draw the graphs of the following chemical compounds:

(a) CH_4 (b) C_2H_6

UNIT-12: Advanced Group Theory

Structure

12.1 Introduction

A graph will be known as the complete bipartite graph if it contains two sets in which each vertex of the first set has a connection with every single vertex of the second set. With the help of symbol $K(X,Y)$, we can indicate the complete bipartite graph. That means the first set of the complete bipartite graph contains the x number of vertices and the second graph contains the y number of vertices.

A wheel and a circle are both similar, but the wheel has one additional vertex, which is used to connect with every other vertex. With the help of symbol Wn, we can indicate the wheels of n vertices with 1 additional vertex. A graph which has no cycle is called an acyclic graph. A tree is an acyclic graph or graph having no cycles. There is only one path between each pair of vertices of a tree. If a graph G there is one and only one path between each pair of vertices G is a tree. A tree T with n vertices has n-1 edges. A graph is a tree if and only if it a minimal connected. The path length of a vertex in a rooted tree is defined to be the number of edges in the path from the root to the vertex.

12.2 Objectives

After reading this unit the learner should be able to understand about:

- Sub graph, Operations on Graphs
- Complement of a Graph,
- Connected Graph, Partitions, Cycle Graph
- Path Graph, Wheel Graph, Bipartite Graph
- Planer Graph, Hamiltonian Graph and Graph Coloring
- Model real world problems using graph theory.

12.3 Sub Graph

Let G and H be two graphs. H is called a subgraph of G if V (H) is a subset of V (G) and E (H) is a subset of E (G).

If H is a subgraph of G then

a) All the vertices of H are in G

b) All the edges of H are in G.

c) Each edge of H has the same end points in H as in G.

Example 1: In fig.1 the graph H is subgraph of G.

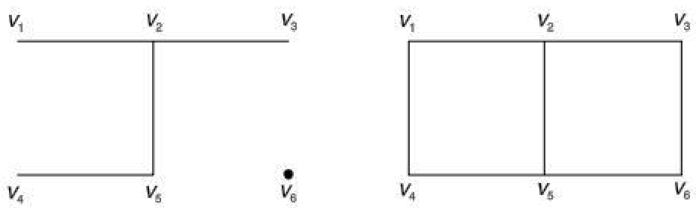


Fig.1

Example 2: In Fig.2 H is a subgraph of G.

Fig.2 Sub graph

Spanning Sub graph:

A sub graph H of a graph G is called a spanning sub graph of G if V (H) = V (G); i.e., H contains all the vertices of G.

In the graphs shown in Fig.3, H is a spanning sub graph of G.

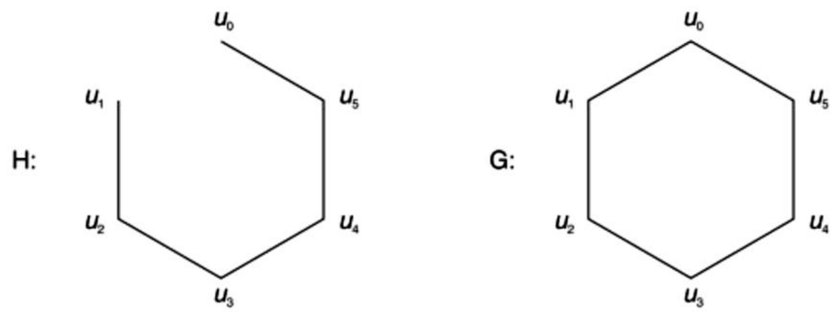


Fig.3 Spanning subgraph

Two sub graphs H_1 and H_2 of a graph G are said to vertex disjoint if $V(H_1) \cap V(H_2) = \emptyset$.

Edge Disjoint Sub graph:

Two sub graphs H_1 and H_2 of a graph G are said to be edge disjoint sub graphs of G if H_1 and H_2 do not share any edges in common.

REMOVAL OF VERTICES AND EDGES FROM A GRAPH:

The removal of a vertex V; from a graph G results in a subgraph of G; consisting of all points of G except v_i and all edges of G not incident with v_i. The obtained is denoted by G-v_i and is the maximal subgraph of G not containing v_i (Fig. 4(a)).

The removal of an edge e_j from a graph results in the spanning subgraph of G which containing all the edges of a G except the edge e_j. It is denoted by G - e_j and is the maximal subgraph of G not containing e_j (Fig.4(b)).

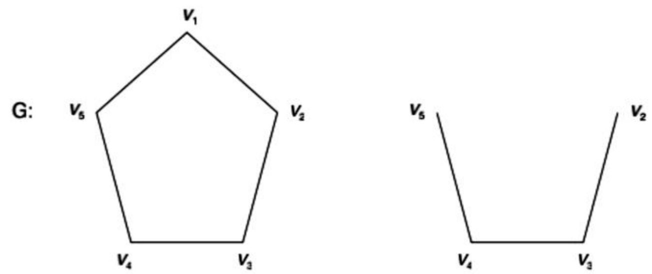


Fig.4(a) Graph minus a vertex v_1

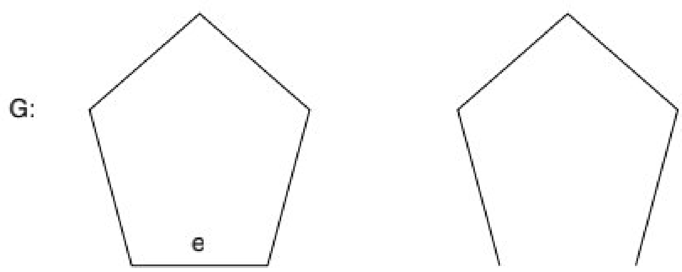


Fig. 4(b) Graph minus edge e

Addition of a Vertex:

Let G be a graph and v be a vertex which is not in G then the graph obtained by joining v with each vertex of G, is the sum graph $G + K_1$• It is denoted by G + v (see Fig.5).

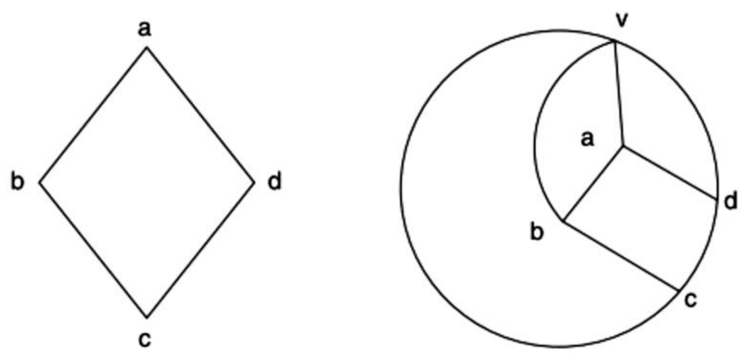


Fig.5

12.4 Operations on Graphs

Union of Graphs:

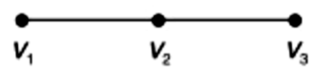
Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs whose vertex sets v_1 and v_2 are disjoint. Then the union of G_1 and G_2 denoted by $G_1 \cup G_2$ is defined as the graph $G = (V, E)$ such that

i. $V(G) = V(G_1) \cup V(G_2) = V_1 \cup V_2$

ii. $E(G) = E(G_1) \cup E(G_2) = E_1 \cup E_2$

Example 1: If

G_1 is the graph



and G_2 is the graph

then $G_1 \cup G_2$

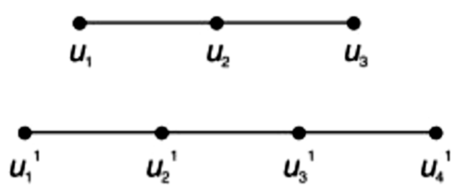
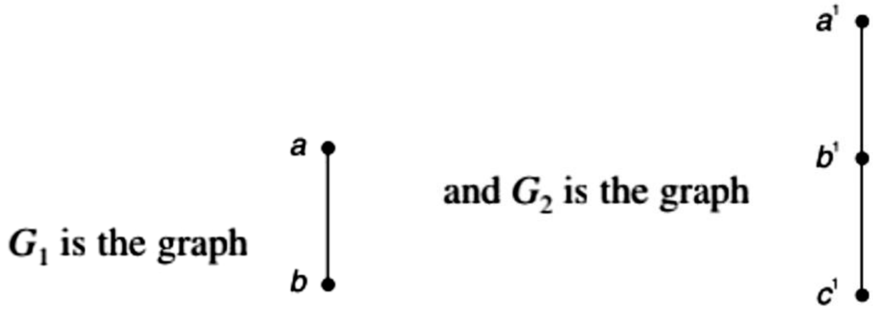


Fig.6

Sum of Two Graphs:

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, V_2)$ denote two vertex disjoint graphs. Then the sum of G_1 and G_2 denoted by $G_1 + G_2$ is defined as $G_1 \cup G_2$ together with all the edges joining vertices of V_1 to vertices of V_2.

Example: If



Then $G_1 + G_2$ is the graph given below

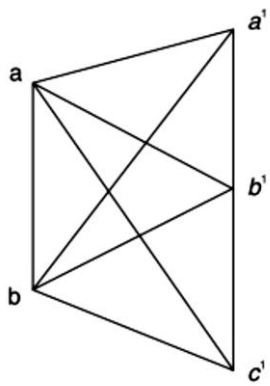


Fig.6 Sum of two graphs

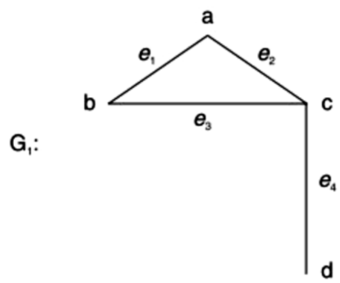
Intersection of Graphs:

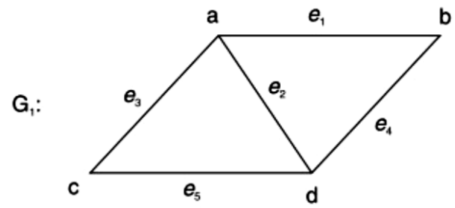
Let G_1 and G_2 be two graphs. Then the intersection of $G_1$and G_2 denoted by $G_1 \cap G_2$ is defined as the graph G such that

(i) $V(G) = V(G_1) \cap V(G_2)$.

(ii) $E(G) = E(G_1) \cap E(G_2)$.

Example: Let





then $G_1 \cap G_2$ is the graph

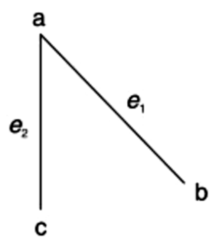


Fig.7 Intersection of two graphs

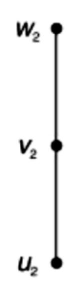
Product of Two Graphs:

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $V_1 \cap V_2 = \emptyset$. Then the product of G_1 and G_2 denoted by G_1 x G_2 is the graph having $V = V_1$ x V_2 and $u = \{u_1, u_2\}$ and $V = \{u_1, u_2\}$ are adjacent if $u_1 = v_1$ and u_2 is adjacent to V_2 is G_2 or u, is adjacent to V_1 and G_1 and $u_2 = v_2$.

Example: If

G_1 is the graph

and G_2 is the graph



then G_1 x G_2 is the graph given below:

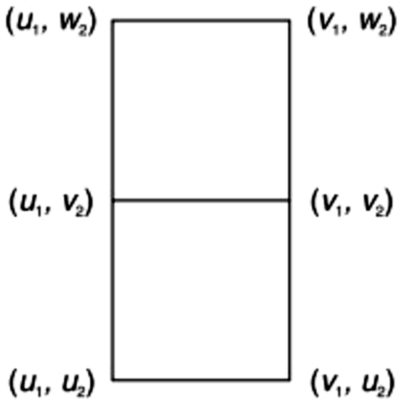


Fig. 8

Composition (Lexicographic Product) of G_1 with G_2:

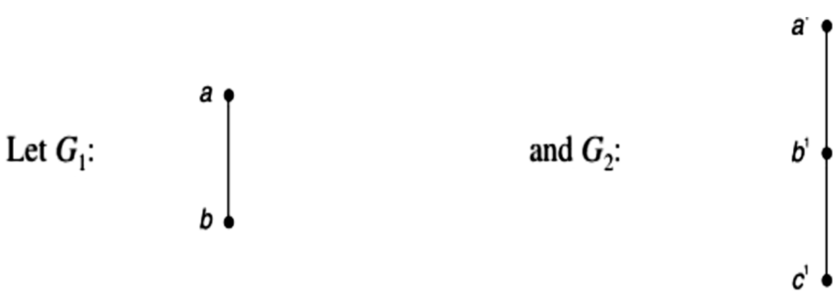
Let G_1 and G_2 be two graphs. The composition of G_1 and G_2 denoted by $G_1[G_2]$ is a graph

$G = G_1[G_2]$ such that

(i) $V(G) = \{(u, v): u \in v(G1), u \in V(G)\}$

(ii) $E(G) = \{((u_1, v_1), (u_2, u_2)\} :$ either $u_2 u_2 \in E(G)$ or $u_1 = u_2$ and $v_1 v_2 \in E(G)\}$

Example:



Then $G = G_1[G_2]$ is the graph

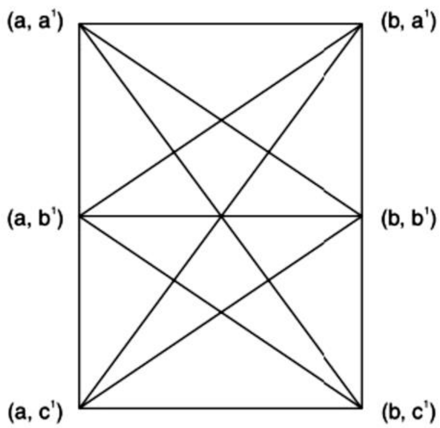


Fig. 9

Now we state the following theorem without proof:

Theorem 12.6: Let G_1 be a (p, q) graph and G_2 be (p_2, q_2) graph then:

(i) $G_1 \cup G_2$ is $(p_1 + p_2, q_1 + q_2)$ graph

(ii) $G_1 + G_2$ is a $(p_1 + p_2, q_1 + q_2 + P_1 P_2)$ graph

(iii) $G_1 \times G_2$ is a $(p_1 p_2, q_1, p_2 + q_2 p_1)$ graph

And (iv) $G_1[G_2]$ is a $(p_1, p_2, p_1, q_2 + p_2^2, q_1)$ graph

The proof is left to the reader as an exercise.

12.5 Complement of a Graph

Let G be a graph with n vertices then K_n - G is called the complement of G. It is denoted by $\bar{G}$.

A graph and its complement are shown in Fig.10 .

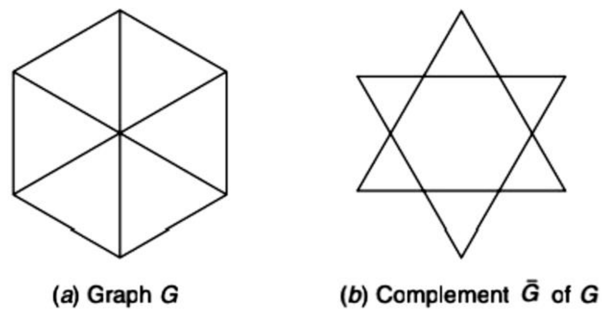


Fig.10 A graph and its complement

Complement of a Sub graph:

Let G be a graph and H be a sub graph of G. The complement H in G is the graph obtained by deleting the edges of H from those of G.

The complement of H in G is denoted by $\bar{H}$ (or $\bar{H}$ (G)). In Fig. 11(a) G is a given graph, and in Fig. 11(b): H is a subgraph of G. The complement of H in G is shown in Fig.11 (c).

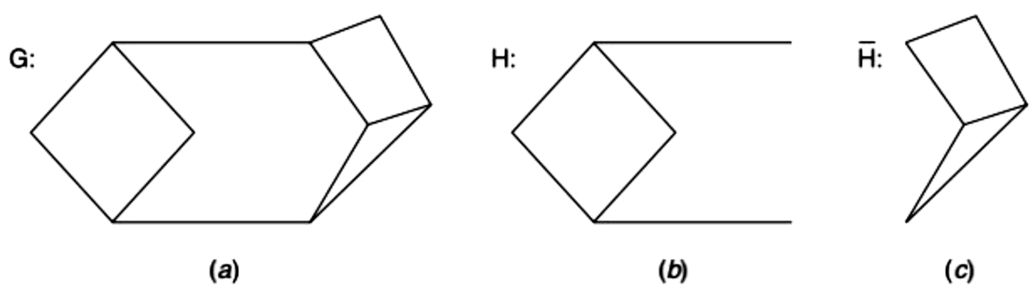


Fig.11 Complement of a subgraph

12.6 Connected Graph

A graph G is said to be connected if every pair of points in G are joined by a path. If G is not connected then G is called a disconnected graph.

A maximal connected subgraph of G is called a component of G. If G is disconnected then G has at least two components.

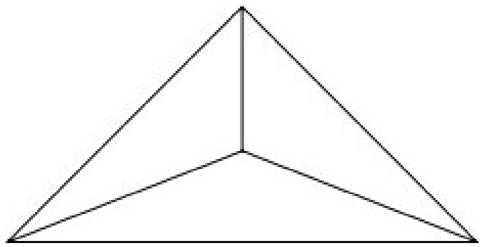


Fig.12 A connected graph

Clearly a graph G is connected iff it has exactly one component.

Theorem 12.1: If G is a graph with n with points and $\delta(G) \geq \frac{n-1}{2}$ then G is connected.

Proof; Let assume that G is not connected. Then G has more than one component. Consider any component $G_1 = (V_1, E_1)$ of G.

Let $v_1 \in v_2$ since $\delta(G) \geq \frac{n-1}{2}$ there exist atleast $\frac{n-1}{2}$ points in G_1 which are adjacent to v_1 in G_1.

Then we have $|V| \geq \frac{n-1}{2} + 1$

Or $|V| \geq \frac{n+1}{2}$

Thus each component of G has atleast $\frac{n+1}{2}$ points and G has least two components.

Hence the number of points (vertices) in G $\geq 2\ \frac{n+1}{2}$.

i.e., $|V(G)| \geq (n+1)$ which is a contradiction.

Thus G is connected.

12.7 Partitions of a graph

Let G = (V, E) be a graph. A partition of the vertex set V(G) is a collection $\{V_i\}_{1 \leq i \leq \alpha}$ of non-empty subsets of V such that

(i) $V_1 \cup V_2 \cup V_3 \cup ... \cup V_\alpha = V$, $(\alpha \neq 1)$

and (ii) $V_i \cap V_j = 0$ whenever $i \neq j$

A partition of the edge set E(G) is a collection $\{E_i\}$ of non-empty subsets of E such that $1 \leq i \leq \beta$.

(i) $E_1 \cup E_2 \cup E_3 \cup ... \cup E_\beta = V$, $(\beta \neq 1)$

(ii) $E_i \cap E_j = 0$ whenever $i \neq j$

The partition of the edge set E is also called edge decomposition of G.

Example: Consider the graph shown in Fig. 13.

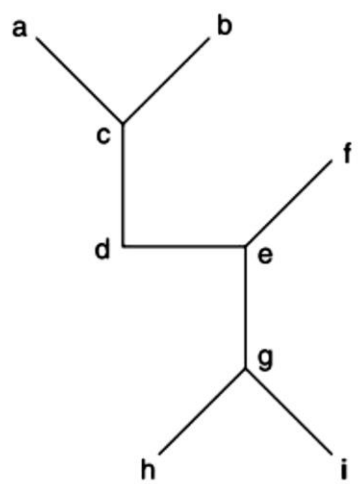


Fig.13

$V_1 = \{a, b, c\}$, $V_2 = \{d, e, f, g\}$, $V_3 = \{h, i\}$ is a vertex partition of the vertex set V (G).

and $E_1 = \{(a, c), (c, b)\}$, $E_2 = \{(c, d), (d, e), (e, f)\}$ $E_3 = \{(e, g)\}$, $E_4 = \{(g, h)\}$, $E_5 = \{(g, i)\}$is an edge decomposition of G.

Theorem 12.2: A graph G is connected if and only if for any partition of V into subsets V_1 and V_2 there is an edge joining a vertex of V_1 to a vertex of V_2.

Proof: Let G be a connected graph and $V = V_1 \cup V_2$ be a partition of V into two subsets.

Let $u \in V_1$ and $v \in V_2$. Since the graph G is connected there exists a u - v path in G say $u = v_0, v_1, v_2, .. v_n = v$. Let i be the least positive integer such that $v_1 \in v_2$. Then $v_{i-1} \in V_1$ and the vertices v_{i-1}, v_i are adjacent. Thus there is an edge joining $v_{i-1} \in V_1$ and $v_1 \in V_2$.

Conversely

Let G be a disconnected graph.

Then G contains atleast two components.

Let V_1 be the set of all vertices of one component and V_2 be the set of remaining vertices of G. clearly $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$.

The collection $\{V_1, V_2\}$ is a partition of V and there is no edge joining any vertex of V_1 to any vertex of V_2.

Hence the theorem.

Theorem 12.3: If G is a simple graph with n vertices and k components; then G can have at most

(n -k) (n + k + 1)/2 edges.

Proof: Let G be a simple graph with n vertices and $G_1, G_2, G_3, ... G_k$ be the k components of G. Let the number of vertices in ith component G_i be n_i.

Then

$|V(G_1)| + |V(G_2)| + \cdots . + |V(G_k)| = n_1 + n_2 + \cdots + n_k = |V(G)| = n$ where $n_i > 1$

and max $|E(G_i)| \leq \frac{n_i(n_i-1)}{2}$

$$\therefore \quad |E(G)| \leq \sum_{i=1}^{k} max\,|E(G_i)|$$

$$= \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2}$$

$$= \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i\right]$$

$$= \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - n\right]$$

$i.e.,\quad |E(G)| \leq \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - n\right]$ ………. (1)

Now

$\sum_{i=1}^{k}(n_1 - 1) = (n_1 – 1) + (n_2 – 1) + \ldots + (n_k – 1)$

$= (n_1 + n_2 + \ldots + n_k) – (1 + 1 + \ldots + k \text{ time})$

$= n – k$

Squaring on both sides

$\left[\sum_{i=1}^{k}(n_1 - 1)\right]^2 = (n – k)^2 = n^2 + k^2 – 2nk$

Or $\sum_{i=1}^{k}(n_1 - 1)^2 + 2$ (non - negative terms) $= n^2 + k^2 – 2nk$

Or $\sum_{i=1}^{k}(n_1 - 1)^2 = n^2 + k^2 – 2nk – 2$

Or $\sum_{i=1}^{k}(n_1 - 1)^2 \leq n^2 + k^2 – 2nk$

Or $\sum_{i=1}^{k} n^2 + \sum_{i=1}^{k} 1 - 2\sum_{i=1}^{k} n_i \leq n^2 + k^2 – 2nk$

Or $\sum_{i=1}^{k} n_i^2 + k - 2n \leq n^2 + k^2 – 2nk$

Or $\sum_{i=1}^{k} n_i^2 - n \leq n^2 – nk + n + k^2 – k$

$= n(n – k + 1) – k(n – k + 1)$

$= (n – k)(n – k + 1)$

i.e., $\sum_{i=1}^{k} n_i^2 - n \leq (n – k)(n – k + 1)$ ………….. (2)

From (1) and (2) we get

$$|E(G)| \leq (n-k)(n-k+1)/2$$

Hence proved

Corollary: If $m > \frac{1}{2}(n-1)(n-2)$, then a simple graph with n vertices and m edges are connected.

Proof: Let G be a simple graph with n vertices and m edges.

Let us assume that G is disconnected. Then we have

$$m > \frac{1}{2}(n-1)(n-2)$$

Since G is a disconnected graph, G has at least two components. Therefore for $k \geq 2$, we have

$$m > \frac{1}{2}(n-k)(n-k+1)$$

Hence $m > \frac{1}{2}(n-2)(n-1)$

Contradicting our assumption that

$$m > \frac{1}{2}(n-1)(n-2)$$

Therefore graph G is a connected graph.

Theorem 12.4: If G is not connected then $\bar{G}$ is connected.

Proof· Let G be disconnected graph. Then G has more than one component.

Let u, v be any two vertices of G. The theorem is proved if we show that there is a u-v path in $\bar{G}$.

If u, v are in different components of G, then u,v are not adjacent in G. Hence, they are adjacent in $\bar{G}$.

If u, v are in the square components of G. Choose a vertex w in a different component of G. Then u - w - v is a u - v path in $\bar{G}$. Hence $\bar{G}$ is connected.

12.8 Cycle Graph

A cycle graph of order n is a connected graph whose edges form a cycle of length n.

Cycle graph of order n is denoted by c_n. The graph shown in Fig.14 is a cycle graph of order 5.

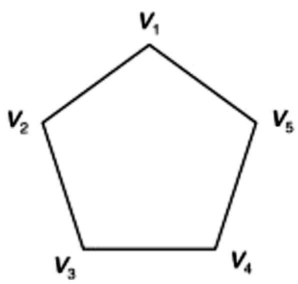


Fig.14 Cycle graph of order 5 (c_5)

12.9 Path Graph

Let G be a cycle graph order n. The graph obtained by removing an edge from G is called a path graph of order n. It is denoted by P_n.

The graph shown in Fig. 15 is a path graph order 5.

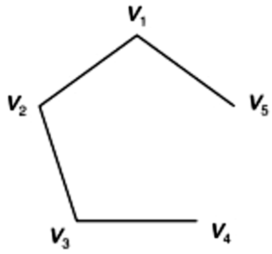


Fig.15 Path graph of order 5 (P_5)

12.10 Wheel Graph

Let G be a cycle graph order (n - 1). The graph obtained by join ing a single new vertex v to each vertex of G is called a wheel graph of order n.

A wheel graph of order n is denoted by w_n. The new vertex v is called the "hub". The graph shown in Fig.16 is a wheel graph.

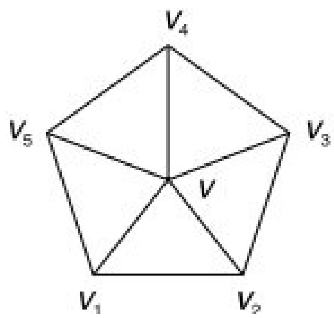


Fig.16 Wheel graph (W_6)

Theorem 12.5: If G is a graph with 6 points then G or $\bar{G}$ contains a triangle.

Proof: Let G = (V, E) be a graph with 6 points and $v \in V$. The vertex v is adjacent either in G or in $\bar{G}$ to the other five points of G. Let us assume u_1, u_2 and u_3 are three adjacent vertices of v in G.

If ant two of these vertices are adjacent then the 2 adjacent vertices and v form a triangle. If no two of the points u_1, u_2, u_3 are adjacent in G. Then they are adjacent in G and form a triangle in $\bar{G}$.

12.11 Bipartite Graph

There are a number of special classes of graph. One example is the bipartite graph. We now introduce few more graphs which are important.

A graph G is called a bipartite graph if its vertex set V can be partitioned into two disjoint subsets A and B such that every edge in G, joins a vertex in A to a vertex in B. The graph shown in Fig.17 is a bipartite graph.

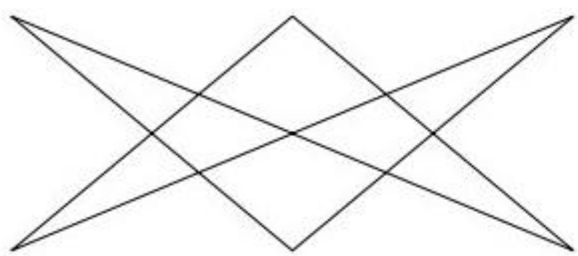


Fig.17 Bipartite graph

A bipartite graph can have no self loop.

Complete Bipartite Graph:

A bipartite graph G in which every vertex of A is adjacent to every vertex in B is called a complete bipartite graph. Where A and B are partitioned subsets of the vertex V of G.

If $|A| = m$ and $|B| = n$, then the complete bipartite graph is denoted by $k_{m,\,n}$ and has m n lines. The graphs in Fig.18 are complete bipartite.

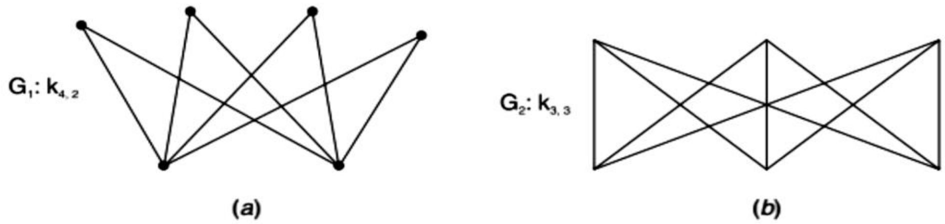


Fig.18 complete bipartite

A complete bipartite graph $k_{1, n}$ is called a star graph.

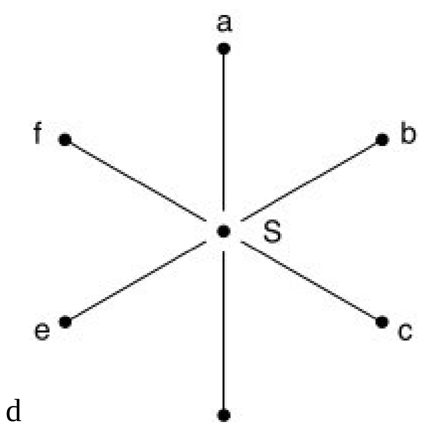


Fig19. Star graph ($k_{1, 6}$)

Example 1: Draw the complement of the graphs G shown in Fig.20:

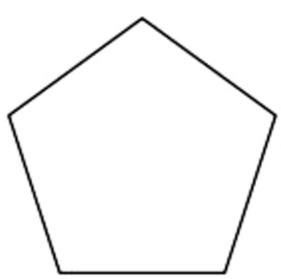


Fig. 20

Solution: The complement of G is shown in Fig.21:

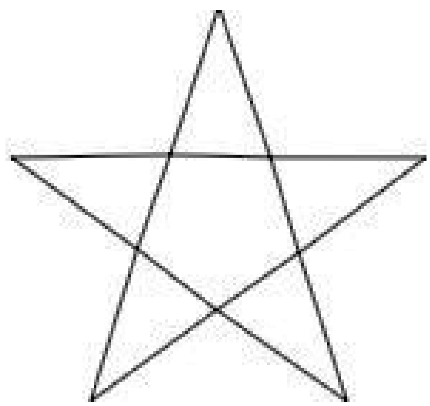


Fig.21 Complement of G

Example 2: Draw simple unlabeled graphs of 3 vertices.

Solution: The graphs shown in Fig. 22 are simple graphs with three vertices.

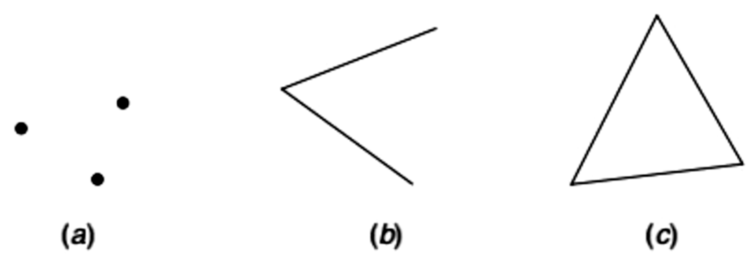


Fig.22 Simple unlabeled graphs

Example 3: Find the number of connected graphs with four vertices and draw them.

Solution: There five connected graphs with four vertices (see Fig. 23):

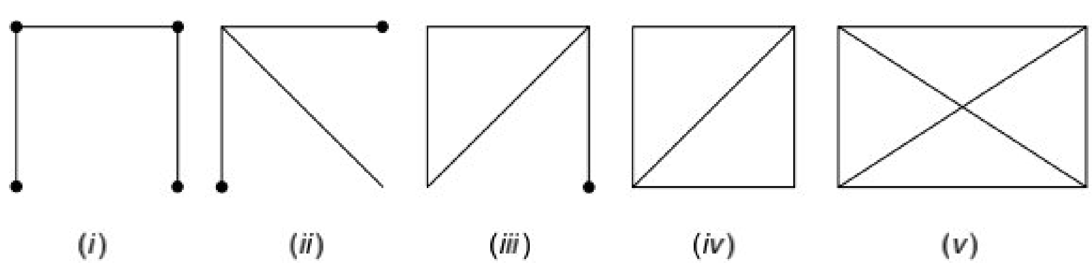


Fig. 23

Example 4: Draw the graph $K_{2,5}$

Solution: The graph $K_{2,5}$ bas 2 x 5 = I0 edges and 7 vertices. It is shown in Fig. 24. The partitioned sets are A = $\{u_1, u_2\}$ and B = $\{v_1, v_2, v_3, v_4, v_5\}$.

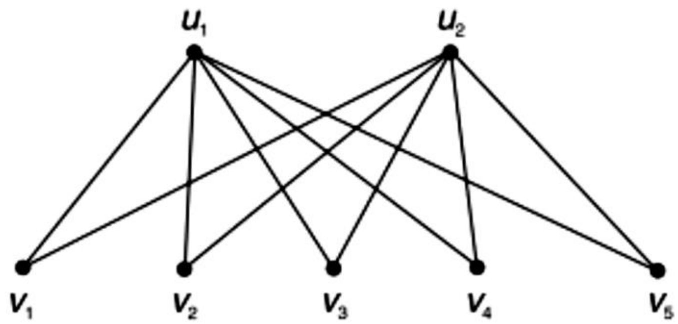
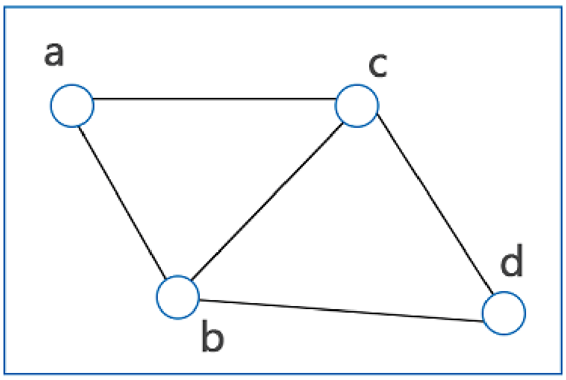


Fig. 24

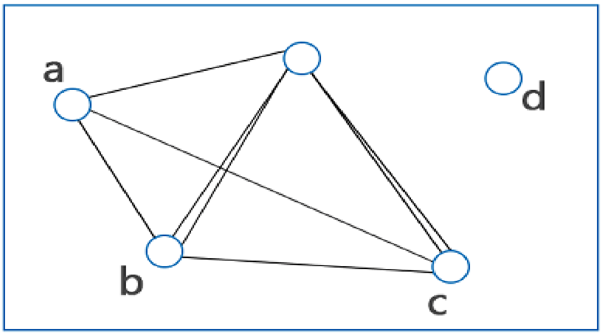
12.12 Planer Graph

A graph will be known as the planer graph if it is drawn in a single plane and the two edges of this graph do not cross each other. In this graph, all the nodes and edges can be drawn in a plane. The diagram of a planer graph is described as follows:



In the above graph, there is no edge which is crossed to each other, and this graph forms in a single plane. So this graph is a planer graph.

Non-planer graph: A given graph will be known as the non-planer graph if it is not drawn in a single plane, and two edges of this graph must be crossed each other. The diagram of a non-planer graph is described as follows:



In the above graph, there are many edges that cross each other, and this graph does not form in a single plane. So this graph is a non-planer graph.

12.13 Hamiltonian Graph

A graph which contains a **Hamiltonian cycle**, i.e. a cycle which includes all the vertices, is said to be **Hamiltonian**.

Walks, Trails, and Circuits:

A walk in a graph is a sequence of adjacent edges. A trail is a walk with distinct edges. A circuit is a trail in which the first and last edge are adjacent.

Eulerian Graph:

A trail which includes all of the edges of a graph and visits every vertex is called an **Eulerian Tour**. If a graph contains an Eulerian tour which is a circuit, i.e. an **Eulerian circuit**, the graph is simply said to be **Eulerian**.

12.14 Graph Coloring

A combinatorial problem named "coloring" was originally posed as follows.

"Given a map, how many colours are necessary and sufficient to paint countries on the map so that two countries sharing boundary must be painted by different colours."

For a given map, we can construct a simple undirected graph G = (V,E), called the **dual graph** of the map, such that each vertex represents a distinct region of the map and an edge between vertices exists iff the regions represented by the vertices share boundary.

A **coloring** of G = (V,E) with k colors is a function $f : V \rightarrow \{1,2,...,k\}$ such that $f(u) \neq f(v)$ for every edge (u,v) in E.

The **chromatic number** of G is the minimum number of colors needed for any coloring of G. It is difficult to determine whether G has a coloring with k colors for given G and k.

A sub graph of G is called a **clique** of G if it is a complete graph. The size of a clique is the number of vertices in the clique.

Theorem 12.6: If G contains a clique of size k, then the chromatic number of G is at least k.

Four Colour Theorem: If a graph is planar, then there is a coloring of the graph with 4 colours.

Other Example Applications

- VLSI Design: Channel Routing by Using Interval Graphs

- Scheduling: Register Allocation

- Resource Allocation: Classroom Assignment etc. Regarding applications of graph coloring,

Graph coloring is a major sub-topic of graph theory with many useful applications as well as many unsolved problems. There are two types of graph colourings we will consider.

Vertex-Colourings and Edge-Colorings: Given a set C called the set of colours (these could be numbers, letters, names, whatever), a function which assigns a value in C to each vertex of a graph is called a vertex- coloring. A proper vertex-coloring never assigns adjacent vertices the same colour. Similarly, a function which assigns a value from a set of colours C to each edge in a graph is called an edge-coloring. A proper edge-coloring never assigns adjacent edges the same colour.

In the case of vertex-colourings, we will primarily be interested in colourings which are proper, and following convention, we will use the word **coloring** to mean a proper vertex-coloring. In contrast, we will want to consider edge-colourings which are not necessarily proper.

Note: That the values of the set C are arbitrary, what is important is the size of C. The most interesting question we will consider regarding colourings is how big the set C must be in order for a coloring of a given graph to exist.

K-Coloring: A coloring of a graph using a set of k colours is called a *k-coloring*. A graph which has a k-coloring is said to be **k-colourable**.

The four-color theorem is equivalent to the statement that all planar graphs are 4- colourable. Note that a graph which is k-colourable might be colourable with fewer than k colours. It is often desirable to minimize the number of colours, i.e. find the smallest k.

Chromatic Number:

The chromatic number of a graph G is the least k for which a k- Colouring of G exists. Thus if a graph G has chromatic number k, then G has a k- coloring, but not a (k-1)- coloring.

For example a path has chromatic number 2, while the complete graph K_n has chromatic number n. We now consider the chromatic number of cycles.

Theorem 12.7: The chromatic number of C_n is 2 if n is even, and 3 if n is odd.

Proof: First note that the chromatic number must be at least 2 for any graph which has an edge in it, including all cycles.

We now prove the theorem by induction on n. We will consider two base cases, C_3 and C_4.

C_3 is isomorphic to K_3 which has chromatic number 3.

C_4 can be colored with two colors by giving opposing corners of the square the same color.

For $n > 4$,

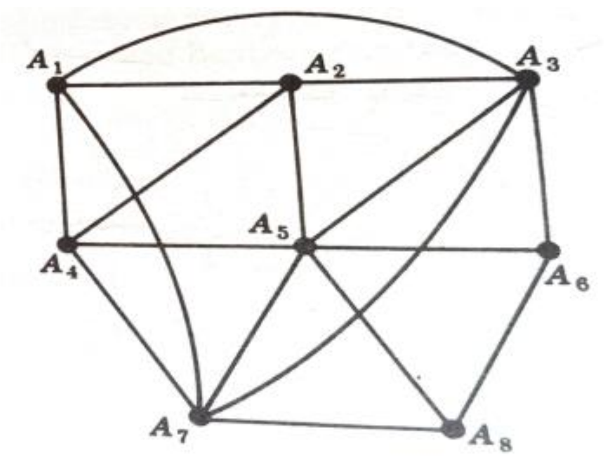
we can take a coloring of C_{n-2} and insert 2 adjacent vertices and edges and then color the new vertices appropriately to get a coloring of C_n.

Thus the chromatic number of C_n is not greater than that of C_{n-2}.

In the case where n is odd, note that if C_n had chromatic number 2, we could remove two adjacent vertices and edges to get a 2-coloring of C_{n-2} which contradicts the inductive hypothesis since n-2 must be odd if n is odd.

Example: Consider the graph G in figure. We use the Welch-Powel Algorithm to obtain a coloring of G. Ordering the vertices according to decreasing degrees yields the following Sequence:

A_5, A_3, A_7, A_1, A_2, A_4, A_6, A_8



The first color is assigned to vertices A_5, A_1. the second color is assigned to vertices A_3, A_4 and A_8 . The third color is assigned to vertices A_7, A_2 and A_6.

All the vertices have been assigned a color , and so G is 3-colorable. Observe that G is not 2-colorable since vertices A_1, A_2, A_3, which are connected to each other, must be assigned different colors. Accordingly, $x(G) = 3$

12.15 Summary

Let G and H be two graphs. H is called a subgraph of G if V (H) is a subset of V (G) and E (H) is a subset of E (G).

If H is a subgraph of G then

d) All the vertices of H are in G

e) All the edges of H are in G.

f) Each edge of H has the same end points in H as in G.

A sub graph H of a graph G is called a spanning sub graph of G if V (H) = V (G):i.e., H contains all the vertices of G.

Let G be a graph with n vertices then K_n - G is called the complement of G. It is denoted by $\bar{G}$.

A graph G is said to be connected if every pair of points in G are joined by a path. If G is not connected then G is called a disconnected graph.

Let G = (V, E) be a graph. A partition of the vertex set V(G) is a collection $\{V_i\} 1 \leq i \leq \alpha$ of non-empty subsets of V such that

(i) $V_1 \cup V_2 \cup V_3 \cup ... \cup V_\alpha = V$, $(\alpha \neq 1)$

and (ii) $V_i \cap V_j = 0$ whenever $i \neq j$

A cycle graph of order n is a connected graph where the edges form a cycle of length n.

Let G be a cycle graph of order n. Removing an edge from G results in a path graph of order

n, denoted by Pn.

There are several special classes of graphs, such as bipartite graphs.

A bipartite graph G where every vertex in one set (denoted A) is adjacent to every vertex in another set (denoted B) is called a complete bipartite graph, where A and B are partitioned subsets of the vertex set V of G.

A graph will be known as the planer graph if it is drawn in a single plane and the two edges of this graph do not cross each other.

In this graph, all the nodes and edges can be drawn in a plane. A graph which contains a **Hamiltonian cycle**, i.e. a cycle which includes all the vertices, is said to be **Hamiltonian**.

12.16 Terminal Questions:

Q.1. Define the subgraph with examples.

Q.2. Write a short note on operations on graphs.

Q.3. What do you mean by Connected and Bipartite graph?

Q.4. Explain Planner and Hamiltonian graph.

Q.5. Write a short not on graph coloring.

UNIT-13 : Tree

Structure

13.1 Introduction

The tree is fundamental structure in mathematics and computer science, some of the terminology of rooted tree, such as, edge, path, branch, leaf, depth, and level number, will also be used for binary trees. However, we will use the term node, rather than vertex, with binary trees.

We emphasize that a binary tree is not a special case of a rooted tree; they are different mathematical objects. The Concept of a tree was discovered by Cayley in the year 1857.

13.2 Objectives

After reading in this unit, learners should be able to

- understand the tree
- directed tree.
- Minimal spanning tree
- Kruskal's Algorithm and Prim's Algorithm

13.3 Definition of a Tree

A tree is a connected graph without any cycles. This means that a tree is acyclic and connected. It has no self-loops or parallel edges. Trees are denoted by the symbol T.

Since trees are acyclic, we adopt a convention similar to that used for Hasse diagrams. Trees can be either directed or undirected.

13.4 Directed Tree

A connected and, a cyclic, directed graph is known as a directed tree.

The graph in Fig. 1(a) is a non-directed tree, and graph shown in Fig. 1(b) is a directed tree:

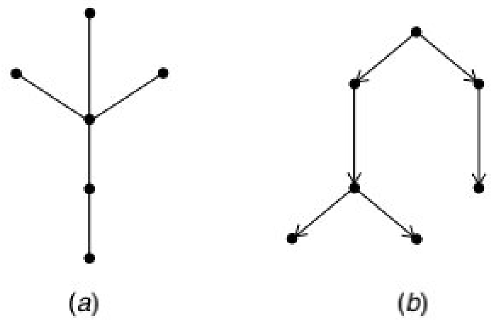


Fig.1 Tree

If T is a tree, then it has a unique simple undirected path between each pair of vertices. A tree with only one vertex is called a trivial tree. If T is not a trivial tree, then it is called a non-trivial tree. The vertex set (i.e., the set of nodes) of a tree is finite. In most cases, the vertices of a tree are labeled.

Theorem 13.1: A simple non-directed graph G is a tree if and only if G is connected and has no cycles.

Proof: Let G be a tree. Then each pair of vertices of G are joined by a unique path, therefore G is connected. Let t1 and v be two distinct vettices of G. Such that G contains a cycle containing u and v. Then u and v are joined by atleast two simple paths, one along one portion of the cycle and the other path completing the cycle. This contradicts our hypothesis that there is a simple unique path between u and v. Hence tree has no cycle.

Conversely let G be a connected graph having no cycles. Let v_1 and v_2 be any pair of vertices of G and let there be two different simple paths say P_1 and P_2 from v_1 to v_2• Then we can find a cycle in G as follows: Since the paths P_1 and P_2 are different, there must be a vertex say u, which is on both P1 and P2 but its successor on P_1 is not on P_2. If u' is the next point on P 1 which is also on P_2, the segments of P_1 and P_2 which are between P_1 and P_2 form a cycle in G. A contradiction. Hence there is almost one path between any two vertices of G, which shows that G is a tree.

Theorem 13.2: Any non-trivial tree has atleast one vertex of degree 1.

Proof: Let G be a non-trivial tree, then G has no circuits. Let v 1 be any ve1tex of G. If deg (V_1) = 1, then the theorem is at once established. Let deg (v_1) $\neq$ 1 move along any edge to a vertex V_2 incident with v_1. If deg (v_2) $\neq$ 1 then continue to another vertex say v_3 along a different edge. Continuing the process, we get a path v_1 - v_2 - v_3 - v_4 - -.. in which none of the v^1_t s is repeated. Since the number of vertices in a graph is finite, the path must end some where. The vertex at which the path ends is of degree are, since we can enter the vertex but cannot leave the vertex.

Theorem 13.3: A tree T with n vertices has exactly (n - 1) edges.

Proof: The theorem will be prove by mathematical induction on the number of vertices of a tree. If n = 1 then there are no edges in T. Hence the result is trivial.

If n = 2 then the number of edges connecting the vertices is one i.e., n - l. Hence the theorem is true for n = 2. Assume that the theorem holds for all trees with fewer than n vertices. Consider a tree T with n vertices. Let V be a vertex in T of degree 1 and let T' denote the graph obtained by removing the vertex v and edge e associated with it from T. Consider T' = T - e.

T' has n - 1, vertices and fewer edges than T. If v_1 and v_2 are any two vertices in T', then there is a unique simple path from v_1 to v_2 which is not affected by the removal of the vertex and edge.

T' is connected and no edges in it, therefore T' is a tree. T' has n -1 vertices and n- 1-1 = n -2 edges. T has are more edge than T'.

∴ Number of edges in T = n - 2 + 1 = n - 1. Hence T has exactly n - 1 edges.

Theorem 13.4: Every non-trivial tree has atleast 2 vertices of degree 1.

Proof: Let m denote the number of vertices of degree 1 (i.e., pendant vertices) and n be the number of vertices in the tree T (where $n \geq 2$).

Let $v_1, v_2, v_3, \dots, v_m$ denote the m vertices of degree 1in T. Then each of the remaining n - m vertices $v_{m+1}, v_{m+2}, \dots$, has degree atleast two.

Thus deg (v_1) = I for i= I, 2, ..., m

≥ 2 for i= m + 1, m + 2, ..., n

We have $\sum_{i=1}^{n} deg\,(v_1) \geq n + 1.(n - m)$

Or $2(n - 1) \geq 2n - m$

Or $2n - 2 \geq 2n - m$

Or $-2 \geq - m$

Or $m \geq 2$

Thus T contains atleast two vertices of degree 1.

Theorem 13.5: A graph G is a tree if and only if G has no cycled and $|E| = |V| - 1$. Conversely, Jet G be a graph such that G has no cycles and $|E| = |V| - 1$ Clearly G is connected.

Let $G_1, G_2, G_3, \dots G_k$ be k components of G where K > 1.

G has no cycles, therefore each G_i, is connected and each G_i has no cycle in it.

Number of edges in each G_i = $|V| - 1$

Hence number of edges in G

$$= |V_1| - 1 + |V_2| - 1 + \cdots . |V_k| - 1$$

$$= |V_1| + |V_2| + \cdots . |V_k| + K$$

$$= |V_k| + K$$

by hypothesis G has $|V|$- ledges

Thus $|V| - k = |V| - 1$

Or K = 1

The number of components in G is one and G is connected.

Hence G is a tree.

13.5 Distance

If u and v are two vertices of a connected graph G, there may be more than one path joining u and v. Various concepts can be defined based on the lengths of such paths between vertices of G. The simplest is given below:

If G is connected graph and u and v are any two vertices of G, the length of the shortest path between u and v is called the distance between u and v and is denoted by d (u, v).

The distance function on defined above has the following properties. If u, v and w are any three vertices of a connected graph then.

(i) d (u, v) $\geq$ and d (u, v) = 0 iff u = v

(ii) d (u, v) = d (v, u)

and (iii) d (u, v) $\leq$ d (u, w) + d (w, v)

from the above, it is clear that distance in a graph is a metric.

Example 1: In the graph shown in Fig.2

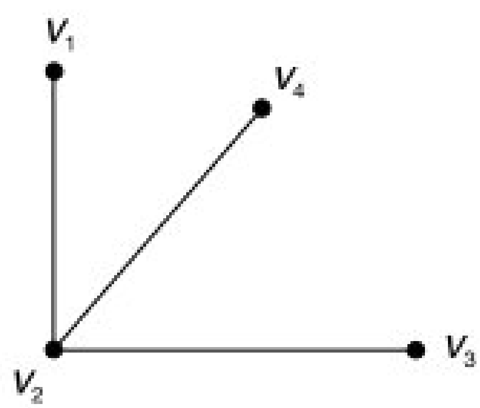


Fig. 2

Let G be a connected graph. For any vertex v on G, the eccentricity of v denoted by

e (v) is

e (v) = max {d (u, v) : u, v $\in$ v}

e (v) is the length of the longest path in G starting from the vertex v.

Example 2: In the graph shown in Fig. 8.93. e (v1) = 3

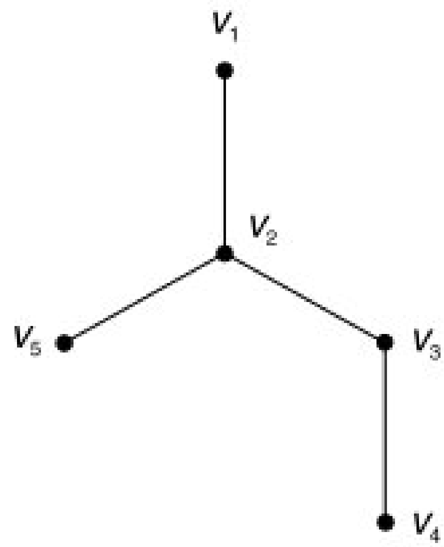


Fig. 3

The diameter of a connected r graph G is defined as the maximum eccentricity among all vertices of the graph G. It is denoted by d.

Hence d = diameter of G = max {e (v): v ∈ V}

The radius of a connected graph G is defined as the maximum eccentricity among all ve1tices of the graph. It is denoted by r.

Thus r = radius of G = min {e (v):v ∈ V}

Note: The radius of connected graph may not be half of its diameter.

Example 3: Consider the tree T shown in Fig.4.

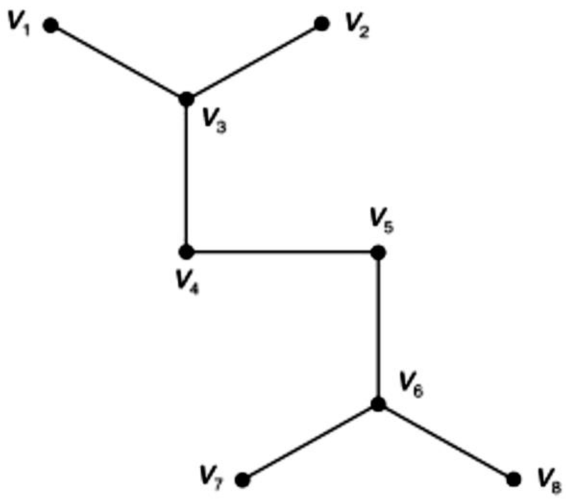


Fig.4

We have

e (v_1) = 5, e (v_2) = 5, e (v_3) = 4,

e (v_4) = 3, e (v_5) = 3, e (v_6) = 4,

e (v_1) = e (v_8) = 5

The radius of T = r = 3

and the diameter of T = 5

The centre of connected graph G is defined as the set of vertices having minimum eccentricity among all vertices of the graph. It is denoted by C or C (G).

C = C (G) = centre of G = {v E V : e(v) = r }

Example 4: Consider the graph shown in Fig.5

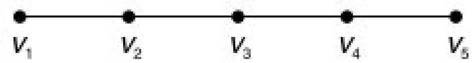


Fig.5

e (v_1) = 4, e (v_{2}) = 3, e (v_3) = 2,

e(v_4) = 3, e(v_5) = 4, radius of G = i = 2

Hence centre of G = {v_3}

Note:

1. Let G be a connected graph and v_1, v_2, •••, v_n be 'n' vertices of G e (v_1), e (v_2), ••• e (v_n) is called the eccentricity sequence of G.

2. The distance between two adjacent vertices of a connected graph G is I.

3. The maximum distance from each vertex of G occurs at a pendant vertices of G.

4. If C (G) = V (G) then G is called self-centred graph.

5. If P is a path of even length the P has only one vertex at the centre.

6. If P is a path of odd length this centre of P contains two adjacent vertices.

Example 5: In the graph shown in Fig. 6:

Fig. 6

Centre of G = {v_3, v_4}

Example 6: In the graph shown in Fig.7

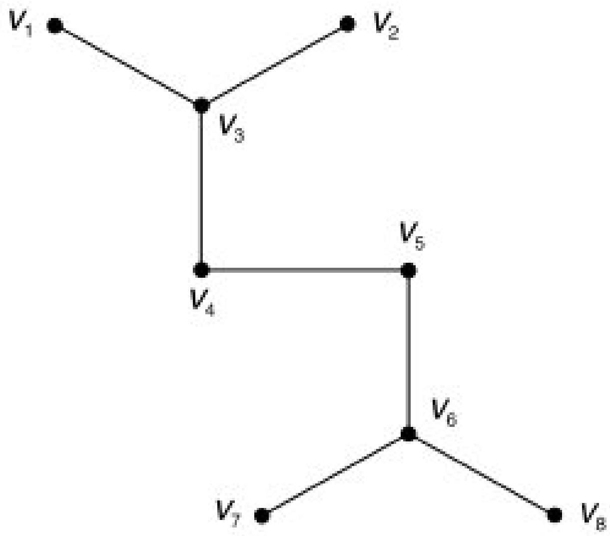


Fig. 7

Theorem 13.6: If r is the radius and d is the diameter of connected graph G then r ≤ d ≤ 2r.

Proof· From the definition of 'r' and 'd', we have r ≤ d …………………. (1)

Let u, v be the ends of a diametral path and w be the central vertex then

D = d (u, v) :::; d (u, v) + d (w, v) :::; r (Triangle inequality)

or d ≤ 2r ……………………… (2)

From equations (1) and (2), we have

r ≤ d ≤ 2r.

13.6 Spanning Tree

Let G be a connected graph. The sub-graph H of G is called a spanning tree of G if

(i) H is a tree

and (ii) H contains all the vertices of G.

A complete graph k_n has n^{n-2} different spanning trees.

Example 7: In the Fig.8, H is spanning tree of G.

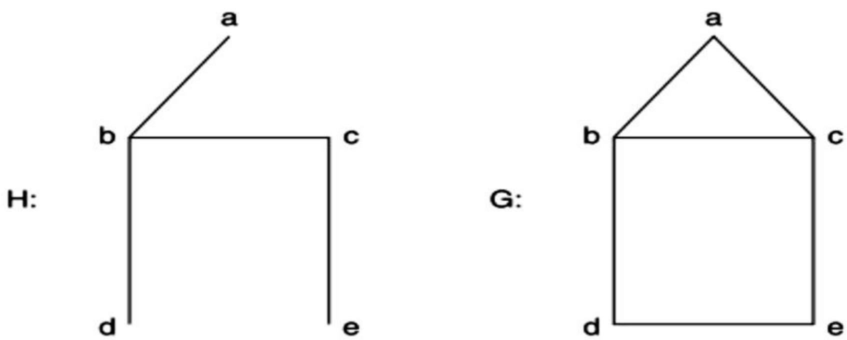


Fig. 8

Example 8: Find all the spanning trees of the graph G shown in the Fig. 9.

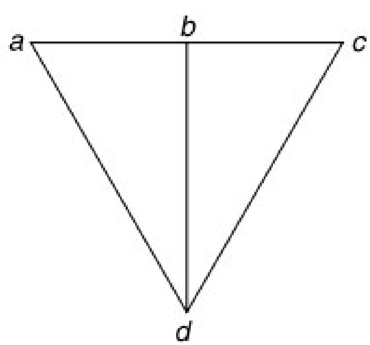


Fig.9

Solution: The spanning trees of G are given below (Fig.10):

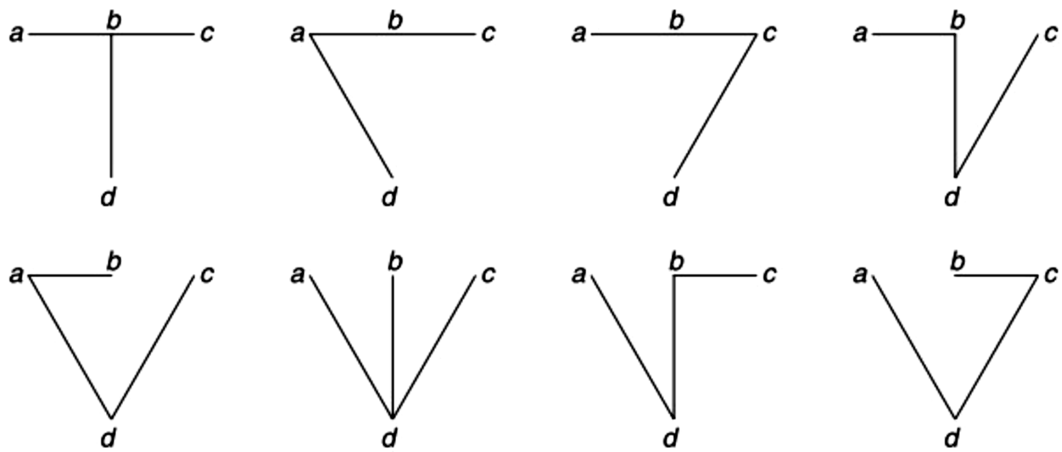


Fig.10

Theorem 13.1: A non-directed graph G is connected if and only if G contains a spanning tree.

Proof: Let T be a spanning tree of G. There exists a path between any pair of vertices in G along the tree T. G is connected.

Conversely let G be a connected graph and K be the number of cycles in G. If K = 0, then G has no cycles and G is connected. Therefore G is a tree when K = 0.

Let us suppose that all connected graphs with fewer than K cycles have a spanning tree. Let G be a connected graph with n cycles. Let e be an edges in one of the cycle. G - e is a connected graph and G - e contains all the vertices of G.

∴ The spanning tree if G - e is also spanning tree for G.

Hence by mathematical induction the result holds for all connected graphs.

If G is a connected graph and T is a spanning tree of G. Edges of G present in T are called the branches of G with respect to and the edges of G which do not belong to T are called the chords of G with respect to T. If G has n vertices and e edges then, the number of branches with respect to the spanning tree T of G is n - 1 and the number of chords is e - n + 1.

The number of branches in a connected graph G is called the rank of G and the number of chords is called the nullity of G. If G has k components then the rank of G is defined as the sum of ranks of the components; i.e.,

Rank (G) = $\sum_{i=1}^{k} \text{rank}(G_i)$

$$= \sum_{i=1}^{k}(n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1$$

$$= n - k$$

where G_i, i= 1, 2, ..., K are K components of G.

and nullity of (G) = $\sum_{i=1}^{k} \text{nullity } (G_i)$

$$= \sum_{i=1}^{k}(e_i - n_i - 1)$$

$$= \sum_{i=1}^{k} e_i - \sum_{i=1}^{k} n_i + \sum_{i=1}^{k} 1$$

$$= e - n + 1.$$

13.7 Minimal Spanning Tree

Let G be a connected weighted graph. A minimal spanning tree of G is a spanning tree of G whose total weight is as small as possible.

There are various methods to find a minimal spanning tree in connected weighted graph. Here we consider algorithms for generating such a minimal spanning tree.

Algorithm

A connected weighted graph with n vertices.

Step 1: Arrange the edges of G in the order of decreasing weights.

Step 2: Proceed sequentially, and delete each edge of G, that does not disconnect the graph G until $n-1$ edges remain.

Step 3: Exit.

Example 9: Consider the graph G given below:

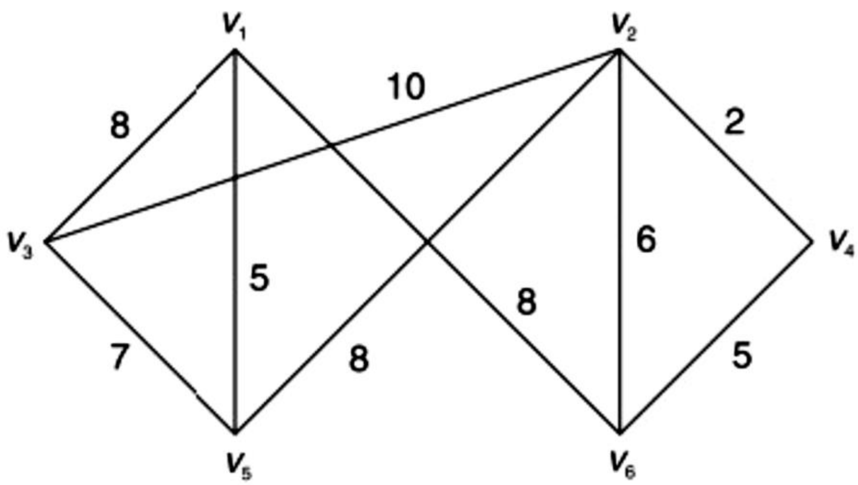


Fig.11

Number of vertices in G = n = 6.

We apply the algorithm given above.

We order the edged by decreasing weights and delete the edges of G until n – 1 = 6 – 1 = 5 edges remain.

Edges	(v_2, v_3)	(v_1, v_6)	(v_1, v_3)	(v_2, v_5)	(v_3, v_5)	(v_2, v_6)
Delete	Yes	Yes	Yes	No	No	Yes
Edges	(v_1, v_5)	(v_4, v_6)	(v_2, v_4)			
delete	No	No	No			

The minimal spanning tree of G is shown in Fig.12:

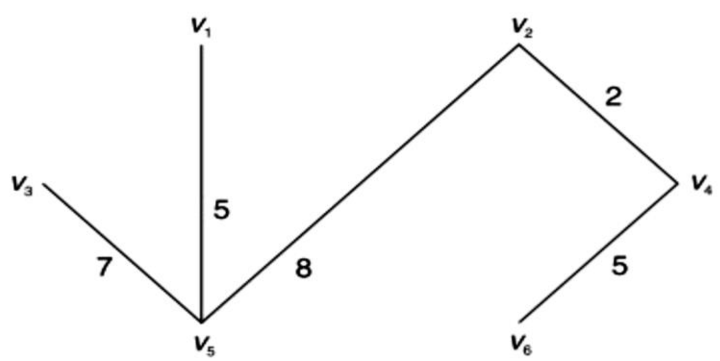


Fig.12

The weight of the minimum spanning tree

$$= 8 + 7 + 5 + 5 + 2 = 27.$$

13.8 Kruskal's Algorithm

Input: A connected weighted graph G with n vertices.

Step 1: Arrange the edges of in order of increasing weights and select the edge with minimum weight.

Step 2: Proceed sequentially, add each edge which does not result in a cycle until n – 1, edges are selected.

Step 3: Exit.

Example 10: Consider the graph in Fig. 13

We have n = 6

We order the edges by increasing weights (v_2, v_4) is edge with minimum weight. Select the edge (v_2, v_4) we successively add edges to (v_2, v_4), without forming cycles until 6 – 1 = 5 edges are selected. This yields:

Edges	(v_2, v_4)	(v_1, v_5)	(v_4, v_6)	(v_2, v_6)	(v_3, v_5)	(v_1, v_3)	(v_1, v_6)	(v_2, v_5)	(v_2, v_3)
Weight	2	5	5	6	7	8	8	8	10
Add?	Yes	Yes	Yes	No	Yes	No	Yes	No	no

Edges in the minimum spanning tree are

(v_2, v_4), $(v_1\ v_5)$, (v_4, v_6), (v_3, v_5), (v_1, v_6).

The resulting minimal (optimal) spanning tree is shown in Fig.

We apply the steps of Kruskal's algorithm to the graph of Fig. as follows:

(v_2, v_4) is the edge with minimum weight, therefore we select the edge (v_2, v_4).

The next edge with minimum weight is (v_1, v_5), selection of (v_1, v_5) does not result in a cycle.

$\therefore$ edge (v_1, v_5) is selected.

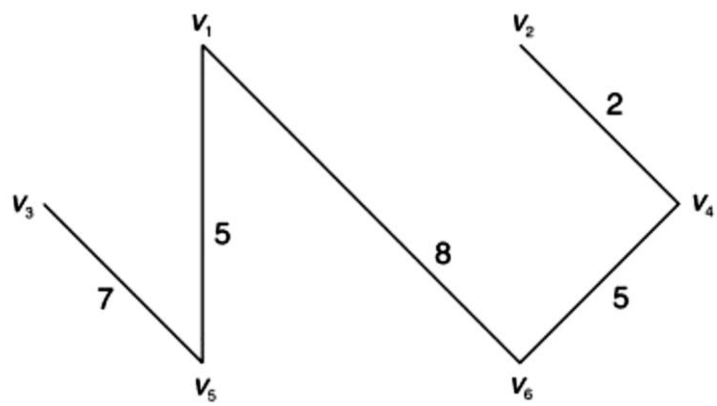


Fig. 13

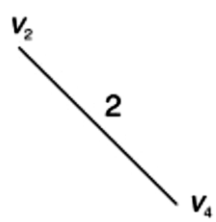


Fig. 13(a)

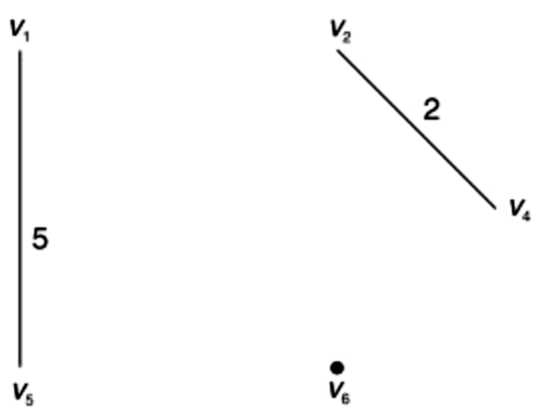


Fig. 13(b)

The edge to be considered, next is (v_4, v_6)

The next edge to be selected is (v_4, v_6)

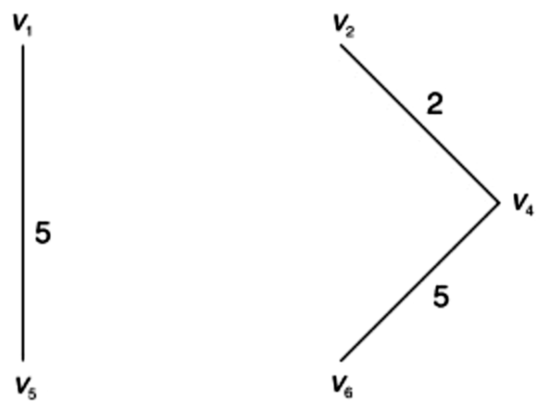


Fig.13 (c)

Selection of the edge (v_2, v_6) for the spanning tree results in a cycle. Therefore (v_2, v_6) is not selected we consider the edge (v_3, v_5) selection of edge (v_3, v_5) does not result in a cycle. Hence (v_3, c_5) is selected.

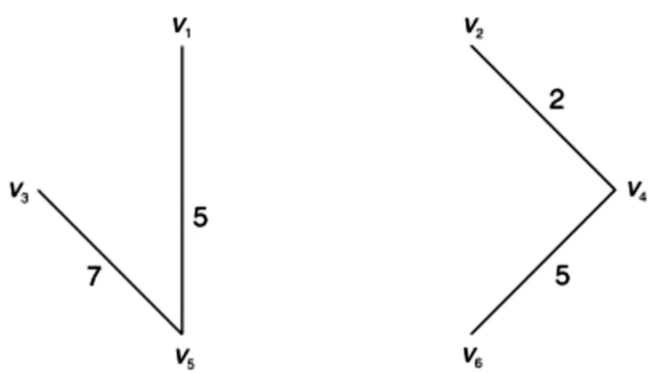


Fig. 13(d)

Next we consider the edge (v_1, v_3) from the list. Selection of the edge (v_1, v_3) results in a cycle. Therefore edge (v_1, v_3) is not selected. Consider the edge (v_1, v_6) selection of edge (v_1, v_6) does not result in a cycle. Hence (v_1, v_6) is selected.

Number of edges selected is 5. We stop, and obtain the spanning trees as shown in Fig.13(e).

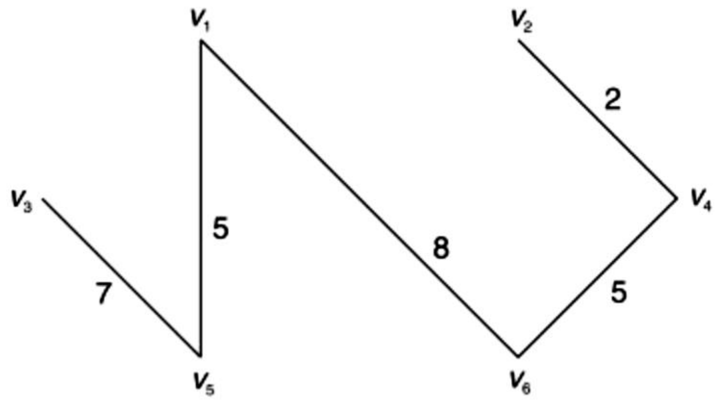


Fig. 13(e)

The weight of the minimal spanning tree.

= 2 + 5 + 5 + 7 + 8

= 27.

13.9 Prim's Algorithm

Input: A connected weighted graph G with n vertices.

Step 1: Select an arbitrary vertex v_1 and an edge e_1 with minimum weight incident with vertex v_1.

Step 2: Having selected the vertices $v_1, v_2, \ldots, v_i$ and $e_1, e_2, \ldots, e_{i-1}$; select an edge e_i such that e_i connects a vertex of the set $(v_1, v_2, .., v_i)$ and a vertex of $V = (v_1, v_2, ..., v_i)$ and of all such edges e_i has the minimum weight.

Step 3: Stop if n - 1, edges are selected, else go to step 2.

Example 11: Consider the graph shown in Fig.14:

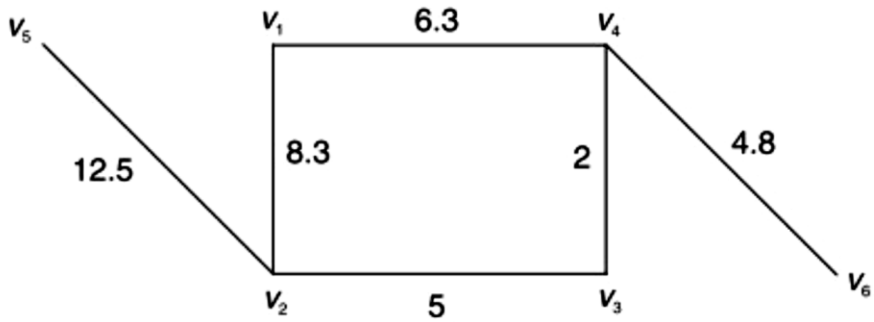


Fig.14

Let $e_1 = (v_1, v_2)$, $e_2 = (v_2, v_3)$

$e_3 = (v_3, v_4)$, $e_4 = (v_4, v_i)$

$e_5 = (v_2, v_5)$ and $e_6 = (v_4, v_6)$.

Denote the edge of G.

We apply Prims algorithm to the graph as follows:

The edge $e_3 = (v_3, v_4)$ is an edge with minimum weight. Hence, we start with the vertex v_3 and select the edge e_3 incident with v_3.

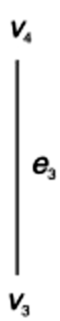


Fig. 14(a)

We next consider the edges connecting a vertex $\{v_3, v_4\}$ with the vertex of the set $V - \{v_3, v_4\}$. We observe that e_6 the edge with minimum weight.

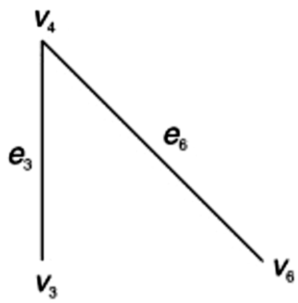


Fig.14 (b)

Consider the edges connecting the vertices of the set {v_3, v_4, v_6} with the vertices of V – { v_3, v_4, v_6}. The edge e_2 has the minimum weight. The edge e_2 is selected.

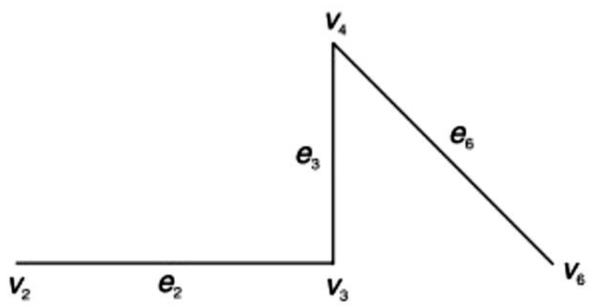


Fig.14 (c)

of the connecting the vertices of {v_2, v_3, v_4, v_6}; with the vertex set V – {v_2, v_3, v_4, v_6}, e_4 has minimum weight, therefore e_4 is selected.

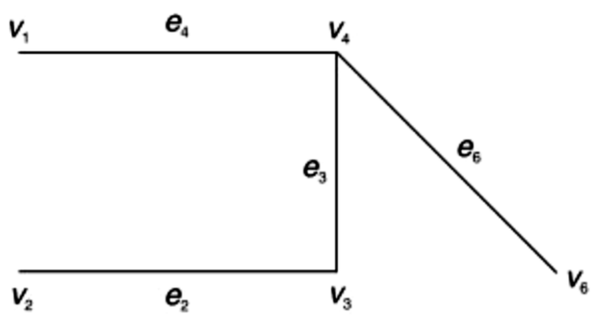


Fig. 14(d)

e_1, e_5 are the edges remaining. e_5 is the only edge connecting {v_1, v_2, v_3, v_4, v_5, v_6} and {v_5} such that the inclusion of e_5 does not result in a cycle. Hence e_5 is selected.

Since number of edges selected is 5 we stop.

The minimal spanning tree obtained is shown in Fig. 14(e).

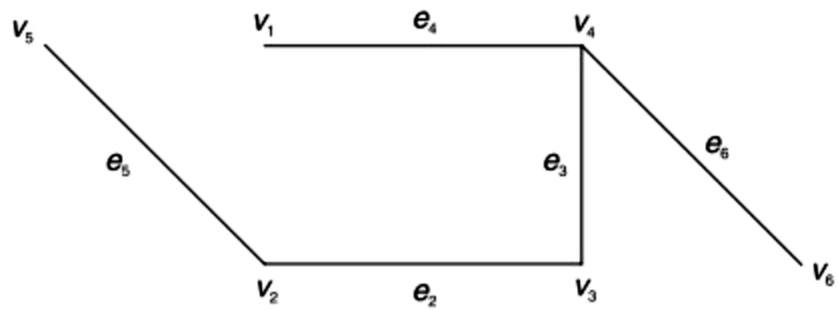


Fig. 14(e)

Weight of the minimal spanning tree

= 2 + 4.8 + 5 + 6.3 + 12.5=30.6.

13.10 Summary

A tree is a connected graph without any circuits.

A connected, a cyclic, directed graph is called a directed tree.

If u and v are two vertices of a connected graph G, there may be more than one path joining u and v. Various concepts can be defined based on the lengths of such paths between vertices of G.

If G is connected graph and u and v are any two vertices of G, the length of the shortest path between u and v is called the distance between u and v and is denoted by d (u, v).

Let G be a connected graph. The sub-graph H of G is called a spanning tree of G if

(i) H is a tree and (ii) H contains all the vertices of G.

A complete graph k_n has n^{n-2} different spanning trees.

Let G be a connected weighted graph. A minimal spanning tree of G is a spanning tree of G whose total weight is as small as possible.

13.11 Terminal Questions:

Q.1. Define the tree with examples.

Q.2. Write a short note on directed tree.

Q.3. What do you mean by minimal spanning tree?

Q.4. Explain Kruskal's algorithm.

Q.5. Write a short not on Prim's algorithm.

UNIT-14: Rooted and Binary Tree

Structure

14.1 Introduction

A rooted tree is a type of tree data structure where one vertex has been designated as the root and every edge is directed away from the root. This structure is often used in computer science for organizing hierarchical data. A binary tree is a type of tree where each node has at most two children, referred to as the left child and the right child. In a binary tree, the order of the children matters.

A binary tree can be viewed as a rooted tree where each node has either zero, one, or two children. The root of the binary tree corresponds to the root of the rooted tree, and each child relationship in the binary tree corresponds to a directed edge in the rooted tree. Binary trees have various applications in computer science, such as in binary search trees, expression trees, and Huffman coding trees. Rooted trees and binary trees have various applications in computer science and other fields of engineering and science.

14.2 Objectives

After reading this unit the learner should be able to understand about:

- Rooted Tree, Expression Tree
- Binary Tree, Complete Binary Tree
- Height Balanced Tree, B-Tree
- Distance between spanning tree of a graph

14.3 Rooted Tree

A rooted tree is a tree with a designated vertex called the root of the tree.

Any tree may be made into a rooted tree by selecting one of the vertices as the root. A rooted is a directed tree if there is a root from which there is a directed path to each vertex of the tree. The graphs in Fig.1 are rooted trees in which the root of each is at the top.

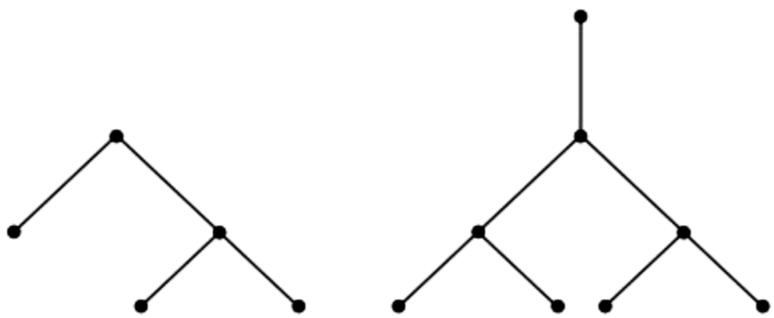


Fig. 1 Rooted trees

The level of a vertex in a rooted tree is the length of the path (number of edges) to v from the root. If T is a rooted tree with designated root v_0 and $v_0 - v_1 - v_2 - \ldots - v_{n-I} - v_n$ is a simple path in T, then v_{n-1} is called the parent of V_n and $v_0, v_1, v_2, \ldots, v_{n-1}$ are called the ancestors of v_n.

v_1 is a child of v_0, v_2 is a child of v_1, ••••

If T is a rooted tree with designated vertex v_0 and u and v are two vertices (nodes) in T, then

a) u is called a leaf of T, if it has no children (i.e., leaves of T are vertices of T with degree 1).

b) y is a descendant of u, if u is an ancestor of v.

c) Is an internal vertex of T, if v is not a leaf of T.

d) The sub-graph of T consisting of v and all its descendants with v as the designated root is a sub-tree of T rooted at v.

If T is a rooted tree then the maximum vertex level of T is called the depth of the tree. We usually adhere to the universal convention of representing the root of the tree as the top vertex (apex) of the tree. Rooted trees are useful in enumerating all the logical possibilities of a sequence of events where each event can occur in finite number f ways. If edges leaving each vertex of a rooted tree T are labeled, then T is called an ordered rooted tree. The vertices of an ordered to rooted can be labeled as follows: we assign 0 to the root of the tree. We next assign 1, 2, 3,4, ... to the vertices immediately following the root of T according as the edges were ordered. The remaining vertices can be ordered as follows: If p is the label of a vertex v of T then p1, p2, p3, ••• are assigned to the vertices immediately following v according as the edges were ordered. The tree in the below Fig. is an ordered rooted tree.

r is the root of the tree in Fig. 2. The vertices of Tare labeled with their addresses. The system is known as universal address system for an ordered rooted tree.

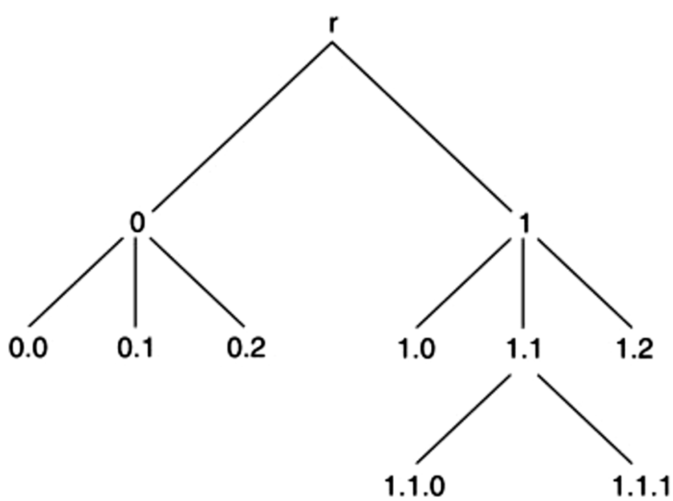


Fig. 2 Ordered rooted tree

14.4 Expression Trees:

Algebraic expressions involving addition, subtraction, multiplication and division can be represented as ordered rooted trees called expression trees. The arithmetic expression $3 + 5 \times 9 - 7 \times 6^2$ can be represented as the tree shown in Fig. 3

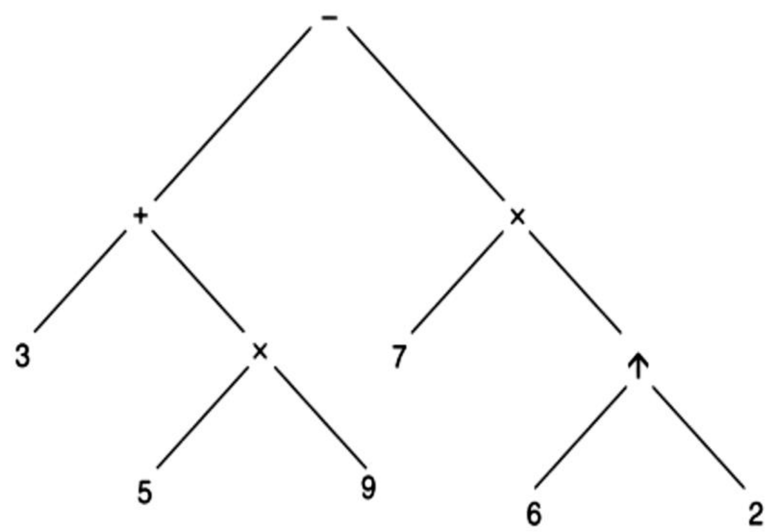


Fig.3

The variables in the algebraic expression appear as the other vertices. In the polish prefix representation, we place the binary operational symbol before the argument and avoid parentheses.

The expression (a – b)/((c x d) + e) can be expressed as I- ab + x cde.

Example.1: Write the following expression as a tree:

[(a x b) x c + (d + e) - (f- - (g x h))]

Solution: The arithmetic expression [(a x b) x c + (d + e) - (f- - (g x h))] can be represented as the tree.

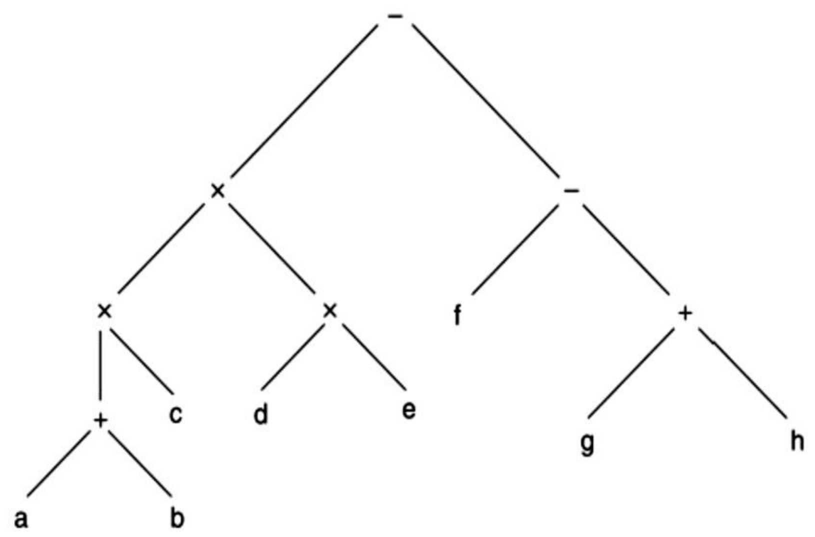


Fig.4

14.5 Binary Tree:

So far we have discussed the tree, and its properties. Now we shall study about a special class of trees known as binary trees. They are special class of rooted tree. Binary trees play an important role in decision - making. They are extensively used in the study of computer search methods, binary identification problems and coding theory.

A tree in which there is exactly one vertex of degree two, and each of the remaining vertices of degree one or three, is called a binary tree.

If Tis a binary tree, the vertex of degree two which is distinct from all the other vertices of T serves as a root of T. Thus every binary tree is a rooted tree. The vertices of degree one in a binary tree are called external vertices and all the remaining vertices are called internal vertices. The number of internal vertices in a binary tree is one less than the number of pendant vertices.

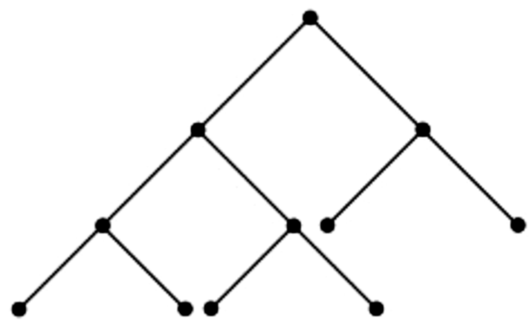


Fig. 5 Binary tree

The leaves of binary tree are ve1tices of degree one. Usually the roots in graph theory are portrayed, with the root and the leaves at the bottom. The direction from the root to leaves is taken as the down direction and the direction from the leaves to the root is taken as the up direction. The number of internal vertices in a binary tree is one less than the number of external vertices (pendant vertices). If v_i is vertex of a binary tree. v_i is said to be at a level I_i if v_i is at a distance l_i from the root of the binary tree. Thus the root a binary tree is at level 0.

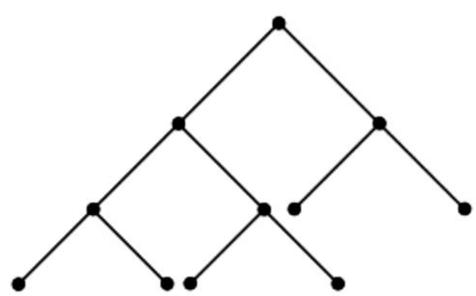


Fig. 6 A 11-vertex 3-level binary tree

The maximum level occurring in a binary tree is called the height of the binary tree. A binary tree with minimum height contains maximum number of vertices at each level. The root of a binary tree is at level 0 and there can be only one vertex at 0 level. The maximum number of vertices at level is 2^1, at level 2 is 2^2 and soon. By induction we can prove that the maximum number of vertices possible at level k in a binary tree is 2^2 We now state the following theorem on the minimum possible height of a binary tree:

Theorem.1: The minimum height of a binary tree on n vertices is $\lceil \log_2(n-1) - 1 \rceil$ (where $\lceil m \rceil$ is the smallest integer > m) and maximum possible height is $\frac{n-1}{2}$.

Proof: The root of T is at level 0. We know that every vertex of T at level k can have 2^1 successors.

Therefore we have

2 vertices at level l

2^2 vertices at level 2

Hence the maximum number of vertices in the binary tree of height l is

$1 + 2 + 2^2 + \ldots + 2^1$

ut T has n vertices, therefore

$1 + 2 + 2^2 + \ldots + 2^l \geq n$

Or $\frac{2^{l+1}-1}{2-1} \geq n$

Or $2^{l+1} - 1 \geq n$

Or $2^{l+1} \geq n + 1$

Hence $l \geq \log_2(n-1) - 1$

But l is an integer

The smallest possible value for 1 is

$$\lceil \log_2(n-1) - 1 \rceil$$

The minimum possible height of a binary tree T is

$$\lceil \log_2(n-1) - 1 \rceil$$

Now Jet denote the maximum possible height of T. We have the root of T at zero level, 2 vertices at level, 2 vertices at level 2, ...

2 vertices at level I.

When T is of height l, we have atleast 1 + (2 + 2 + ...I times) vertices in T.

i.e., 1 + 21 vertices in T

Hence $1 + 2l \leq n$

$\rightarrow 2l \leq n - 1$

$\rightarrow l \leq \frac{n-1}{2}$

but n is odd

$\frac{n-1}{2}$ is an integer. Hence

The maximum possible value of I is $\frac{n-1}{2}$

Thus, we have $\max l = \frac{n-1}{2}$

A binary tree can also be defined as follows:

A binary tree is a directed tree, T = (V, E) together with an edge labeling f; E $\rightarrow$ {0, 1}, such that every vertex of T has at most one edge incident from it is labeled 0, and at most are edge incident from it labeled with 1.

If T is a binary tree, then each edge (u, v) labeled with 0 is called a left edge. u is called the parent of V and v is called the left child of 11. Each edge (u, v) labeled with 1 is called a right edge in T. The vertex u is called the parent of v and u is called the right child of u.

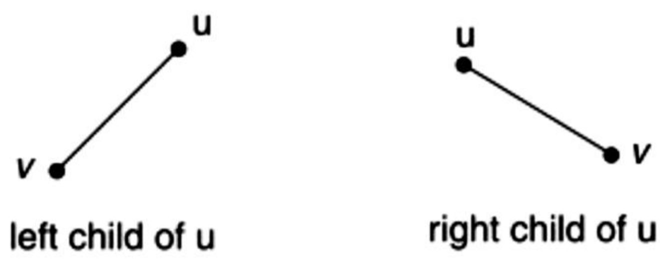


Fig. 7

Example.2: Show that the number of vertices in a binary tree is odd.

Solution: Let T be a binary tree with n vertices. T contains exactly one vertex of degree 2 and the remaining vertices of Tare of degree one or three. Therefore number of odd degree vertices in T is n – 1. But the number of odd degree vertices in a graph is even. Therefore n - 1 is even. Hence n is odd.

Example.3: T is a binary tree on n vertices and p is the number of pendant vertices in T. Show that the number of vertices of degree 3 in T is n - p - 1.

Solution: T has p vertices of degree one and one vertex in T is of degree two. Hence the number of remaining vertices (i.e., vertices of degree 3) is n -p - 1.

Example.4: T.1s a binary tree on n vertices. Show that the number of pendant vertices in T is $\frac{n-1}{2}$

Solution: Let p denote the number of pendant vertices in T.

The number of edges in T is n - I

The degree sum in T = 2 (n - 1)

Therefore $p \times 1 + 3(n - p - 1) + 2 = 2(n - 1)$

or $p + 3n — 3p - 3 + 2 = 2n - 2$

or $n + 1 = 2p$

Hence $p = \frac{n-1}{2}$

Example.5: Find the maximum possible height of a binary with 13 vertices and draw graph of the tree.

Solution: We have n = 13

Maximum possible height of the binary tree $\frac{n-1}{2} = \frac{13-1}{2} = 6$

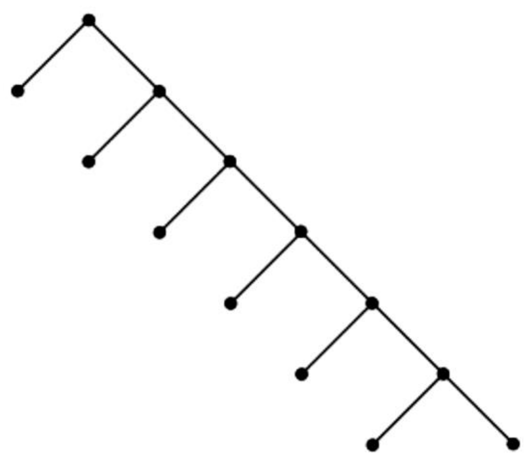


Fig. 8 Binary tree of maximum height with 13 vertices

Example.6: Find the minimum height of the tree with 9 vertices.

Solution: We have n = 9

The minimum height of the binary tree

$= \lceil \log_2(n-1) - 1 \rceil = \lceil \log_2(9-1) - 1 \rceil = 3.$

14.6 Complete Binary Tree:

A binary for which the level order indices of the vertices form a complete interval 1, 2, ..., n of the integers is called a complete binary tree.

If T is a complete binary tree, then all its levels except possibly the last, will have maximum number of possible vertices, and all the vertices at the last level appear as far left as possible. The tree shown in Fig.9 is a complete binary tree.

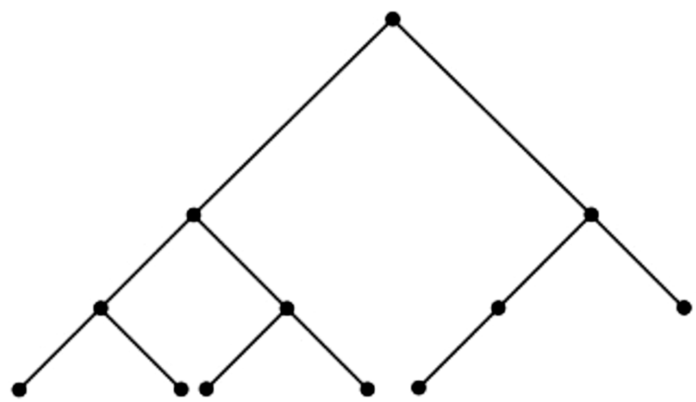


Fig.9 A complete binary tree

If T is a complete binary tree with n vertices, then the vertices at any level l are given the label numbers ranging from 2^l to 2^{l+1} or from 2^l to n if n is less than $2^{l+1} - 1$.

14.7 Height Balanced Binary Tree

A binary tree T in which the heights of left and right subtrees of every vertex differ by at most one is called a height balanced binary tree.

Every complete as a height balanced binary tree. Height balanced trees are important in computer science and are more general than complete binary tree. We state the following theorem without proof on the number of vertices in a height balanced binary tree.

Theorem: There are atleast $\frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^{h+3} - 2$ vertices in any height balanced binary tree with height h.

14.8 B-Tree

Let T be a directed tree of order k.

T is said to said to be a B-tree of order k, if

a) all the leaves are at the same level;

b) every internal vertex, except possibility the root has atleast $\lceil k/2 \rceil$ children (where $\lceil x \rceil$ means the least integer $\geq$ x);

c) The root is a leaf or bas atleast two children; and

d) no vertex bas more than k children.

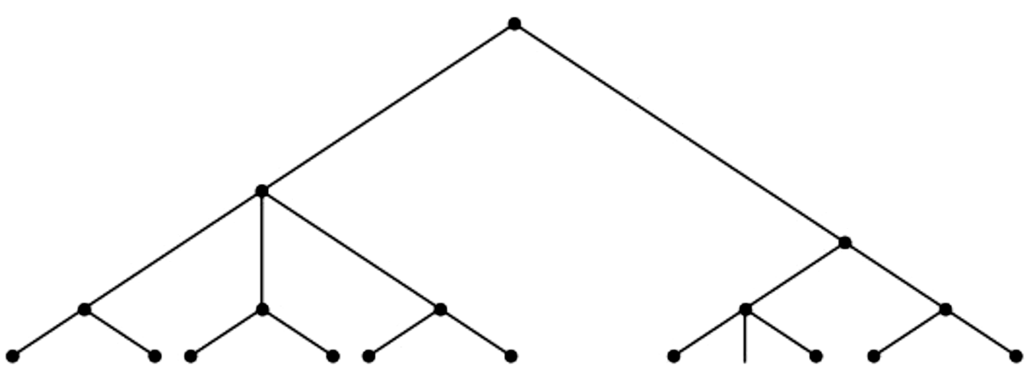


Fig. 10 B-tree

If the height of B-tree of order k is h $\geq$1, then the B-tree atleast $2\lceil k/2 \rceil^{h-1}$ leaves.

14.9 Distance between Spanning Tree of a Graph

Let T_i and T_j be two spanning trees of a graph G. The distance between T_i and T_j is defined as the number of edges of group G, present in T_i but not in T_j.

Example: Consider the graph G as shown in the Fig.11 below:

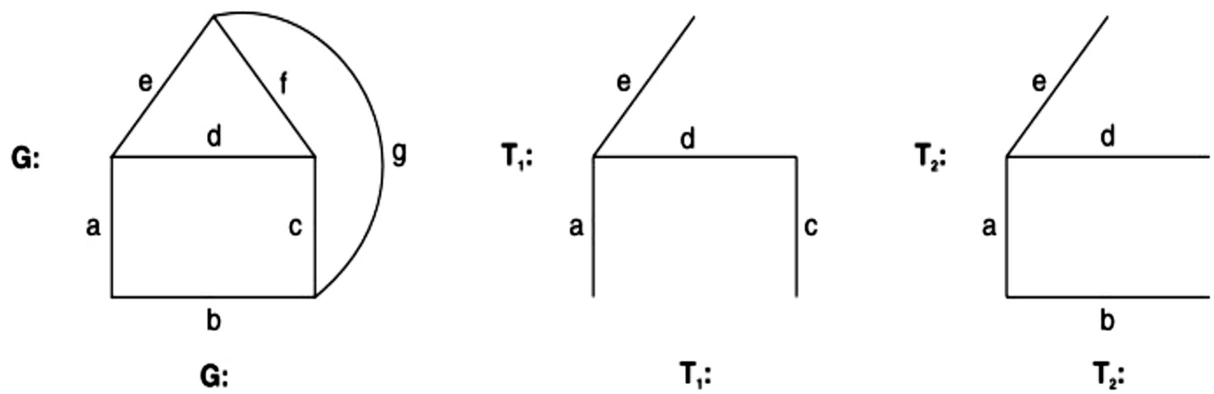


Fig.11 Graph G and two spanning trees T_1 and T_2.

The distance between the spanning trees T_1 and T_2 is one.

14.10 Summary

A rooted tree is a tree with a designated vertex called the root of the tree.

Algebraic expressions involving addition, subtraction, multiplication and division can be represented as ordered rooted trees called expression trees.

A tree in which there is exactly one vertex of degree two, and each of the remaining vertices of degree one or three, is called a binary tree.

A binary for which the level order indices of the vertices form a complete interval 1, 2, ..., n of the integers is called a complete binary tree.

A binary tree T in which the heights of left and right subtrees of every vertex differ by at most one is called a height balanced binary tree.

Let T_i and T_j be two spanning trees of a graph G. The distance between T_i and T_j is defined as the number of edges of group G, present in T_i but not in T_j.

14.11 Terminal Question

1. Define the Rooted Tree, with gives examples.

2. Prove that a tree with n vertices has exactly $(n-1)$ edges.

3. Define the term

(i) Spanning Tree **(ii)** Binary Tree, with gives examples.

4. Show that number of vertices of a binary tree is odd.

5. Show that number of pendant vertices in a binary tree with n vertices is $\frac{n+1}{2}$.

6. Define the term minimal Spanning Tree of a graph.

7. State Kruskal's algorithm for find the minimal Spanning Tree.