॥ सरस्वती नः सुभगा मयस्करत् ॥

Uttar Pradesh Rajarshi Tandon Open University

PGMM-101N/ MAMM-101N

Advanced Real Analysis And Integral Equations

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Uttar Pradesh Rajarshi Tandon Open University

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Advanced Real Analysis And Integral Equations

BLOCK

1

REAL ANALYSIS

UNIT-1

Riemann Integral

UNIT-2

Integration and Differentiation

UNIT-3

Uniform Convergence of Sequences

UNIT-4

Power Series

BLOCK INTRODUCTION

Real analysis is a mathematical discipline focused on the study of real numbers and real-valued functions. It involves a rigorous examination of fundamental concepts like limits, continuity, differentiation, integration, and the behavior of sequences and series of real numbers. Real analysis serves as the theoretical foundation of calculus and finds applications across various fields, including physics, engineering, and economics. The Riemann-Stieltjes integral is a significant extension of the Riemann integral, allowing for the integration of functions that are not necessarily continuous but possess bounded variation. Named after Bernhard Riemann and Thomas Stieltjes, this integral broadens the scope of integration to a wider class of functions, providing a more adaptable approach to integration.

In addition to its role in extending integration concepts, the Riemann-Stieltjes integral plays a crucial role in probability theory and statistics. It is employed to define the expectations of random variables with respect to specific distribution functions, facilitating a deeper understanding of various probabilistic concepts and statistical methods. The Riemann-Stieltjes integral finds application in the theory of differential equations, where it is utilized to define the integral of functions multiplied by distributions. This application is particularly valuable in the study of systems characterized by discontinuous or singular inputs.

In the first unit, we shall have discussed the Partition, lower and upper Riemann-Stieltjes sums, lower and upper Riemann-Stieltjes integrals, Definition of Riemann-Stieltjes integral, necessary and sufficient condition for Riemann-Stieltjes integrability, algebra of Riemann-Stieltjes integrable functions. In the second unit we shall discuss the Integral Function, primitive, fundamental theorem of integral calculus, integration by parts, Integration of vector-valued functions. In the third unit we shall discuss about Uniformly bounded sequence, uniform convergence of sequences, Uniform convergence of a series of function, Cauchy’s general principle of uniform convergence, test for uniform convergence. Power series, Cauchy’s theorem on limits, Radius of convergence, Uniform convergence of power series. Abel’s and Tauber’s theorems are discussed in details in unit fourth.

UNIT 1 RIEMANN INTEGRAL

Structure

1.1 Introduction

1.2 Objectives

1.3 Partition of a closed Interval

1.4 Lower Riemann Sum and Upper Riemann Sum

1.5 Lower and Upper Riemann Integral

1.6 Another definition of Riemann Integrable

1.7 Lower and Upper Riemann-Stieltjes Sums

1.8 Lower Riemann-Stieltjes Integral and Upper Riemann-Stieltjes Integral

1.9 The Riemann-Stieltjes Integral

1.10 The Riemann-Stieltjes Integral as a limit Sums

1.11 Properties of Riemann-Stieltjes Integral

1.12 Algebra of Riemann-Stieltjes Integrable Function

1.13 Summary

1.14 Terminal Questions

1.1 INTRODUCTION

In 1850 the german mathematician G.F.B. Riemann (1826-1866) gave a purely arithmatic approach to formulate and independent theory of integration. Riemann Theory lead others to invent others to invent other integration theories. The most significant being legesgue theory of integration. The Riemann integral is a fundamental concept in calculus that defines the definite integral of a function over a closed interval. It is named after the mathematician Bernhard Riemann, who introduced the integral in the mid-19th century. The Riemann integral is based on the idea of approximating the area under a curve by dividing the interval into smaller subintervals and forming rectangles whose areas approximate the area under the curve. As the width of the subintervals approaches zero, the sum of the areas of these rectangles converges to the Riemann integral of the function over the interval. In this unit we shall discuss the Riemann integral of real valued bounded functions defined on some closed interval. We shall also discuss refinemonts and extension of Riemann theory due to Thomas Jan stieltjes known Riemann-stieltjes integration.

1.2 OBJECTIVES

After reading this unit the learner should be able to understand about the:

- Partition of a closed interval
- lower Riemann sum and upper Riemann sum
- lower and upper Riemann integral
- lower and upper Riemann-Stieltjes Sums
- Riemann –Stieltjes Integral and properties
- Algebra of Riemann=Stieltjes Integrable Function

1.3 PARTITION OF A CLOSED INTERVAL

Let $I=[a,b]$ be a closed and bounded interval then a finite set of real number $P=\{x_0, x_1, x_2, \ldots\ldots, x_{n-1}, x_n\}$ having the property that $a=x_0<x_1<x_2<\ldots\ldots\ldots<x_{n-1}<x_n=b$ [Here $x_0=a$ and $x_n=b$] is called the partition of $[a,b]$.

The closed sub-intervals $I_1=[x_0,x_1], I_2=[x_1,x_2], I_3=[x_2,x_3], \ldots\ldots\ldots, I_n=[x_{n-1},x_n]$ are called segments of the partition Δx_r where $\Delta x_r = x_r - x_{r-1}$ *i.e.*, length of I_r [Here $I_r=[x_{r-1},x_r]$]The norm of a partition P is the maximum of Δx_r defined as $\|P\|=\max\{\Delta x_r : r=1,2,3,\ldots,n\}$

i.e., $\|P\|=\max\{(x_1-x_0),(x_2-x_1),(x_3-x_2),\ldots.,(x_n-x_{n-1})\}$.

A portition P^* is called a refinement of another partion P if and only if $P^* \supset P$ *i.e.*, every point P is used to build P^*.

If P_1 and P_2 are any two partitions of $[a,b]$ then $P^*=P_1 \cup P_2$ is called common refinement of P_1 and P_1.

1.4 LOWER RIEMANN SUM AND UPPER RIEMANN SUM

Let f be a bounded real valued function defined on $[a, b]$ and let $P = \{a = x_0, x_1, x_2, ..., x_{n-1}, x_n\}$ be a partition of $[a, b]$. Let $m_r =$ infimum of f in I_r (Here $I_r = [x_{r-1}, x_r]$)

$M_r =$ supremum of f in I_r respectively, then

$L(P, f) = \sum_{r=1}^{n} m_r \Delta x_r = (m_1 \Delta x_1 + m_2 \Delta x_2 + ... + m_n \Delta x_n)$ is called the lower-Riemann sum

and $U(P, f) = \sum_{r=1}^{n} M_r \Delta x_r = [M_1 \Delta x_1 + M_2 \Delta x_2 + ... + M_n \Delta x_n]$ is called the upper-Riemann sum.

Note: (1) Riemann sum is also known on Darbou x sum.

(2) $U(P, f) - L(P, f) = \sum_{r=1}^{n} (M_r - m_r) \Delta x_r$ is called oscillatory sum for the function f corresponding to the partition P of $[a,b]$. It is denoted by $w(P, f)$ i.e., $w(P, f) = [U(P, f) - L(P, f)]$.

Here we are giving some important theorems with and without proof and definitions of riemann integral.

Theorem.1. Let f be a bounded function on $[a,b]$. Let P be a partition of $[a,b]$. If P^* is a refinement of P then $L(P^*, f) \geq L(P, f)$ and $U(P^*, f) \leq U(P, f)$

Theorem.2: Let P be a partion of $[a,b]$. Let P_1 and P_2 be any two partion of $[a,b]$ such that $P = P_1 \cup P_2$ then $U[P_1, f] \geq L(P_2, f)$

Theorem.3: Let f, g be a bounded function on $[a,b]$ and let P be a partition of $[a,b]$. Then $L[P, f+g] \geq L[P, f] + L(P, g)$ and $U[P, f+g] \leq U[P, f] + U[P, g]$.

1.5 LOWER AND UPPER RIEMANN INTEGRAL

Lower Riemann integral

$$\int_{\underline{a}}^{b} f(x)\,dx = \sup.\text{ of } L(P.f) \text{ for all partitions } [a,b]$$

$$\Rightarrow \quad \int_{\underline{a}}^{b} f(x)\,dx \geq L(P.f) \quad \text{or} \quad \lim_{\|P\| \to 0} L(P, f) = \int_{\underline{a}}^{b} f.$$

Upper Riemann integral

$$\int_{a}^{\bar{b}} f(x)\,dx = \text{ infimum of } U(P, f) \text{ of all partitions of } [a,b]$$

$$\Rightarrow \quad \int_a^{\bar{b}} f(x)dx \le U(P, f) \quad \text{or} \quad \int_a^{\bar{b}} f = \lim_{\|P\|\to 0} U(P, f).$$

Note: (I). If $\int_{\underline{a}}^{b} f = \int_a^{\bar{b}} f \quad \Rightarrow \quad f \in R[a,b]$ *i.e.*, f is Riemann integrable

(2) $\int_{\underline{a}}^{b} -f = -\int_a^{\bar{b}} f$

(3) $\int_a^{\bar{b}} -f = -\int_{\underline{a}}^{b} f$

(4) $\int_{\underline{a}}^{b} f \le \int_a^{\bar{b}} f$

Theorem.4. Let f be a bounded function defined on $[a,b]$ then for every $\varepsilon > 0, \exists \delta > 0$ such that $U[P, f] < \int_a^{\bar{b}} f + \varepsilon$ and $L[P, f] > \int_{\underline{a}}^{b} f - \varepsilon$ for a partition P of $[a,b]$ with $\|P\| < \delta$.

Theorem.5: A necessary and sufficient condition for R-integrability of a function $f:[a,b] \to R$ on $[a,b]$ is for $\varepsilon > 0, \exists$ a partition P of $[a,b]$ such that for P and all its refinements $0 \le U(P, f) - L(P, f) < \varepsilon$.

1.6 ANOTHER DEFINITION OF RIEMANN INTEGRABLE

A function f define on $[a,b]$ is said to be Riemann integrable over $[a,b]$ if and only if for every $\varepsilon > 0$ there exists a $\delta > 0$ and a number I such that for every partition $P = \{a = x_0, x_1, \ldots\ldots, x_n = b\}$ of $[a,b]$ with $\|P\| < \delta$ and for every ε hence of $\varepsilon_r \in [x_{r-1}, x_r]$

$$\left| \sum_{r=1}^{n} f(\varepsilon_r)(x_r - x_{r-1}) - I \right| < \varepsilon$$

$$\Rightarrow \quad I = \int_a^b f(x)dx$$

i.e., I is R- integrable.

Now, we shall discuss the riemann-stieltjes integral which is a refinement and extension of riemann theory.

1.7 LOWER AND UPPER RIEMANN-STIELTJES SUMS

Let f be a real valued function defined on a closed interval $[a,b]$ and let g be a monotonically increasing (real valued) function on $[a,b]$. Let $P = \{a = x_0, x_1, x_2, \ldots., x_n = b\}$ be a partition of $[a,b]$ with $I_r = [x_{r-1}, x_r], r = 1, 2, 3, \ldots., n$

We taken $\Delta g_r = g(x_r) - g(x_{r-1}), r = 1, 2, ..., n \Longrightarrow \Delta g_r >, 0$ as g is monotonically increasing function.

Let $m = \inf \{ f(x) | a \le x \le b \}$

$M = \sup \{ f(x) | a \le x \le b \}$

$m_r = \inf \{ f(x) | x \in I_r \}$

$M_r = \sup \{ f(x) | x \in I_r \}$

Then $L[P, f, g] = \sum_{r=1}^{n} m_r \Delta g_r$ is called lower Riemann-stieltjes sums

And $U[P, f, g] = \sum_{r=1}^{n} M_r \Delta g_r$ is called upper Riemann-stieltjes sums.

Theorem.6. Let f be a bounded function on $[a,b]$ and let g be a monotonically increasing (real valued) function on $[a,b]$. Let P be a partition of $[a,b]$. If $p*$ is refinement of Pthen

$L(P, f, g) \le L(P*, f, g)$

$U(P*, f, g) \le U(P, f, g)$

Proof: Let $P = \{ a = x_0, x_1, x_2, ..., x_{n-1}, x_n = b \}$ be a partition of $[a,b]$. Let $P^* = \{ x_0, x_1, x_2, ..., x_{r-1}, y, x_r, ...x_n \}$ be a refinement of P.

Let $m_r = \inf. \text{ of } f(x) \text{ in } \mathrm{I}_r$

$M_r = \sup. \text{ of } f(x) \text{ in } \mathrm{I}_r$

$m'_r = \inf. \text{ of } f(x) \text{ in } [x_{r-1}, y]$

$M'_r = \sup. \text{ of } f(x) \text{ in } [x_{r-1}, y]$

$m''_r = \inf. \text{ of } f(x) \text{ in } [y, x_r]$

$M''_r = \sup. \text{ of } f(x) \text{ in } [y, x_r]$ respectively

Then we have

$$M_r \ge M'_r, M''_r$$

And $m_r \le m'_r, m''_r$

Here P^* has two subintervals more that P namely $[x_{r-1}, y]$ and $[y, x_r]$ respectively

$\therefore U[P, f, g] - U[P*, f, g] = \sum_{r=1}^{n} M_r [g(x_r) - g(x_{r-1})]$

$$-\left[\sum_{r=1}^{n} M'_r \left(g(y) - g(x_{r-1})\right)\right] + \sum_{r=1}^{n} m''_r \left[g(x_r) - g(y)\right] \underline{\qquad\qquad\qquad} (1)$$

But $M'_r \left[g(y) - g(x_{r-1})\right] + M''_r \left[g(x_r) - g(y)\right]$

$$\leq M_r \left[g(y) - g(x_{r-1})\right] + M_r \left[g(x_r) - g(y)\right]$$

$$\leq M_r \left[g(x_r) - g(x_{r-1})\right]$$

$$\Rightarrow \sum M'_r \left[g(y) - g(x_{r-1})\right] + \sum M''_r \left[g(x_r) - g(y)\right] \leq \sum M_r \left[g(x_r) - g(x_{r-1})\right]$$

$$\Rightarrow U\left[P^*, f, g\right] \leq U\left[P, f, g\right] \underline{\qquad\qquad\qquad} (2)$$

Similarly

$$L\left[P, f, g\right] \leq L\left[P^*, f, g\right] \underline{\qquad\qquad\qquad} (3)$$

Therefore, equation (2) and (3) give the required result.

Theorem.7. Let P_1 and P_2 be any two partitions of $\left[a, b\right]$ such that $P = P_1 \cup P_2$. Let g be monotonically function on $\left[a, b\right]$ then $U\left[P_1, f, g\right] \geq L\left[P_2, f, g\right]$

Proof: Let $P = P_1 \cup P_2$

It means P contains more subintervals then P_1 and P_2 separately.

$$\therefore U\left[P, f, g\right] \leq U\left[P_1, f, g\right] \underline{\qquad\qquad\qquad} (1)$$

$$U\left[P, f, g\right] \leq U\left[P_2, f, g\right] \underline{\qquad\qquad\qquad} (2)$$

And $\quad L\left[P, f, g\right] \geq L\left[P_1, f, g\right] \underline{\qquad\qquad\qquad} (3)$

$$L\left[P, f, g\right] \geq L\left[P_2, f, g\right] \underline{\qquad\qquad\qquad} (4)$$

We know that

$$L\left[P, f, g\right] \leq U\left[P, f, g\right] \underline{\qquad\qquad\qquad} (5)$$

$\therefore$ Equations (1), (2),(3),(4) and (5) imply

$$L\left[P_2, f, g\right] \leq L\left[P, f, g\right] \leq U\left[P, f, g\right] \leq U\left[P_1, f, g\right]$$

$\Rightarrow L\left[P_2, f, g\right] \leq U\left[P_1, f, g\right]$ i.e., $U\left[P_1, f, g\right] \geq L\left[P_2, f, g\right]$ Proved.

Note: Here f is defined on $\left[a, b\right]$ we mean (i) $f : \left[a, b\right] \rightarrow R, R$ be a set of reals.

(ii) In future g will be understood monotonocally increasing (real valued) on $\left[a, b\right]$ unless otherwise stated.

1.8 LOWER RIEMANN-STIELTJES INTEGRAL AND UPPER

RIEMANN-STIELTJES INTEGRAL

Let f be a bounded function and g be a monotonocally increasing function on $[a,b]$ then lower-Riemann stieltjes integral of f relative to g on $[a,b]$ is the supermum of $L[P, f, g]$ i.e., $\underline{\int_a^b} f\, dg = \sup . L[P, f, g]$.

And upper Riemann-stieltjes integral of relative to g on $[a,b]$ is the infimum of $U[P, f, g]$

i.e., $\overline{\int_a^b} f\, dg = \inf . U[P, f, g]$

Theorem.8. Let f be a bounded function and g be monotonically increasing function on $[a,b]$. Then $\underline{\int_a^b} f\, dg \le \overline{\int_a^b} f\, dg$

Proof: We know if P_1 and P_2 be any two partitions of $[a,b]$ such that $P = P_1 \cup P_2$ then we have

$$L[P_1, f, g] \le U[P_2, f, g] \_\_\_\_\_\_\_\_ (1)$$

Taking the supermum over all P_1, taking P_2 fixed, we get

$$\underline{\int_a^b} f\, dg \le U[P_2, f, g] \left[\because \underline{\int_a^b} f\, dg = \sup L[P_1, f, g]\right] \qquad \ldots\ldots\ldots(2)$$

Again, Taking the infimum over all P_2 ,we have from (2)

$\underline{\int_a^b} f\, dg \le \overline{\int_a^b} f\, dg \left[\because \overline{\int_a^b} f\, dg = \inf U[P_2, f, g]\right]$ Proved.

1.9 THE RIEMANN-STIELTJES INTEGRAL

Let f be a bounded function and g be monotonically increasing function on $[a,b]$. Then f is said to be Riemann-stieltjes integrable (Or RS-integrable) if and only if $\underline{\int_a^b} f\, dg = \overline{\int_a^b} f\, dg$ i.e., $f \in RS$ integrable $[[a,b], g] \Leftrightarrow \underline{\int_a^b} f\, dg = \overline{\int_a^b} f\, dg$

i.e. $f \in RS-$ integral $[[a,b], g]$ we mean $\underline{\int_a^b} f\, dg = \overline{\int_a^b} f\, dg$. Or $f \in RS(g)$

Note. $\int_a^b f\, dg$ we mean $\int_a^b f(x)\, dg(x)$.

Theorem.9: Let f be a bounded function and g be a monotonically increasing function on $[a, b]$. Then for every $\varepsilon > 0$, There exists $\delta > 0$ such that $U[P, f, g] < \overline{\int_a^b} f\, dg + \varepsilon$ and $L[P, f, g] > \underline{\int_a^b} fdg - \varepsilon$ for all partition p with $\|P\| < \delta$.

Proof: Let f be a bounded function and g be a monotonically increasing function on $[a,b]$ Let $P = \{a, x_0, x_1, ..., x_n = b\}$ be a partition of $[a,b]$ then for $\varepsilon > 0$, we have $\overline{\int_a^b} f\,dg =$ infimum of $U[P, f, g]$ for all P. $\underline{\int_a^b} f\,dg =$ supremum of $L(P, f, g)$ for all P.

So for given $\varepsilon > 0$, There exists partitions P_1 and P_2 of P such that

$$U[P_1, f, g] < \overline{\int_a^b} f\,dg + \varepsilon \text{\_\_\_\_\_\_\_\_\_\_} (1)$$

And $$L[P_2, f, g] < \underline{\int_a^b} f\,dg - \varepsilon \text{\_\_\_\_\_\_\_\_\_\_} (2)$$

Let P_3 be the common refinement of P_1 and P_2 then

$$U[P_3, f, g] \le U[P_1, f, g] \text{\_\_\_\_\_\_\_\_\_\_} (3)$$

$$L[P_4, f, g] \ge L[P_2, f, g] \text{\_\_\_\_\_\_\_\_\_\_} (4)$$

From equation (1), (2), (3) &(4), we have

$$U[P, f, g] \le U[P_1, f, g] < \overline{\int_a^b} f\,dg + \varepsilon$$

$$\Rightarrow U[P, f, g] < \overline{\int_a^b} f\,dg + \varepsilon \text{\_\_\_\_\_\_} (5)$$

Similarly $$L[P, f, g] > \underline{\int_a^b} f\,dg - \varepsilon \text{\_\_\_\_\_\_} (6)$$

For all partitions P of $[a,b]$ with $\|P\| < \delta$ where $\delta > 0$.

Theorem.10: Let f be a bounded function and g be a monotonically increasing function on $[a,b]$. Then $f \in RS(g)$ if and only if for every $\varepsilon > 0$ there exists a partition P such that

$$U[P, f, g] - L[P, f, g] < \varepsilon$$

Proof: Let f be a bounded function and g be a monotonically increasing function on $[a,b]$.

Let $P = \{a = x_0, x_1, ..., x_n = b\}$ be a partition of $[a,b]$ suppose for $\varepsilon > 0$, we have

$$U[P, f, g] - L[P, f, g] < \varepsilon \text{\_\_\_\_\_\_\_\_} (1)$$

To prove $f \in RS(g)$

We know for every partition P

$$L[P,f,g] \le \underline{\int_a^b} f\,dg \le \overline{\int_a^b} f\,dg \le U[P,f,g] \_\_\_\_\_\_(2)$$

From (1) and (2), we get

$$0 \le \overline{\int_a^b} f\,dg - \underline{\int_a^b} f\,dg \le U[P,f,g] - L[P,f,g] < \varepsilon$$

$\Rightarrow \overline{\int_a^b} f\,dg - \underline{\int_a^b} f\,dg \le \varepsilon$ Take $\varepsilon \to 0$ as ε is arbitrary

$\Rightarrow \overline{\int_a^b} f\,dg = \underline{\int_a^b} f\,dg \Rightarrow f \in \mathrm{RS}(g)$

"Only of Part." Let $f \in RS(g)$ To prove $U[P,f,g] - L[P,f,g] < \varepsilon$ given $f \in RS(g) \Rightarrow \underline{\int_a^b} f\,dg = \overline{\int_a^b} f\,dg$ Let $\varepsilon > 0$ be given. Since $\underline{\int_a^b} f\,dg$ is the supremum of $L[P,f,g]$ over all partition P, there exists a partition P_1 such that

$$\underline{\int_a^b} f\,dg < L[P_1,f,g] + \frac{\varepsilon}{2}$$

Similarly, since $\overline{\int_a^b} fdg$ is the infimum of $U[P,f,g]$ over all partition P, there exists a partition P_2 such that $U[P_2,f,g] < \overline{\int_a^b} f\,dg + \frac{\varepsilon}{2}$

Let $P = P_1 \cup P_2$ so that P is the common refinement of P_1 nd P_2 then we have

$$\underline{\int_a^b} f\,dg < L[P,f,g] + \frac{\varepsilon}{2}$$

And $\quad \overline{\int_a^b} f\,dg > U[P,f,g] - \frac{\varepsilon}{2} \_\_\_\_\_(3)$

$\Rightarrow -\underline{\int_a^b} f\,dg > -L[P,f,g] - \frac{\varepsilon}{2} \_\_\_\_\_\_(4)$

Adding (3) & (4) , we get $\overline{\int_a^b} f\,dg - \underline{\int_a^b} f\,dg > U[P,f,g] - L[P,f,g] - \varepsilon$

$\Rightarrow U[P,f,g] - L[P,f,g] < \varepsilon \left[\because \overline{\int_a^b} f\,dg = \underline{\int_a^b} f\,dg\right]$, proved.

Examples-

Example.1. Let f be a constant function on $[a,b]$ defined by $f(x) = k$ and g be monotonically increasing function on $[a,b]$ then

$$\int_a^b f\,dg \text{ exists and } \int_a^b f\,dg = k\left[g(b) - g(a)\right]$$

Solution: Let $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ be any partition of $[a,b]$.

Let $I_r = [x_{r-1}, x_r], r = 1, 2, ..., n$.

Let $m_r = \inf$ of $f(x)$ in $I_r = k$ as $f(x) = k$

$m_r = \sup$ of $f(x)$ in $I_r = k$ as $f(x) = k$

$\therefore \underline{\int_a^b} f\, dg = \sup L(P, f, g)$

$$= \sup \sum_{r=1}^{n} m_r \,\Delta g_r, \; [\Delta g_r = g(x_r) - g(x_{r-1})]$$

$$= \sup \sum_{r=1}^{n} k[g(x_r) - g(x_{r-1})]$$

$$= \sup k[g(x_1) - g(x_0) + g(x_2) - g(x_1) + ... + g(x_n) - g(x_{n-1})]$$

$$= \sup k[g(x_n) - g(x_0)]$$

$$\underline{\int_a^b} f\, dg = k[g(b) - g(a)]$$

Similarly, we have

$$\overline{\int_a^b} f\, dg = \inf U[P, f, g]$$

$$= \inf \sum_{r=1}^{n} k\,\Delta g_r$$

$$= k[g(b)g(a)] \text{ Since } \underline{\int_a^b} fdg = \overline{\int_a^b} f\, dg \Rightarrow f \in RS(g)$$

$$\Rightarrow \int_a^b f\, dg = k[g(b) - g(a)]$$

Note: $\sum_{r=1}^{n} \Delta gr = \sum_{r=1}^{n} g(x_r) - g(x_{r-1})$

Put $r = 1, 2, 3, ...n.$, we have

$$\Delta g_1 = g(x_1) - g(x_0)$$
$$\Delta g_2 = g(x_2) - g(x_1)$$
$$\Delta g_3 = g(x_3) - g(x_2)$$

$$\Delta g_{n-1} = g(x_{n-1}) - g(x_{n-2})$$
$$\Delta g_n = g(x_n) - g(x_{n-1})$$

________________ Adding qerlically

$$\sum_{r=1}^{n} \Delta g_r = g(x_n) - g(x_0)$$

1.10 THE RIEMANN-STIELTJES INTEGRAL AS A LIMIT SUMS

Let f be a bounded function and g be a monotonic increasing function on $[a,b]$. Let $P = \{a = x_0, x_1, ..., x_n = b\}$ be a partition of $[a,b]$ and let $Q = \{\varepsilon_1, \varepsilon_2, ..., \varepsilon_r, ..., \varepsilon_n\}$ be an intermedeate partition of P such that

$$x_{r-1} \le \varepsilon_r \le x_r$$

Then the sum

$$S[P, Q, f, g] = \sum_{r=1}^{n} f(\varepsilon_r) \Delta g_r$$

Where $\Delta g_r = g(x_r) - g(x_{r-1})$

Is called the riemann-stieltjes (or RS -sum) of f relative to g on $[a,b]$ corresponding to the partition P and the intermedeate partition Q.

Theorem.11: If $\lim\limits_{\|P\| \to 0} S(P, Q, f, g)$ **exists, then** $f \in RS(g)$

And $\lim\limits_{\|P\| \to 0} S(P, Q, f, g) = \int_a^b f \, dg$

Proof: Suppose $\lim\limits_{\|P\| \to 0} S(P, Q, f, g) = \lim\limits_{\|P\| \to 0} \sum_{r=1}^{n} f(\varepsilon_r) \Delta g_r$ exists and is equal to A.

Therefore, for $\varepsilon > 0$, There exists $\delta > 0$ such that for every partition P of $[a,b]$ with $\|P\| < \delta$, we have

$$|S(P, Q, f, g) - A| < \frac{\varepsilon}{2}$$

Or $\left(A - \frac{\varepsilon}{2}\right) < S(P, Q, f, g) < A + \frac{\varepsilon}{2}$ ________ (1)

This gives $U(P, f, g) - L(P, f, g) < \varepsilon$ hence $f \in RS(g)$ therefore $\int_{\bar{a}}^{b} f \, dg = \int_a^{\bar{b}} f \, dg = \int_a^b f \, dg$ more over, since $S(P, Q, f, g)$ and $\int_a^b f \, dg$ lie between $U(P, f, g)$ and $L(P, f, g)$, therefore

$$\left|S(P,Q,f,g)-\int_a^b f\,dg\right|\le U(P,f,g)-L(P,f,g)<\varepsilon$$

$$\Rightarrow\left|S(P,Q,f,g)-\int_a^b f\,dg\right|<\varepsilon\Rightarrow\lim_{\|P\|\to 0}S(P,Q,f,g)=\int_a^b f\,dg$$

Note: The existance of $\Rightarrow\lim_{\|P\|\to 0}S(P,Q,f,g)$ is a sufficient condition for $f\in RS(g)$, but it is not a necessary condition i.e. there exist a function which are integrable but for which $\lim_{\|P\|\to 0}S(P,Q,f,g)$ does not exist. Thus whenever $\dim S(P,Q,f,g)$ exist t will be equal to $\int_a^b f\,dg$ but when $f\in RS(g)$ nothing can be said about the existance of $\lim_{\|P\|\to 0}S(P,Q,f,g)$.

For example, Let f be an arbitrary function and $g=k$, a constant function on $[a,b]$. Then for any

$$\sum_{r=1}^{n} f(\varepsilon_r)\Delta g_r=\sum_{r=1}^{n} f(\varepsilon_r)[k-k]\left[\because \Delta g_r=g(x_r)-g(x_{r-1})\right]=k-k$$

$$= 0$$

t implies that $\int_a^b f\,dg=0\left[\because\int_a^b f\,dg=\sum_{r=1}^{n} f(\varepsilon_r)\Delta g_r\right]$

Theorem.12: If f is continuous and g is monotonically increasing function on $[a,0]$, then $f\in RS(g)$. Moreover, to every $\varepsilon>0$ there corresponds a $\delta>0$ such that $\left|\sum_{r=1}^{n} f(\varepsilon_r)\Delta g_r-\int_a^b f\,dg\right|<\varepsilon$ for every partition $P=\{x_0=a,x_1,x_2,....,x_n=b\}$ with $\|P\|<\delta$ and for every intermediate partition $Q=\{\varepsilon_1,\varepsilon_2,....,\varepsilon_n\}$ of P, that is $\lim_{\|P\|\to 0}RS(P,Q,f,g)=\int_a^b f\,dg$.

Proof: It is given g is monotonically increasing function on $[a,b]$, so for $\varepsilon>0$, choose $\eta>0$ such that

$\eta|g(b)-g(a)|<\varepsilon$ ________(1)

Also t is given f is a continuous function on $[a,b]$

$\Rightarrow f$ is uniformly continuous on $[a,b]$

$\Rightarrow$ for $\eta>0$, There exist $\delta>0$ such that

$|f(x)-f(y)|<\eta$ whenever $|n-y|<\delta$, $\forall n,y\in[a,b]$ _____(2)

Choose $P=\{a=x_0,x_1,x_2,....,x_{n-1},x_n=b\}$ such that $\|P\|<\delta$, Let $I_r=[x_{r-1},x_r]$ since f is continuous on $[a,b]$

$\Rightarrow f$ is also continuous in I_r

$\Rightarrow f$ has its bounds m_r,m_r on I_r

That is there exist points c,d $\in I_r$ such that

$$m_r = f(c), M_r = f(d) \quad \_\_\_\_\_\_\_\_\_\_ (3)$$

From equations (2) and (3), we have

$$|f(d) - f(c)| < \eta$$

$$\Rightarrow |M_r - m_r| < \eta \qquad , \quad r = 1,2,3,.....,n \, .$$

Hence $U(P, f, g) - L(P, f, g)$

$$= \sum_{r=1}^{n} (M_r - m_r) \Delta g_r$$

$$= U(P, f, g) - L(P, f, g) < \eta . \sum_{r=1}^{n} \Delta g_r \quad [\because (M_r - m_r) < \eta]$$

$$< \eta [g(b) - g(a)]$$

$\Rightarrow U(P, f, g) - L(P, f, g) < \varepsilon$ from (1)

$$\Rightarrow f \in RS(g) \quad \_\_\_\_\_\_ (4)$$

For second part

(4) $\Rightarrow U(P, f, g) - \varepsilon < L(P, f, g) \quad \left[\because f \in RS(g) \Leftrightarrow \int_a^{\bar{b}} f \, dg = \int_{\underline{a}}^{b} f \, dg = \int_a^{\bar{b}} f \, dg\right]$

$$\Rightarrow \int_a^{\bar{b}} f \, dg - \varepsilon < L(P, f, g) \_\_\_\_\_\_ (5)$$

And $U(P, f, g) < L(P, f, g) + \varepsilon \;\; \Rightarrow U(P, f, g) < \int_{\underline{a}}^{b} f \, dg + \varepsilon \;\; \Rightarrow U(P, f, g) < \int_a^b f \, dg + \varepsilon$

$$\left[\because \int_a^b f \, dg = \int_{\underline{a}}^{b} f \, dg\right]$$

Thus $\int_a^b f \, dg - \varepsilon < L(P, f, g) \le U[P, f, g] < \int_a^b f \, dg + \varepsilon$ ______(6)

But $L(P, f, g) \le RS(P, Q, f, g) \le U(P, f, g)$ ______(7)

From (6) & (7), we have

$$\int_a^b f \, dg - \varepsilon < RS(P, Q, f, g) < \int_a^b f \, dg + \varepsilon$$

$$\Rightarrow \left| RS(P, Q, f, g) - \int_a^b f \, dg \right| < \varepsilon$$

$\Rightarrow \lim_{\|P\| \to 0} RS(P, Q, f, g) = \int_a^b f \, dg$, Proved.

Theorem.13. If f is monotonic on $[a,b]$ and g is monotonic and continuous on $[a,b]$ then $f \in RS(g)$.

Proof: Let $P = \{a = x_0, x_1, ..., x_n = b\}$ be a partition of $[a,b]$ then for $\varepsilon > 0$, we have

$$\Delta g_r = \frac{g(b) - g(a)}{n}, r = 1, 2, 3, ..., n.$$

Which is possible as g is continuous and monotonically increasing let f be monotonically increasing on $[a,b]$.

Let $\quad m_r = \inf \text{ of } f(n) \text{ in } I_r \quad , I_r = [x_{r-1}, x_r]$

$\quad M_r = \sup \text{in } f(x) \text{ in } I_r$

Then $f(x_{r-1}) = m_r, \quad r = 1, 2, ..., n$

$\quad f(x_r) = m_r$

Therefore $U(P, f, g) - L(P, f, g) = \sum_{r=1}^{n} (M_r - m_r) \Delta g_r$

$$= \sum_{r=1}^{n} [f(x_r) - f(x_{r-1})] \frac{g(b) - g(a)}{n}$$

$$= \frac{g(b) - g(a)}{n} [f(b) - f(a)]$$

$U[P, f, g] - L[P, f, g] < \varepsilon,$ for large value of n

$\Rightarrow f \in RS(g)$ on $[a,b]$.

Example.2: Let $f(x) = x, g(x) = x^2$. Does $\int_0^1 f\, dg$ exist?

If it exists, find its value

Solution: Let $f(x) = x, x \in [0,1], g(x) = x^2$

Let $P = \left\{ \frac{0}{n}, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \quad \frac{r-1}{n}, \frac{r}{n}, ...\left(\frac{n-1}{n}\right), \frac{n}{n} \right\}$ be a partition of $[0,1]$.

Let $m_r = \inf \text{ of } f(x)$ in I_r, Here $I_r = [x_{r-1}, x_r]$ $m_r = \left(\frac{r-1}{n}\right) \begin{bmatrix} \because f(x) = x \\ f(x_{r-1}) = \frac{r-1}{n} \end{bmatrix}$

$\quad m_r = \sup \text{ of f}(x) \text{ in } I_r$

$M_r = \frac{r}{n}$ $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $\left[\begin{array}{l} \because f(x) = x \\ f(x_r) = \frac{r}{n} \end{array}\right]$

Now, $g(n) = x^2$ is monotonically increasing in I_r

$$\therefore \quad \Delta g_r = g[x_r] - g[x_{r-1}] = g\left[\frac{r}{n}\right] - g\left[\frac{r-1}{n}\right]$$

$$= \left(\frac{r}{n}\right)^2 - \left(\frac{r-1}{n}\right)^2 = \frac{(r-r+1)(r+r-1)}{n^2}$$

$$\Delta g_r = \left(\frac{2r-1}{n^2}\right)$$

$$\therefore U[P, f, g] - L(P, f, g) = \sum_{r=1}^{n} [M_r \Delta g_r - m_r \, \Delta \, g_r]$$

$$= \sum_{r=1}^{n} (M_r - m_r) \Delta \, g_r$$

$$= \sum_{r=1}^{n} \left(\frac{r}{n} - \frac{r-1}{n}\right)\left(\frac{2r-1}{n^2}\right)$$

$$= \frac{1}{n^3} \sum_{r=1}^{n} (2r-1) = \frac{1}{n^3}\left[2\sum_{r=1}^{n} r - n\right]$$

$$= \frac{1}{n^2}\left[2.\frac{n(n+1)}{2} - n\right] = \frac{1}{n}$$

$$U[P, f, g] - L[P, f, g] < \varepsilon \; \left[\text{ Take } \frac{1}{n} < \varepsilon\right]$$

$$\Rightarrow f \in RS(g) \text{ on } [0,1]$$

Now we have

$$\int_0^1 f \, dg = \int_0^1 f \, dr^2 \quad \left[\because g = x^2\right] = \dim_{\|P\| \to 0} S(P, Q, f, g)$$

$$, \; Q = \{\varepsilon_1, \varepsilon_2, ..., \varepsilon_{r-1}, \varepsilon_x, ...\varepsilon_n\} \text{ where } x_{r-1} \le \varepsilon_r \le x_r = \dim_{n \to \infty} \sum_{r=1}^{n} f(\varepsilon_r) \Delta g_r$$

$$= \dim_{n \to \infty} \sum f\left(\frac{r}{n}\right)\left[\frac{2r-1}{n^2}\right] \quad\quad \left[\text{Take } \varepsilon_r = x_r = \frac{r}{n}\right]$$

$$= \dim_{n\to\infty} \sum_{r=1}^{n} \frac{r}{n}\left(\frac{2r-1}{n^2}\right) \left[\because f(n) = x\right]$$

$$= \dim_{n\to\infty} \frac{1}{n^3}\left[2.\sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r\right]$$

$$= \dim_{n\to\infty} \frac{4n^2+3n-1}{6n^2}$$

$$\int_0^1 f\, dg = \dim_{n\to\infty} \frac{1}{6}\left[4+\frac{3}{2n}-\frac{1}{6n^2}\right] = \frac{2}{3}$$

$$\therefore \int_0^1 f\, dg = \frac{2}{3}$$

Example.3: $\int_0^3 x\, d(x)$ where $[x]$ is the integral value of x .

Sol. Let $f(x) = x,\ g(x) = [x], x \in [0,3]$.

Let $P = \left\{\frac{0}{n}, \frac{1}{n}, \frac{2}{n}, \frac{3n}{n}\right\}$ $P = \left\{\frac{0}{n}, \frac{1}{n}, \frac{2}{n}, \ldots.., \frac{\varepsilon-1}{n}, \frac{r}{n}, \ldots\ldots., \frac{3n}{n}\right\}$ be a partition of $[0,3]$

$$\Delta g_r = g(x_r) - g(x_{r-1}) = g\left(\frac{r}{n}\right) - g\left(\frac{r-1}{n}\right) \ \Delta g_r = \left[\frac{r}{n}\right] + \left[\frac{r-1}{n}\right]$$

$$\therefore \int_0^3 x\, d[x] = \lim_{\|P\|\to 0} S\left[P, Q, x, [x]\right]$$

$$= \lim_{n\to\infty} \sum_{r=1}^{3n} f(\varepsilon_r)\Delta g_r,\ x_{r-1} \le \varepsilon_r \le x_r$$

$$= \lim_{n\to\infty} \sum_{r=1}^{3n} \frac{r}{n}\left[\left[\frac{r}{n}\right] - \left[\frac{r-1}{n}\right]\right]$$

$$= \lim_{n\to\infty} \left[0 + \frac{n}{n}.1 + \frac{2n}{n}.1 + \frac{3n}{n}.1\right]$$

$$= 1+2+3 \quad \left[\left[\frac{r}{n}\right] - \left[\frac{r-1}{n}\right]\right] = 0, r \ne n$$

$$\int_0^3 x\, d[x] = 6 \qquad \left[\frac{r}{n}\right] - \left[\frac{r-1}{n}\right] = 1, r = n, r = 2n, r = 3n.$$

Example.4; Evalute $\int_0^{\pi/2} \cos x(\sin x)$

Solution: Here $f(x) = \cos n, g(n) = \sin x, I = \left[0, \frac{\pi}{2}\right]$

Let $P = \left\{\frac{0}{2n}, \frac{1\pi}{2n}, \frac{2\pi}{2n}, ..., \frac{(r-1)\pi}{2n}, \frac{r\pi}{2n}, ... \frac{n\pi}{2n}\right\}$

Let $\varepsilon_r = \frac{\pi r}{2n}$

Also $\Delta g_r = \sin\frac{\pi r}{2n} - \sin\frac{\pi(r-1)}{2n}$ $\left[\Delta gr = g(x_r) - g(x_{r-1})\right]$

$$\Delta g_r = 2\cos\frac{\pi}{4n}(2r-1)\sin\frac{\pi}{4n} \int_0^{\frac{\pi}{2}} \cos x\, d(\sin x) = \lim_{\|P\|\to 0} S(P, Q, f, g)$$

$$\int_0^{\frac{\pi}{2}} \cos x\, d(\sin x) = \lim_{\|P\|\to 0} \sum_{r=1}^{n} f(\varepsilon_r)\Delta g_r$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \cos\frac{\pi r}{2n}.2.\cos\frac{\pi(2r-1)}{4n}\sin\frac{\pi}{4n}$$

$$= \lim_{n\to\infty} \sin\frac{\pi}{4n}\left[\sum_{r=1}^{n} 2\cos\frac{\pi r}{2n}\cos\frac{\pi(2r-1)}{4n}\right]$$

$$= \lim_{n\to\infty} \sin\frac{\pi}{4n}\left[\sum_{r=1}^{n} \cos\frac{\pi}{4n}(4r-1) + \cos\frac{\pi}{4n}\right]$$

$$= \lim_{n\to\infty}\left[\sin\frac{\pi}{4n}\sum_{r=1}^{n}\cos\frac{\pi(4r-1)}{4n} + \sin\frac{\pi}{4n}\sum_{r=1}^{n}\cos\frac{\pi}{4n}\right]$$

$$= \lim_{n\to\infty}\left[\sin\frac{\pi}{4n}\left\{\cos\left\{\frac{3\pi}{4n}\right\} + \cos\left\{\frac{7\pi}{4n}\right\} + ... + \cos\left\{\frac{(4n-1)\pi}{4n}\right\}\right\}\right]$$

$$= \lim_{n\to\infty}\left[\cos\left[\frac{3\pi}{4n} + (n-1)\frac{\pi}{2n}\right]\frac{\sin\frac{n\pi}{2n}}{\sin\frac{\pi}{2n}}\sin\frac{\pi}{4n} + n\sin\frac{\pi}{4n}.\cos\frac{\pi}{4n}\right]$$

$$= \lim_{n\to\infty}\left[\left\{\cos\left(\frac{\pi}{2} + \frac{\pi}{4n}\right)\frac{\sin\frac{\pi}{2}}{\sin\frac{\pi}{2n}}\sin\frac{\pi}{4n}\right\} + n\sin\frac{\pi}{4n}\cos\frac{\pi}{4n}\right]$$

$$= \lim_{n\to\infty}\left[-\sin\frac{\pi}{4n}.\frac{1}{\sin\frac{\pi}{2n}}\sin\frac{\pi}{4n}+n\sin\frac{\pi}{4n}\cos\frac{\pi}{4n}\right]$$

$$= 0+\frac{\pi}{4} \qquad \left[\text{ Use } \lim_{n\to\infty}\frac{\sin\theta}{\theta}=1 \text{ etc}\right]$$

$$= \frac{\pi}{4} \therefore \int_0^{\frac{\pi}{2}}\cos d\left(\sin x\right)=\frac{\pi}{2}.$$

1.11 PROPERTIES OF RIEMANN-STIELTJES INTEGRAL

Let $f \in RS\left(g\right)$ on $\left[a,b\right]$ then

$$m\left[g\left(b\right)-g\left(a\right)\right] \le \int_a^b f\,dg \le m\left[g\left(b\right)-g\left(a\right)\right]$$

Where m and M are the lower and upper bounds of f on $\left[a,b\right]$

Proof. Let $P=\left\{a=x_0,x_1,x_2,....,x_{r-1},x_r,.....,x_n\right\}$ be a partition of $\left[a,b\right]$ where $I_r=\left[x_{r-1},x_r\right]$

Then with usual notations, we have

$$m \le m_r \le M_r \le M$$

$$\Rightarrow m\Delta g_r \le m_r\Delta g_r \le M_r\Delta g_r \le M\Delta g_r$$

$$\Rightarrow \sum_{r=1}^{n} m\Delta g_r \le \sum_{r=1}^{n} m_r\Delta g_r \le \sum_{r=1}^{n} M_r\Delta g_r \le \sum_{r=1}^{n} M\Delta g_r$$

$$\Rightarrow m\left[g\left(b\right)-g\left(a\right)\right] \le L\left(P,f,g\right) \le U\left[P,f,g\right] \le M\left[g\left(b\right)-g\left(a\right)\right] \qquad \text{\_\_\_\_\_\_(1)}$$

But $L\left(P,f,g\right) \le \int_{\underline{a}}^{b} f\,dg$

And $\quad U\left(P,f,g\right) \ge \int_a^{\bar{b}} f\,dg$

Also we know that $\int_{\underline{a}}^{b} f\,dg \le \int_a^{\bar{b}} f\,dg \;\therefore m\left[g\left(b\right)-g\left(a\right)\right] \le \int_{\underline{a}}^{b} f\,dg \le \int_a^{\bar{b}} f\,dg \le M\left[g\left(b\right)-g\left(a\right)\right]$

Since f is RS- integrable relative to g, we have

$$\int_{\underline{a}}^{b} f\,dg = \int_{a}^{\bar{b}} f\,dg = \int_{a}^{b} f\,dg \;\therefore m\left[g(b)-g(a)\right] \le \int_{a}^{b} f\,dg \le M\left[g(b)-g(a)\right]$$

Cor. 1 If $f \in RS(g)$, then there exists a number is, lying between m and M such that

$$\int_{a}^{b} f\,dg = \mu\left[g(b)-g(a)\right]$$

Cor. 2 If f is continuous on $[a,b]$ then there exist a number c lying between a and b such that

$$\int_{a}^{b} f\,dg = f(c)\left[g(b)-g(a)\right]$$

Proof: since f is continuous on $[a,b]$, must take all values between m and M and in particular there exists $c \in [a,b]$ such that $f(c) = \mu$ where $m \le \mu \le M$

, m is lower bound of f on $[a,b]$

And M is upper bound of f on $[a,b]$

Cor 3. If $f \in RS(g)$ and of $|f(x)| \le K$ on $[a,b]$, then $\left|\int_{a}^{b} f\,dg\right| \le K\left[g(b)-g(a)\right]$

Proof: we have for all $x \in [a,b]$ $|f(x)| \le K \Rightarrow -K \le f(x) \le K$

$$\Rightarrow -K \le -m \le f(x) \le m \le K$$

$$\Rightarrow -K \le m \le m_r \le f(x) \le M_r \le M \le K$$

$$\Rightarrow -K \le m_r \le M_r \le K$$

$$\Rightarrow -\sum_{r=1}^{n} K\Delta g_r \le \sum_{r=1}^{n} m_r\Delta g_r \le \sum_{r=1}^{n} M_r\Delta g_r \le \sum_{r=1}^{n} K\Delta g_r$$

$$\Rightarrow -K\left[g(b)-g(a)\right] \le L(P,f,g) \le U[P,f,g] \le K\left[g(b)-g(a)\right] \Rightarrow \left|\int_{a}^{b} f\,dg\right| \le K\left[g(b)-g(a)\right]$$

1.12 ALGEBRA OF RIEMANN-STIELTJES INTEGRABLE

Theorem.14. Let $f \in RS(g)$ on $[a,b]$. Then $c\,f \in RS(g)$ on $[a,b]$ for every constant c and

$$\int_{a}^{b} (cf)\,dg = c\int_{a}^{b} f\,dg$$

Proof: Let $f \in RS(g)$ on $[a,b]$ to prove $c\,f \in RS(g)$ and $\int_{a}^{b} (cf)dg = c\int_{a}^{b} f\,dg$

Case I if $c = 0$, then theorem is obrious.

Case II if $c > 0,\ f \in RS(g) \Rightarrow \int_a^b f\, dg = \int_a^{\bar{b}} f\, dg = \int_a^b f\, dg$

Also for $\varepsilon > 0$, ther exists $P = \{x_0, x_1, x_2, ..., x_n\}$ such that $U[P, f, g] - L[P, f, g] < \frac{\varepsilon}{c}$

Let $m_r = \inf$ of $f(n)$ in $I_r, I_r = [x_{r-1}, x_r]$

$M_r = \sup$ of $f(n)$ in $I_r \Rightarrow c\, m_r$ and $c\, M_r$ are inf of $c\ f(x)$ and sup of $c\ f(x)$ in I_r respectively

Therefore, $U[P, cf, g] = cU[P, f, g]$
and $L[P, cf, g] = c\, L[P, f, g]$

Hence $U[P, cf, g] - L[P, cf, g] = c[U(P, f, g) - L(P, f, g)] < c, \frac{\varepsilon}{c} < \varepsilon \Rightarrow cf \in RS(g)$

Case III c < 0, Take $c = -c_1 \Rightarrow c_1 > 0$

So

$U[P, c_1 f, g] = (-a)L[P, f, g]$
$L[P, c_1 f, g] = (-a)U[P, f, g]$

Using these relations, we have

$U[P, cf, g] - L[P, cf, g] < \varepsilon$

Hence $cf \in RS(g)$ if $c < 0$ $\left[\begin{array}{l} \because U[P, c_1 f, g] - L[P, -c_1 f, g] \\ \quad = -c_1 L(P, f, g) \\ \qquad + c_1 U(P, f, g) \\ \quad = c_1[U(P, f, g) - L(P, f, g)] \\ \quad = c_1 . \frac{\varepsilon}{\varepsilon_1} \end{array}\right.$

Hence $\int_a^{\bar{b}} cf\, dg = \int_{\underline{a}}^b cf\, dg = \int_a^b cf\, dg$

Now, For $c > 0\ (c = -c_1)$, we have

$\int_a^b (cf)\, dg = \int_a^{\bar{b}} (cf)\, dg$

$= \int_a^{\bar{b}} (-c_1)\, f\, dg = (-c_1)\int_{\underline{a}}^b f\, dg = c\int_a^b fdg$

Hence $\int_a^b (cf)\, dg = c\int_a^b f\, dg$

Cor. If $f \in RS(g)$ then $(-f) \in RS$ $\int_a^b (-f)\,dg = -\int_a^b f\,dg$

Theorem.14. Let $f_1, f_2 \in RS(g)$ on $[a,b]$ then $(f_1 + f_2) \in RS(g) = \int_a^b f_1 dg + \int_a^b f_2\,dg$

Proof: Let $f_1, f_2 \in RS(g)$ on $[a,b]$

To prove $(f_1 + f_2) \in RS(g)$

Since $f_1, f_2 \in RS(g)$, t means for $\varepsilon > 0$ there exists partitions P_1 and P_2 of $[a,b]$ such that

$$U[P_1, f_1, g] - L[P_1, f_1, g] < \frac{\varepsilon}{2} \quad \_\_\_\_\_\_(1)$$

And $$U[P_2, f_2, g] - L[P_2, f_2, g] < \frac{\varepsilon}{2} \quad \_\_\_\_\_\_(2)$$

If P is a common refinement of P_1 and P_2 then we have

$$U[P, f_1, g] - L[P, f_1, g] < \frac{\varepsilon}{2} \_\_\_\_\_\_(3)$$

And $$U[P, f_2, g] - L[P, f_2, g] < \frac{\varepsilon}{2} \_\_\_\_\_\_(4)$$

Now , if $f = f_1 + f_2$ and P is any partition of $[a,b]$ then

$$L[P, f_1, g] + L[P, f_2, g] \le L[P, f, g] \le U[P, f, g] \le I[P, f_1, g] + U[P, f_2, g] \_\_\_\_\_\_(5)$$

$\therefore (5)$ implies

$$U[P, f, g] - L[P, f, g] \le U[P, f_1, g] + U[P, f_2, g] - \{L[P, f, g] + L[P, f_2, g]\}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \ [\text{ from } (3) \& (4)] < \varepsilon$$

$\Rightarrow f \in RS(g) \Rightarrow (f_1 + f_2) \in RS(g)$ because $f = f_1 + f_2$

Again $U[P, f_1, g] < \int_a^b f_1 dg + \frac{\varepsilon}{2} \_\_\_\_\_\_(6)$

And $U[P, f_2, g] < \int_a^b f_2 dg + \frac{\varepsilon}{2} \_\_\_\_\_\_(7)$

Now, $f \in RS(g) \Rightarrow \int_a^b f\,dg = \int_a^{\bar{b}} f\,dg \le U(P, f, g)$

$$\Rightarrow \int_a^b f\,dg \le U[P, f_1, g] + U[P, f_2, g], \text{ from } (5)$$

$$< \int_a^b f_1 dg + \frac{\varepsilon}{2} + \int_a^b f_2 dg + \frac{\varepsilon}{2}$$

$$\int_a^b f\,dg < \int_a^b f_1\,dg + \int_a^b f_2 dg + \varepsilon$$

$\int_a^b f\,dg \le \int_a^b f_1\,dg + \int_a^b f_2 dg$, Taking $\varepsilon \to 0$ ________(8)

Since $f_1 \in Rs(g) \Rightarrow -f_1 \in RS(g)$

$f_2 \in Rs(g) \Rightarrow -f_2 \in RS(g)$

Now $\int_a^b (-f)\,dg = \int_a^b -[f_1 + f_2]dg = \int_a^b (-f_1)\,dg + \int_a^b (-f_2)\,dg$

$\Rightarrow -\int_a^b f\,dg \le -\left[\int_a^b f_1 dg + \int_a^b f_2 dg\right]$,(we prove it)

$\Rightarrow \int_a^b f\,dg \ge \int_a^b f_1\,dg + \int_a^b f_2 dg$ ______(9)

From equations (8) & (9), we have

$\Rightarrow \int_a^b f\,dg = \int_a^b f_1 dg + \int_a^b f_2 dg$

$\Rightarrow \int_a^b (f_1 + f_2)dg = \int_a^b f_1 dg + \int_a^b f_2 dg$, Prove

Theorem.15. If $f \in RS(g_1)$ and $f \in RS(g_2)$ on $[a,b]$ then $f \in RS(g_1 + g_2)$ on $[a,b]$

And $\int_a^b fd(g_1 + g_2) = \int_a^b f\,dg_1 + \int_a^b f\,dg_2$

Proof: Let $P = \{a = x_0, x_1, x_2, ..., x_{r-1}, x_r, ..., x_n\}$ be a partition of $[a,b]$. Let $I_r = [x_{r-1}, x_r]\, r = 1, 2, ..., n.$

$m_r = \inf$ of f in I_r

$M_r = \sup$ of f in I_r

$\Delta g_r = g(x_r) - g(x_{r-1})$ where $g = g_1 + g_2$

$= (g_1 + g_2)(x_r) - (g_1 + g_2)(x_{r-1})$

$\Delta g_r = [g_1(n_r) - g_1(x_{r-1})] + [g_2(x_r) - g_2(x_{r-1})] = \Delta g_{1r} + \Delta g_{2r}$

$$\therefore U[P, f, g] = \sum_{r=1}^{n} M_r \Delta g_r$$

$$= \sum M_r [\Delta g_{1r} + \Delta g_{2r}]$$

$$= \sum_{r=1}^{n} M_r \Delta g_{1r} + \sum_{r=1}^{n} M_r \Delta g_{2r}$$

$U[P, f, g] = U[P, f, g_1] + U[P, f, g_2]$

Similarly, we have

$L[P, f, g] = L(P, f, g_1) + L(P, f, g_2)$

Therefore, we have

$U[P, f, g] - L[P, f, g] = \{U[P, f, g_1] - L[P, f, g_1]\} + \{U[P, f, g_2] - L[P, f, g_2]\}$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \qquad \begin{bmatrix} \because f \in RS(g_1) \\ \text{and } f \in RS(g_2) \end{bmatrix}$$

$$< \varepsilon$$

$\Rightarrow f \in RS(g)$ on $[a,b]$

$\Rightarrow f \in RS(g_1 + g_2) \quad [\because g = g_1 + g_2]$

Further $\overline{\int_a^b} f\, dg = \inf U[P, f, g]$

$$= \inf U\left[U(P, f, g_1) + U[P, f, g_2]\right]$$

$$\geq \overline{\int_a^b} f\, dg_1 + \overline{\int_a^b} f\, dg_2 \text{ \_\_\_\_ } (1)$$

Similary $\underline{\int_a^b} fdg = \sup L(P, f, g)$

$$= \sup\left[L(P, f, g_1) + L(P, f, g_2)\right]$$

$$\leq \sup\left[L(P, f, g_1) + \sup L(P, f, g_2)\right]$$

$$\underline{\int_a^b} fdg \leq \underline{\int_a^b} fdg_1 + \underline{\int_a^b} f\, dg_2 \text{ \_\_\_\_ } (2)$$

From equations (1) & (2), we have

$$\int_a^b fdg = \int_a^b fdg_1 + \int_a^b fdg_2 \quad \left[\because f \in RS(g) \Rightarrow \overline{\int_a^b} f\, dg = \underline{\int_a^b} fdg\right]$$

$$\Rightarrow \int_a^b fd(g_1 + g_2) = \int_a^b fdg_1 + \int_a^b fdg_2 \text{ ,}$$

Theorem.16. Let $f \in RS(g)$ on $[a,b]$. If $a < c < b$, then $f \in RS(g)$ on $[a,c]$, $f \in RS(g)$ on $[c,d]$ **and** $\int_a^b fdg = \int_a^c fdg + \int_c^b fdg$

Proof: since $f \in RS(g)$ on $[a,b]$ for a given $\varepsilon < 0$, there exists a partition P of $[a,b]$ such that

$$w[P, f, g] = U[P, f, g] - L[P, f, g] < \varepsilon \text{ \_\_\_\_\_\_ } (1)$$

Let P_1, P_2 be the sets of those points of P which constante the partitions of $[a,c]$ and $[c,b]$ respectively Then the inequality (1) implies that

$$w[P_1, f, g] < \varepsilon \text{ and } w[P_2, f, g] < \varepsilon$$

Let implies

$f \in RS(g)$ on $[a,b]$

and $f \in RS(g)$ on $[c,d]$

Thus $\underline{\int_a^b} fdg = \overline{\int_a^b} fdg = \int_a^b fdg$ ______ (2)

$$\underline{\int_a^c} fdg = \overline{\int_a^c} fdg = \int_a^c fdg \text{ ______ } (3)$$

And $\underline{\int_c^b} fdg = \overline{\int_c^b} fdg = \int_c^b fdg$ ______ (4)

Now, Let P_1 be any partition of [a,c] and P_2 be any partition of [c,b], then $(P_1 \cup P_2)$ is a partition of [a,b] whose component subintervals are those of P_1 and P_2.

Hence $L(P_1, f, g) + L[P_2, f, g] = L[P_1 \cup P_2, f, g] \le \underline{\int_a^b} fdg$

$$\Rightarrow L[P_1, f, g] + L[P_2, f, g] \le \int_a^b fdg \quad \left[\because \underline{\int_a^b} fdg = \int_a^b fdg\right] \text{ from (2)}$$

By taking the supremum on the left over all P_1keeping P_2 as fixed, we have on [a,b]

$$\int_a^c f\,dg + L[P_2, f, g] \le \int_a^b fdg \quad \text{(using (3))}$$

Now, taking the supremum over all P_2 and Using (3)

$$\int_a^c fdg + \int_c^b fdg \le \int_a^b fdg \text{ ______ } (5)$$

Similarly, we have

$$U[P_1, f, g] + U[P_2, f, g] = U[P_1 \cup P_2, f, g] \ge \overline{\int_a^b} fdg$$

$$\Rightarrow U[P_1, f, g] + U[P_2, f, g] \ge \int_a^b fdg$$

$$\text{Inf } U[P_1, f, g] + \inf U[P_2, f, g] \ge \int_a^b fdg$$

$$\Rightarrow \int_a^c fdg + \int_c^b fdg \ge \int_a^b fdg \text{ ______ } (6)$$

From equations (5) and (6), we have

$$\int_a^c fdg + \int_c^b fdg = \int_a^b fdg$$

Cor. If $\int_a^b fdg$ exists and if $[c,d] \subset [a,b]$ then

$\int_c^d fdg$ exists.

1.13 SUMMARY

Let $I = [a,b]$ be a closed and bounded interval then a finite set of real number $P = \{x_0, x_1, x_2, \ldots\ldots, x_{n-1}, x_n\}$ having the property that $a = x_0 < x_1 < x_2 < \ldots\ldots\ldots < x_{n-1} < x_n = b$ [Here $x_0 = a$ and $x_n = b$] is called the partition of $[a,b]$.

A portition P^* is called a refinement of another partion P if and only if $P^* \supset P$ *i.e.*, every point P is used to build P^*.

If P_1 and P_2 are any two partitions of $[a, b]$ then $P^* = P_1 \cup P_2$ is called common refinement of P_1 and P_1.

Let f be a bounded real valued function defined on $[a, b]$ and let $P = \{a = x_0, x_1, x_2, ..., x_{n-1}, x_n\}$ be a partition of $[a, b]$. Let $m_r =$ infimum of f in I_r (Here $I_r = [x_{r-1}, x_r]$)

$M_r =$ supremum of f in I_r respectively, then

$L(P, f) = \sum_{r=1}^{n} m_r \Delta x_r = (m_1 \Delta x_1 + m_2 \Delta x_2 + ... + m_n \Delta x_n)$ is called the lower-Riemann sum

and $U(P, f) = \sum_{r=1}^{n} M_r \Delta x_r = [M_1 \Delta x_1 + M_2 \Delta x_2 + ... + M_n \Delta x_n]$ is called the upper-Riemann sum.

A function f define on $[a, b]$ is said to be Riemann integrable over $[a, b]$ if and only if for every $\varepsilon > 0$ there exists a $\delta > 0$ and a number I such that for every partition $P = \{a = x_0, x_1, ..., x_n = b\}$ of $[a, b]$ with $\|P\| < \delta$ and for every ε hence of $\varepsilon_r \in [x_{r-1}, x_r]$

$$\left| \sum_{r=1}^{n} f(\varepsilon_r)(x_r - x_{r-1}) - I \right| < \varepsilon \quad \Rightarrow \quad I = \int_a^b f(x)\,dx$$

i.e., I is R- integrable.

Let $f \in RS(g)$ on $[a, b]$ then

$$m[g(b) - g(a)] \leq \int_a^b f\,dg \leq m\,[g(b) - g(a)]$$

Where m and M are the lower and upper bounds of f on $[a, b]$.

1.14 TERMINAL QUESTIONS

Q.1 What do you mean by Partition of a closed interval?

Q.2 Explain the Riemann Integral.

Q.3 Write a short note on Riemann-Stieltjes integral.

Q.4 Let f be a bounded function on $[a, b]$. Let P be a partition of $[a, b]$. If $P*$ is a refinement of P then $L(P*, f) \geq L(P, f)$ and $U(P*, f) \leq U(P, f)$

Q.5 Let P be a partion of $[a, b]$. Let P_1 and P_2 be any two partion of $[a, b]$ such that $P = P_1 \cup P_2$ then $U[P_1, f] \geq L(P_2, f)$

Q.6 Let f, g be a bounded function on $[a,b]$ and let P be a partition of $[a,b]$. Then $L[P, f+g] \geq L[P, f] + L(P, g)$ and $U[P, f+g] \leq U[P, f] + U[P, g]$.

Q.7 Let f be a bounded function defined on $[a,b]$ then for every $\varepsilon > 0, \exists \delta > 0$ such that $U[P, f] < \int_a^{\bar{b}} f + \varepsilon$ and $L[P, f] > \int_{\underline{a}}^{b} f - \varepsilon$ for a partition P of $[a,b]$ with $\|P\| < \delta$.

Q.8 A necessary and sufficient condition for R-integrability of a function $f : [a,b] \to R$ on $[a,b]$ is for $\varepsilon > 0, \exists$ a partition P of $[a,b]$ such that for P and all its refinements $0 \leq U(P, f) - L(P, f) < \varepsilon$.

UNIT 2 INTEGRATION AND DIFFERENTIATION

Structure

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2.1 INTRODUCTION

In Riemann integration, integration and differentiation are closely related, as both involve understanding a function's behavior over an interval. The goal of Riemann integration is to find the area under a curve $y=f(x)$ over an interval $[a, b]$. This is accomplished by dividing the interval into smaller subintervals and approximating the area under the curve in each subinterval. The Riemann sum is then calculated by adding these approximations. As the width of the subintervals approaches zero (i.e., as the partition becomes finer), the Riemann sum converges to the Riemann integral of $f(x)$ over $[a, b]$.

Differentiation is not directly involved in finding the Riemann integral. However, differentiation can be used to analyze a function and its derivative's behavior over an interval. This analysis can help understand the properties of the function and its integral. Although integration and differentiation are distinct concepts in calculus, they are closely related in Riemann integration, where both involve analyzing a function's behavior over an interval to determine properties such as the area under a curve or the relationship between a function and its antiderivative.

2.2 OBJECTIVES

After studying this unit, the learner will be able to understand:

- the integral function
- the continuity of integral function
- the differentiability of integral function
- the fundamental theorem of integral calculus

2.3 INTEGRAL FUNCTION

Let f be a function integrable over $[a,b]$ then th4e function F on $[a,b]$ given by $F(x)=\int_a^{\pi} f(t)\,dt$, $a \le \pi \le b$ is called the integral function of f.

2.4 COUNTINUITY OF INTEGRAL FUNCTION

Theorem 1. Let $f \in R[a,b]$. Than the integral function F of f is given by

$F(x)=\int_a^{\pi} f(t)\,dt,\ a \le \pi \le b$ **is continuous on** $[a,b]$

Proof: Since f is $R-$integrable over $[a,b]$. It means f is bounded on $[a,b]$

$\Rightarrow$there exists a positive integer M such that $|f(t)| \le M$, for all $t \in [a,b]$

Let $a < x_1 < x_2 \le b$. Then we have

$$\left|F(x_2)-F(x_1)\right| = \left|\int_a^{x_2} f(t)\,dt - \int_a^{x_1} f(t)\,dt\right|$$

$$= \left|\int_a^{x_2} f(t)\,dt + \int_{x_1}^{a} f(t)\,dt\right|$$

$$= \left|\int_{x_1}^{x_2} f(t)\,dt\right|$$

$$\leq M.\left|\int_{x_1}^{x_2} dt\right| \qquad \left[\because \left|f(t)\right| \leq M\right]$$

$$\leq M.\left|x_2 - x_1\right|$$

Now, for a given $\varepsilon > 0$, let $\left|x_2 - x_1\right| < \dfrac{\varepsilon}{M}$

Thus we have $\left|F(x_2) - F(x_1)\right| < M.\dfrac{\varepsilon}{M}$ whenever $\left|x_2 - x_1\right| < \delta$

Which shows that F is uniformly continuous on $[a,b]$ and hence x is continuous $[q,b]$.

2.4 DIFFERENTIABILITY OF INTEGRAL FUNCTION

Theorem 2. Let f be a continuous function on $[a,b]$ and let $F(x) = \int_a^x f(t)\,dt$, for all $x \in [a,b]$ then $F'(x) = f(x)$

Proof: Let f be a continuous function on $[a,b]$

Let $x \in [a,b]$. Choose $h \neq o$ such that $x + h \in [a,b]$, Then we have

$$F(x+h) - F(x) = \int_a^{x+h} f(t)dt - \int_a^x f(t)\,at$$

$$= \int_a^{x+h} f(t)\,dt + \int_x^a f(t)dt$$

$$F(x+h) - F(x) = \int_x^{x+h} f(t)\,dt \qquad \ldots(1)$$

Now, t is given $t(x)$ is continuous on $[a,b]$

t means There exists a number $c \in [x, x+h]$ such that

$$\int_x^{x+h} f(t)\,dt = (x+h-x)f(c) \qquad \ldots(2)$$

Clearly $c \to x$ as $h \to 0$

From equation (1) and (2), we have

$$\therefore \quad F(x+h) - F(x) = h\ f(c)$$

$$\lim_{h \to o} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} f(c)$$

$= f(x) \qquad [\because c \& x \text{ as } h \to 0]$

$\Rightarrow F^1(x) = f(x),\ \forall x \in [a,b]$.

2.6 FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

Theorem.3: Let f be a continuous function on $[a,b]$ and let & be a differentiable function on $[a,b]$ such that $\phi'(x) = f(x);\ x \in [a,b]$.

Then $\int_a^b f(t)dt = \phi(b) - \phi(a)$.

Proof: Since f is continuous on $[a,b]$ then $F(x) = \int_a^x f(t)dt, x \in [a,b]$ is differentiable

i.e., $F'(x) = f(x)$ for all $x \in [a,b]$ ….(1)

But it is given

$$\phi^1(x) = f(x), \text{ for all } x \in [a,b] \qquad ….(2)$$

Therefore from (1) and (2), we have

$F'(x) = \phi'(x)$, for all $x \in [a,b]$

$\Rightarrow F'(x) - \phi'(x) = 0,$ for all $x \in [a,b]$

$\Rightarrow \frac{d}{dx}[F(x) - \phi(x)] = 0$

$\Rightarrow F(x) - \phi(x) = c,$ c is some constant ….(3)

Now we have

$$F(a) = \phi(a) + c$$

$$F(b) = \phi(b) + c$$

$$F(b) - F(a) = \phi(b) + \phi(a)$$

But $F(b) = \int_a^b f(t)dt$

And $F(a) = \int_a^a f(t)dt = o$

$\therefore F(b) - F(a) = \phi(b) - \phi(a) \Rightarrow \int_a^b f(t)dt - o = \phi(b) - \phi(a)$

i.e., $\int_a^b f(t)dt = \phi(b) - \phi(a)$

Alternate method: Let $P=\{a=x_0,x_1,.....,x_{r-1},x_r=b\}$ be a partition of $[a,b]$.

Let $I_r=[x_{r-1},x_r],r=1,2,...,r$ be the sub intervals of $\Delta x_r=x_r-x_{r-1}$

Since. ϕ is differentiable on $[a,b]\Rightarrow\phi$ is differentiable on I_r

So by mean value theorem

$$\frac{\phi(x_r)-\phi(x_{r-1})}{x_r-x_{r-1}}=\phi'(\varepsilon_r),x_{r-1}<\varepsilon_r<x_r$$

$$\Rightarrow\sum_{r=1}^{x}\left[\phi(x_r)-\phi(x_{r-1})\right]=\sum_{r=1}^{x}f(\varepsilon_r)\Delta x_r\quad[\because\phi'(x)=f(x)\Rightarrow\phi'(\varepsilon)=f(\varepsilon)$$

$$\Rightarrow\phi(b)-\phi(a)=\sum_{r=1}^{x}f(\varepsilon_r)\Delta x_r\left[\because\sum_{r=1}^{x}\left[\phi(x_r)-\phi(x_{r-1})\right]=\phi(b)-\phi(a);x_n=b,x_0=a\right]$$

Taking linit $\|P\|\to o$, we have

$$\phi(b)-\phi(a)=\int_a^b f(x)dx\left[\because\lim_{\|P\|\to o}\sum_{r=1}^{x}f(\varepsilon_r)\Delta x_r=\int_a^b f(x)dx\right]$$

Hence $\int_a^b f(x)dx=\phi(b)-\phi(a)$.

Examples

Example.1: Evaluate $\int_o^1\left(2x\sin\frac{1}{x}-\cos\frac{1}{x}\right)dx$

Sol. Let $g'(x)=f(x)=\begin{cases}x^2\sin\frac{1}{x} & ,x\in]0,1[\\ o & ,x=0\end{cases}$

Whene $g(x)=\left[2x\sin\frac{1}{x}-\cos\frac{1}{x}\right]$

Hence $f(x)$ is not continuous on $[0,1]$ but bounded on $]0,1[$ so f is Riemann integral on $[0,1]$.

Also $f(x)$ is differentiable on $[0,1]$ such that

$$g'(x)=f(x),\forall x\in[0,1]$$

$$\therefore\qquad\int\left(2x\sin\frac{1}{x}-\cos\frac{1}{x}\right)dx=g(1)-g(0)=\sin 1.$$

$$\left[\int x^2\sin\frac{1}{x}dx=2x\sin\frac{1}{x}-\cos\frac{1}{x}\right]$$

2.7 ABSOLUTE VALUE OF RIEMANN-STIELTJES INTEGRAL

Theorem 4. If $f \in \mathrm{RS}(g)$ on $[a,b]$ there $|f| \in \mathrm{RS}(g)$ on $[a,b]$ and $\left|\int_a^b f\, \mathrm{dg}\right| \leq \int_a^b |f| dg$

Proof: (i) If $\int_a^b f\, dg \geq 0$ then $\left|\int_a^b f\, dg\right| = \int_a^b fdg \qquad [\because |x| = x, \text{ if } x \geq 0]$

$$\left|\int_a^b f\, dg\right| \leq \int_a^b |f| dg \qquad [\because f \leq |f|] \qquad ....(1)$$

(ii) If $\int_a^b f\, dg < 0$

Then $-\int_a^b f\, dg > 0$

$$\left|\int_a^b f\, dg\right| = -\int_a^b f\, dg$$

$$= \int_a^b (-f) dg$$

$$\left|\int_a^b f\, dg\right| \leq \int_a^b |f| dg \qquad [\because -f \leq |f|] \qquad ....(2)$$

From equations (1) and (2) , we have

$$\left|\int_a^b f\, dg\right| \leq \int_a^b |f| dg\ .$$

2.8 RELATION BETWEEN RIEMANN INTEGRAL AND RS-INTEGRAL

Theorem 8. If f is continuous on $[a,b]$ and g has a continuous derivative on $[a,b]$ such that $g^1(x) \neq o$ for all $x \in [a,b]$, then $\int_a^b f\, dg = \int_a^b f(x) g^1(x)\ dx$

Proof: Let $f(x)$ is continuous on $[a,b]$. Let g has a continuous derivative on $[a,b]$ such that $g^1[x] \neq 0, \forall x \in [a,b]$

$\Rightarrow g^1(x)$ has same sign positive or negative for all $x \in [a,b]$ suppose $g^1(x) > 0$ for all $x \in [a,b]$

$\Rightarrow g(x)$ will be monotonically increasing

Also fg^1 is continuous on $[a,b]$ as product of two continuous function is also continuous on $[a,b]$.

$\Rightarrow (fg^{'1}) \in R[a,b]$

Let $P = \{a = x_0, x_1, x_2, ..., x_n\}$ be a partition of $[a,b]$. Let $I_r = [x_{r-1}, x_r], r = 1, 2, ..., x$ be rth subinterval of P.

Since g is differentiable on $\left[a,b\right]$, so by Lagrange's mean value theorem, we have

$$\frac{g\left[x_r\right]-g\left[x_{r-1}\right]}{x_r-x_{r-1}}=g^1\left(\varepsilon_r\right),x_{r-1}\le\varepsilon_r\le x_{r-1}\_\_\_\_\_\_\_\_(1)$$

As $\int_a^b f\,dg$ exists.

$$\int_a^b f\,dg=\lim_{\|P\|\to 0}\sum_{r=1}^{x} f\left(\varepsilon_r\right)\left[g\left(x_r\right)-g\left(x_{r-1}\right)\right]$$

$$=\lim_{\|P\|\to 0}\sum_{r=1}^{r} f\left(\varepsilon_r\right)g^1\left(\varepsilon_r\right)\Delta_r,\text{ from equation (1)}$$

$$=\lim_{\|P\|\to 0}\sum\left(fg^1\right)\left(\varepsilon_r\right)\Delta_r$$

$$=\int_a^b f\left(x\right)g^1\left(x\right)dx.$$

2.9 INTEGRATION OF VECTOR VALUED FUNCTON

Let $f_1, f_2,\ldots\ldots, f_P$ be a real valued functions defined on $\left[a,b\right]$ and let $\overline{f}\left(x\right)=\left(f_1\left(x\right), f_2\left(x\right),\ldots, f_p\left(x\right)\right)$, for all $x\in\left[a,b\right]$

If g is a monotonically increasing function on $\left[a,b\right]$ then $\overline{f}\in\text{RS}\left(g\right)$ on $\left[a,b\right]$ iff $f_r\in\text{RS}\left(g\right)$ on $\left[a,b\right]$ for $\{=1,2,\ldots,p$

In this case, we shall define

$$\int_a^b\overline{f}\,dg=\left(\int_a^b f_1dg,\int_a^b f_2\,dg,\ldots,\int_a^b f_ndg\right)$$

Theorem 9. Let $\overline{f}$ be a mapping on $\left[a,b\right]$ into R^p and let $\overline{f}\in RS\left(g\right)$ on $\left[a,b\right]$ for some monotonically increasing function g on $\left[a,b\right]$. Then $\left|\overline{f}\right|\in\text{RS}\left(g\right)$ on $\left[a,b\right]$ and $\left|\int_a^b\overline{f}dg\right|\le\int_a^b\left|\overline{f}\right|dg.$

Proof: Let $\overline{f}=\left(f_1, f_2,\ldots., f_p\right)$, we have

$$\left|\overline{f}\right|=\left[f_1^2+f_2^2+\ldots\ldots+f_p^2\right]^{\frac{1}{2}}$$

Clearly each $f_i^2\left(\Sigma=1,2,\ldots,p\right)$ is RS – integral relative to g and hence so is their sum. Since x^2 is a continuous function of x, The square root function is continuous on $\left[o,K\right]$ for every real K Let $\phi=\left|f\right|$, f = identity function I on $\left[a,b\right]$, $g\left(x\right)=x$ such that $\left|f\right|oI=\left|f\right|$.

$\Rightarrow\left|f\right|\in RS\left(g\right)$ on $\left[a,b\right]$ $\left(\because I\text{ is continuous on }\left[a,b\right]=I\in RS\left(g\right)\right)$

Let $\bar{u} = (u_1, u_2, ..., u_p)$ and $\bar{u}_i = \int_a^b f_i dg, \Sigma = 1, 2, ..., x$

Then $\bar{u} = \int_a^b f\ dg$ ...(1)

And $|\bar{u}|^2 = \sum_{\Sigma=1}^{p} u_i^2 = \sum_{\Sigma=1}^{b} u_i \int_a^b f_i\ dg$

$$|\bar{u}|^2 = \int_a^b \sum_{i=1}^{p} u_i\ f_i\ dg \qquad ....(2)$$

Applying schwarz inequality, we have

$$\left(\sum a_i b_i\right)^2 \leq \sum_{\sum=1}^{p} a_i^2 \sum_{\sum=1}^{p} p_i^2$$

With $a_i = u_i, b_i = f_i(x)$ for $a \leq x \leq b$ we get $a \leq x \leq b$

$$\Rightarrow \int_a^b \left(\sum_{i=1}^{p} u_i f_i(x)\right) dg \leq \int_a^b |\bar{u}| |\bar{f}(x)| dg$$

$$\leq |\bar{u}| . \int_a^b |\bar{f}(x)| dg \qquad ......(3)$$

From equation (2), we have

$$|\bar{u}|^2 \leq |\bar{u}| \int_a^b |f| dg \qquad \text{[from equation (3)]} \qquad ...(4)$$

If $\bar{u} = 0$, The theorem holds trirally

If $\bar{u} \neq 0$, equation (4) imphis

$$|\bar{u}| \leq \int_a^b |f| dg$$

or $\int_a^b f\ dg \leq \int_a^b |f| dg$ [from equation (1)].

2.10 FUNCTION OF BOUNDED VARIATION

In this section, we will expand the integration theory developed so far by replacing the class of monotonic functions with the class of functions of bounded variation. We will define this new concept as follows:

Let f be a mapping of $[a,b]$ inro R^p and let $p = \{a = x_0, x_1, ..., x_r = b\}$ be any partition of $[a,b]$.Let

$$\Delta f_r = f(x_r) - f(x_{r-1})$$

Define $V[[a,b], P, f] = \sum_{r=1}^{x} \left|\sum \bar{f}(r)\right|$

$$P\left[\left[a,b\right],\overline{f}\right]=\sup V\left[\left[a,b\right],P,\overline{f}\right],$$

The supremum being taken over all partitons of $\left[a,b\right]$.

Then $V\left[\left[a,b\right],\overline{f}\right]$ is called the total variation of $V\left[\left[a,b\right],\overline{f}\right]$ on $\left[a,b\right]$. The function $\overline{f}$ is said to be of bounded variation on $\left[a,b\right]$ if and only if $V\left[\left[a,b\right],f\right]<\infty$ we shall use $V\left[\left[a,b\right],\overline{f}\right]$ as $V\left(\overline{f}\right)$.

Note: Most of the properties of vector valued functions of bounded variation can be reduced to the case of real valued function. We shall prove therefore most of the theorems for vector valued functions only.

Theoerm 10. Let $\overline{f}=\left(f_1,f_2,...,f_p\right)$ be a mapping of $\left[a,b\right]$ into R^p. Then $\overline{f}$ is of bounded variation on $\left[a,b\right]$ if and only if each of the function f_i is of bounded variation on $\left[a,b\right]$. For $i\le i\le p$, we have $V\left(f_i\right)\le V\left(\overline{f}\right)\le \sum_{\sum=1}^{p} V\left(f_i\right)$

Proof. Let $P=\left\{a=x_0,x_1,x_2,...,x_n=b\right\}$ be any partition of $\left[a,b\right]$. Then it is easy to see that

$$\left|f_i\left(x_r\right)-f_i\left(x_{r-1}\right)\right|\le\left|\overline{f}\left(x_r\right)-\overline{f}\left(x_{r-1}\right)\right|\le\sum_{r=1}^{n}\left|f\left(x_r\right)-f_i\left(x_{r-1}\right)\right|$$

Adding These inequalities for $r=1,2,....,n$

$\left|f_i\left(x_n\right)-f_i\left(x_0\right)\right|\le\sum_{r=1}^{n}\left|\overline{f}\left(x_r\right)-\overline{f}\left(x_{r-1}\right)\right|\le\sum_{r=1}\sum_{i=1}^{p}\left|f_i\left(x_r\right)-f_i\left(x_{r-1}\right)\right|$ Taking the supremum, we obtain

$$V\left(f_i\right)\le V\left(\overline{f}\right)<\sum_{i=1}^{p}V\left(f_i\right) \qquad \text{\_\_\_\_\_\_}(1)$$

The inequality (1) shows that $\overline{f}$ is bounded vareation iff each of the functions f_i is of bounded variation.

Remarks. (i) Every real function of bounded variation on $\left[a,b\right]$ is bounded.

Since f is of bounded variation on $\left[a,b\right]$, there exists positive number M such that $V\left(f\right)\le M$.

Also $\left|f\left(x\right)-f\left(a\right)\right|\le V\left(f\right),\forall x\in\left[a,b\right]$

If follows that $\left|f\left(x\right)-f\left(a\right)\right|\le M$

But $\left|f\left(x\right)\right|-\left|f\left(a\right)\right|\le\left|f\left(x\right)-f\left(a\right)\right|$

Hence $\left|f\left(x\right)\right|\le M+f\left(a\right),\forall x\in\left[a,b\right]$ Thus f is bounded on $\left[a,b\right]$

(ii) A real function f defined on $\left[a,b\right]$ may be continuous with out being of bounded variation.

Consider the function f defined on $[0,2]$ as follows $f(x)=\begin{cases} x\sin\dfrac{\pi}{x} & , \quad 0<x\le 2 \\ 0 & \quad x=0 \end{cases}$

Consider the partition

$$P=\left\{0,\frac{2}{2n-1},\frac{2}{2n-3},.....,\frac{2}{5},\frac{2}{3},\frac{2}{1}\right\}$$

Here $x_0=0,\ x_r=\dfrac{1}{2n-2r-1}$ for $r=1,2,...,n$

Then

$$\sum_{r=1}^{n}\left|f(x_r)-f(x_{r-1})\right|=\frac{2}{2n-1}+\left(\frac{2}{2n-3}+\frac{2}{2n-1}\right)+....+\left(\frac{2}{3}+\frac{2}{5}\right)$$

$$+......\left(2+\frac{2}{3}\right)>\frac{1}{n}+\frac{1}{n-1}+...+\frac{1}{3}+\frac{1}{2}$$

And this can be made arbitrarily large by taking n sufficiently larg, since $\sum\frac{1}{n}$ is divergral

$\Rightarrow \sum_{r=1}^{n}\left|f(x_r)-f(r_{r-1})\right|$ must be divergral because any series grealer than div is div.

When shows that a bounded function need not be of bounded variation.

Theorem 11: Let f be a monotonic and bounded on $[a,b]$ then f is of bounded variation on $[a,b]$ and $V(f)=\left|f(b)-f(a)\right|$.

Proof: Let f be monotonically increasing function on $[a,b]$.

Then for any partion $P=\{a=x_0,x_1,...,x_n=b\}$, we have

$$\sum_{r=1}^{n}\left|f(x_r)-f(x_{r-1})\right|=\sum_{r=1}^{n}\left|f(x_r)-f(x_{r-1})\right|=f(x_n)-f(x_0)=f(b)-f(a)$$

Therefore $V(f)=\sup\sum_{r=1}^{n}\left|f(x_r)-f(x_{r-1})\right|=f(b)-f(a)$

Therefore, we have $V(f)=\sup\sum_{r=1}^{n}\left|f(x_r)-f(x_{r-1})\right|$

$V(f)=f(b)-f(a)$

Supermum being taken over all partition of $[a,b]$ since f is bounded, $f(b)-f(a)$ is finite and hence f is of bounded variation on $[a,b]$

Theorem 12. Let $\overline{f}$ a mapping defined on $[a,b]$ into R^p. Let $\overline{f}^1$ exist on $[a,b]$. If $\overline{f}^1$ is bounded on $[a,b]$, then $\overline{f}$ is of bounded variation.

Proof: Let $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$

Let $I_r = [x_{r-1}, x_r]$ be a subinterval of P.

Since $\overline{f}^1$ exists on $[a,b]$ Then by mean value theorem There exists $\varepsilon_r \in [x_{r-1} x_r]$ such that

$\left|\overline{f}(x_r) - \overline{f}(x_{r-1})\right| = \overline{f}^1(\varepsilon_r)(x_r - x_{r-1})$ __________(1)

Also $\left|\overline{f}(x_r)\right| \quad \leq M, \forall x_r \in [a,b]$ as $\overline{f}$ is bounded on $[a,b]$

From equations (1) and (2), we have

$\left|\overline{f}(x_r) - \overline{f}(x_{r-1})\right| \leq \sum_{r=1}^{n} M(x_r - x_{r-1}) \leq M(b-a)$ If follows that

$$V\left(\overline{f}\right) = \sup \sum_{r=1}^{n} \left|\overline{f}(x_r) - \overline{f}(x_{r-1})\right| \leq M(b-a) \Rightarrow f \text{ is of bounded variation on } [a,b]$$

Theorem 13. Let f and g be any complex valued function of bounded variation on $[a,b]$. Then $(f+g)$ and fg are also of bounded variation on $[a,b]$

Proof: Let $P\{a = x_0, x_1, x_2, ..., x_n = b\}$ be any partition of $[a,b]$, we have

$$\sum_{r=1}^{n} \left|\Delta(f+g)_r\right| = \sum_{r=1}^{n} \left|(f+g)(x_r) - (f+g)(x_{r-1})\right|$$

$$= \sum_{r=1}^{n} \left|f(x_r) - f(x_{r-1}) + g(x_r) - g(x_{r-1})\right|$$

$$= \sum_{r=1}^{n} |\Delta f_r + \Delta g_r|$$

$$\leq \sum_{r=1}^{n} |\Delta f_r| + \sum_{r=1}^{n} |\Delta g_r|$$

$$\leq V(f) + V(g) \qquad ...(1)$$

Since the inequality (1) holds for any partition P,

Taking supermum, we have

$V(f+g) \leq V(f) + V(g) < \infty \qquad [\because f, g$ are of and variation $]$

If follows that $(f+g)$ is of bounded variation.

To prove fg are of bounded variation.

It is given f and g are of bounded variation

$\Rightarrow \left| f(x) \right| \le M$ and

$\left| g(x) \right| \le N, \ \forall x \in [a,b]$

Let $h = fg,$ then $\Delta h_r = fg(x_r) - fg(x_{r-1})$

$$= f(x_r)g(x_r) - f(x_{r-1})g(x_{r-1})$$

$$= f(x_r)g(x_r) - f(x_r)g(x_{r-1}) + f(x_r)g(x_{r-1})$$

$$= f(x_r)\left[g(x_r) - g(x_{r-1})\right] + g(x_{r-1})\left(fx_r - f(x_{r-1})\right)$$

$$\Delta h_r = f(x_r)\Delta g_r + g(x_{r-1})\Delta f_r$$

$$\sum_{r=1}^{n} |\Delta h_r| = \sum_{r=1}^{n} \left| f(x_r)\Delta g_r + g(x_{r-1})\Delta f_r \right|$$

$$\le \sum_{r=1}^{n} \left| f(x_r) \right| . |\Delta g_r| + \sum_{r=1}^{n} |gx_{r-1}| . |\Delta f_r|$$

$$\le M.\sum_{r=1}^{n} |\Delta g_r| + N.\sum_{r=1}^{n} |\Delta f_r|$$

$$\le M.V(g) + NV(f)$$

Hence $V[fg] \le m.v(g) + NV(f) < +\infty$

$\left[\because M\right.$ and N are finite and f and g are of bounded variation$\left.\right]$

$\Rightarrow fg$ is of bounded variation.

Cor. Let f and g be monotonically increasing on $[a,b]$. Then $(f - g)$ is of bounded variation on $[a,b]$.

2.11 TOTAL VARIATION FUNCTION

Let $\bar{f} : [a,b] \to R^p$

Let $\bar{f}$ be a bounded variation on $[a,b]$ Define $V_{\bar{f}} : [a,b] \to \bar{R}$

Such that $V_{\bar{f}}(x) = V\left[[a,x], f\right]$, for all $x \in [a,b]$

Then $V_{\bar{f}}$ is called the total variation of $\bar{f}$

Note: It is clear $V_{\bar{f}}$ is monotonically increasing on $[a,b]$ and $V_{\bar{f}}(0)$.

Theorem 14: let $\bar{f}$ be a mapping of $[a,b]$ into R^p and let $\bar{f}$ be bounded variation on $[a,b]$

(i) if $a \le x \le b$, then $V\left[[a,y],\bar{f}\right] = v\left[[a,x],\bar{f}\right] + v\left[[x,y],\bar{f}\right]$

(ii) If $\bar{f}$ is also continuous on $[a,b]$, then so is $V_{\bar{f}}$.

Proof: (i) If $x=0$ or $y=x$ then the result is obrious, Now let $a < x < y$ and $\varepsilon > 0$ then there exists a partition $P = \{a = x_0, x_1, x_2, ..., x_n = y\}$ of $[a,y]$ such that

$$V_{\bar{f}}(y) - \varepsilon \le \sum_{r=1}^{n} \left|\bar{f}(x_r) - \bar{f}(x_{r-1})\right| \le v_{\bar{f}} - \varepsilon \text{ ____(1)}$$

In case $x \le y$, we adjoin it to p and thus get a new partition p ^{x} for which (1) still holds.

Then $V\left[[a,n],\bar{f}\right] + V\left[[x,y],\bar{f}\right]$ is the supremum of $\sum_{r=1}^{n} \left|f(x_r) - f(x_{r-1})\right|$, for all partitions on $[a,b]$.

$\Rightarrow v_{\bar{f}}(y) - \varepsilon \le v_f(x) + v\left[[x,y],\bar{f}\right] \le v_f(y)$

Taking $\varepsilon \to 0$, we get $\Rightarrow v_{\bar{f}}(y) - v_{\bar{f}}(x) + V\left[[x,y],\bar{f}\right]$

(ii) Assume That $\bar{f}$ is continuous on $[a,b]$ Let a < y $\le b$. We shall show that v_f is continuous at y from the left. That is we shall show $\lim_{x\to y}{+} v_{\bar{f}}(x) = v_{\bar{f}}(y)$ __________(2)

Now for every $x \in [a,y]$, we have by (i)

$V\left[[a,y],\bar{f}\right] = v\left[[a,x],\bar{f}\right] + v\left[[x,y],\bar{f}\right]$

or $v_{\bar{f}}(y) = v_{\bar{f}}(x) + v\left[[x,y],\bar{f}\right]$ __________(3)

Equality of (2) will hold if $v\left[[x,y],\bar{f}\right] = 0$

To show $v\left[[x,y],\bar{f}\right] = 0$ __________(4)

Suppose if possible

$$v\left[[x,y],\bar{f}\right] > \delta \text{ for some } \delta > 0$$

And for every $x \in [a,b]$. If we take $x = a$ in (4)

We find that there is a position

$$\left[a = x_0, x_1, ..., x_{n,y}\right] \text{ of } [a,y]$$

Such that $\sum_{r=1}^{n} \left|\bar{f}(x_r) - \bar{f}(x_{r-1})\right| > \delta$ ______(5)

Remember that $x_n = y, x_{n-1} < y$. Since $\bar{f}$ is continuous, there exists a point a_1 such that

$x_{n-1} \le a_1 < y$,such that

$\left|\bar{f}(y)-\bar{f}(x_{n-1})\right|$ and $\left|\bar{f}(a_1)-\bar{f}(x_{n-1})\right|$ different by as little as we please.

In particular (5) will hold if y is replaced by some such a_1. Thus we have proved that there exists $a_1 < y$ such that $v\left[[a, a_1], \bar{f}\right] > \delta$. We continuous this process with a_1 in place of a and so on and obtain the number $a = a_0 < a_1 < a_2 < ......, a_m < y,$ for every m. such that

$$v\left[[a_{r-1}, a_r], \bar{f}\right] > \delta(1 \le r \le m)$$

Putting $r = 1, 2, ..., m$ and adding, we get

$$V\left[[a, a_1], \bar{f}\right] + v\left[[a_1, a_2], \bar{f}\right] + ...v\left[[a_{m-1}, a_m] + \right] > m\delta$$

$$\Rightarrow v\left[[a, a_m], \bar{f}\right] > m\delta \qquad \text{[using (i) part]}$$

$$\Rightarrow V\left[[a, y], \bar{f}\right] = v\left[[a, a_m], \bar{f}\right] + v[a_m, y] > m\delta$$

$\Rightarrow v_{\bar{f}}(y) > m\delta$ for all m , for all m.

When is inpossible since $v_{\bar{f}}$ is bounded

So we reach to contradiction

Therefore, we have

$$\lim_{x \to y} v\left[[x, y], \bar{f}\right] = 0$$

Hence $\lim_{x \to y} v_{\bar{f}}(x) = v_{\bar{f}}(y)$

$v_{\bar{f}}$ is continuous from the left in the same manner, h can be proved v_f is continuous from the right on $y \in [a, b]$ Hence $v_{\bar{f}}$ is continuous on $[a, b]$.

Theorem 15. Let f be a real valued function of bounded variation on $[a, b]$. Then there exist monotonically increasing functions g and h on $[a, b]$ with $g(a) = h(a) = 0$ such that

(i) $\quad f(x) - f(a) = g(x) - h(x)$

(ii) $\quad v_f(x) = g(x) + h(x)$ **for every** $x = [a, b]$

Proof. We define g and h by

and $\qquad \left.\begin{aligned} g &= \frac{1}{2}\left[v_f + f - f(a)\right] \\ h &= \frac{1}{2}\left[v_f - f + f(a)\right] \end{aligned}\right\} - (1)$ as f is real valued $f = \bar{f}$

Since $v_f(a)=0,$

implies $g(a)=h(a)=0$

Also for every $x\in[a,b]$, we have

$$g(x)=\frac{1}{2}\left[v_f(x)+f(x)-f(a)\right] \quad \_\_\_\_(2)$$

And $$h(x)=\frac{1}{2}\left[v_f(x)-f(x)+f(a)\right]\_\_\_\_\_(3)$$

Now from equation (3)– (2), we have

$g(x)-h(x)=f(x)-f(a)$

Now from equation (3)+ (2), we have

$g(x)+h(x)=v_f(x)$

Finally if $a\le x\le y\le b,$ then

$$2g(y)-2g(x)-v_f(y)+f(y)-v_f(x)-f(x)=v_f(x)+v\big[[x,y],f\big]+f(y)-v_f(x)-f(x)$$

$$=v\big[[x,y],f\big]+\big[f(y)-f(x)\big] \quad \_\_\_\_(4)$$

Similarly, we have

$$2h(y)-2h(x)=\big[[x,y],f\big]-\big[f(y)-f(x)\big]\_\_\_\_\_\_\_(5)$$

Since $|f(y)-f(x)|\le v\big[[x,y],f\big]$

From equations (4) & (5) $\Rightarrow g$ and h are increasing

i.e., $v_f(x)=g(x)+h(x),\forall x\in[a,b]$

Corrollary. Let $\bar{f}$ be of bounded variation on $[a,b]$. Then $\bar{f}(x+0)$ exists for $a\le x<b$ and $\bar{f}(x-0)$ exists for $a<x\le b,$ and the set of discontinuous at most countable.

Note: If α is a real valued function of bounded variation on $[a,b]$. Then there exist monotonically increasing function β and v such that

$$\alpha=\beta-v$$

$$\Rightarrow\int_a^b f\,d\alpha=\int_a^b f\,d\beta-\int_a^b f\,dv$$

Theorem 16. Let f and $\propto$ be complex valued functions defined on $[a,b]$ such that (i) f is continuous and $\propto$ is of bounded variation

(ii) Let f is of bounded variation and $\propto$ continuous function. Let v be the total variation of $\propto$ on $[a,b]$

. Than

$$\left|\int_a^b f\,dx\right| \le \int_a^b |f|\,dv$$

Proof: Clearly $|f|$ is continuous function and $|f|$ is of bounded variation.

$\Rightarrow \int_a^b |f|\,dv$ exists in each case.

Let $P=\{a=x_0,x_1,x_2,...,x_n=b\}$ be a partition of $[a,b]$ then

$$\left|\sum f/\varepsilon_r)\Delta \propto_r\right| \le \sum |f/\varepsilon_r)|.|\Delta \propto_r| \le \sum |f/\varepsilon_r|.\Delta v_r \_\_\_\_\_\_ (1)$$

Where $x_{r-1} \le \varepsilon_r \le x_r$, for $1 \le r \le x$ As $\|p\| \to 0$ and The sum of right side of (1) tends to $\int_a^b f\ d\propto$ and the sum on the right side of (1) tends to $\int_a^b |f|dv$

$$\Rightarrow \left|\int_a^b f\ dx\right| \le \int_a^b |f|\,dv$$

Theorem 17. Let f and g be complex valued functions of bounded variation on $[a,b]$, and let f be also continuous on $[a,b]$ then $\int_a^b fdg = f(b)g(b)-f(a)g(a)-\int_a^b g\,df$

Proof: Let $P=\{a=x_0,x_1,.....,x_n=b\}$ be any partition of $[a,b]$ and let $Q=\{a=\varepsilon_0,\varepsilon_1,\varepsilon_2,..,\varepsilon_{n+1}=b\}$ be any intermediate partition of p so that $x_{r-1}<\varepsilon_r<x_r$ for $r=1,2,...,n$.

Then with usual notations, we have

$$\begin{aligned}\text{RS}(P,Q,f,g) &= \sum_{r=1}^{n} f(\varepsilon_r)\left[g(x_r)-g(x_{r-1})\right]\\ &= f(\varepsilon_1)g(x_1)-f(\varepsilon_1)g(x_0)+f(\varepsilon_2)g(x_2)-f(\varepsilon_2)g(x_1)\\ &\qquad +......+f(\varepsilon_n)g(x_n)-f(\varepsilon_n)g(x_{n-1})\\ &= f(x_0)g(x_0)-g(x_0)\left[f(\varepsilon_1)+f(\varepsilon_0)\right]\\ &\qquad -g(x_0)\left[f(\varepsilon_2)-f(\varepsilon_1)\right]..........-g(x_{n-1})\left[f(\varepsilon_n)-f(\varepsilon_{n-1})\right]+f(\varepsilon_n)g(x_n)\end{aligned}$$

[adding and subtracting the term $f(x_0)g(x_0)$ and rearranging]

$$\begin{aligned}&= f(b)g(b)-f(a)g(a)-\sum_{r=1}^{n} g(x_{r-1})\left[f(\varepsilon_r)+f(\varepsilon_{r-1})\right]\\ &= f(b)g(b)-f(a)g(a)-\text{RS}[Q,P,g,f] \qquad ...(1)\end{aligned}$$

$$\left[\because \varepsilon_{r-1} \le x_{r-1} \le \varepsilon_r\right]$$

If $\|P\| \to 0$, then $\|Q\| \to 0$ and consequently RS $(P,Q,f,g) \to \int_a^b f\,dg$ and RS

$$(P,Q,f,g)\to\int_a^b g\,df$$

From equation (1), we have

$$\int_a^b fdg = f(b)g(b)-f(a)g(a)-\int_a^b g\,df\,.$$

Theorem 18 (First Mean value theorem) If f is continuous and real and g is monotonically increasing on $[a,b]$ then there exists a point ε such that $a\le\varepsilon\le b$, and $\int_a^b f\,dg = f(\varepsilon)[g(b)-g(a)]$

Proof: Consider

$$m=\inf\{f(x):\ a\le x\le b\}$$

$$M=\sup\{f(x):a\le x\le b\}$$

Then we have

$$m\le f(x)\le M \quad (a\le x\le b)$$

$$\Rightarrow m\int_a^b dg\le\int_a^b f\,dg\le M\int_a^b dg$$

$$\Rightarrow m[g(b)-g(a)]\le\int_a^b f\,dg\le M[g(b)-g(a)]$$

Hence There exists a number μ, $m\le\mu\le M$ such that

$$\int_a^b f\,dg=\mu[g(b)-g(a)]$$

Since f is continuous on $[a,b]$, t takes all values between its infmum m and supremum M on $[a,b]$ in other words, $\exists$ a point ε in $[a,b]$ for which

$$f(\varepsilon)=\mu$$

$$\Rightarrow\int_a^b f\,dg=f(\varepsilon)[g(b)-g(a)]$$

Theorem.19: (Second Mean value theorem):

Let f be monotonic and let g be real, continuous and of bounded variation on $[a,b]$. There exists a point $\varepsilon\in[a,b]$ such that

$$\int_a^b f\,dg=f(a)[g(\varepsilon)-g(a)]+f(b)[g(b)-g(\varepsilon)]$$

Proof: We know if f is monotonic, g is real, continuous and of bounded variation on $[a,b]$ then

$$\int_a^b f\,dg=f(b)g(b)-f(a)g(a)-\int_a^b g\,df \quad \text{\_\_\_\_\_}(1)$$

Since g is continuous so by lst mean value theorem

$$\int_a^b g\,df=g(\varepsilon)[f(b)-f(a)] \quad \text{\_\_\_\_\_}(2)\ \text{ for } \varepsilon\in[a,b]$$

From equations (1) and (2), we have

$$\int_a^b f\,dg = f(b)g(b) - f(a)g(a) - g(\varepsilon)\left[f(b) - f(a)\right]$$

$$= f(a)\left[g(\varepsilon) - (a)\right] + f(b)\left[g(b) - g(\varepsilon)\right]$$

Theorem.20: Let f and ϕ be continuous on $[a,b]$ and ϕ be increasing on $[a,b]$. If F is the increase function of ϕ, then

$$\int_a^b f(x)\,dx = \int_{\phi(a)}^{\phi(b)} f\left[F(y)\right]dF(y)$$

Proof: Let $P = \{a = x_0, x_1, \ldots\ldots, x_n = b\}$ be any partition of $[a,b]$ Let $y_r = \phi(x_r)$ so that $r_r = F(y_r)$

$$r = 0,1,2,\ldots.,n$$

Consider the partition $Q = \{\phi(a) = y_0, y_1, \ldots., y_n = \phi(b)\}$ of $[a,b]$

We put $h(y) = f\left[F(y)\right]$

$$\Rightarrow \sum_{r=1}^{n} f(x_r)(x_r - x_{r-1}) = \sum_{x=1}^{n} f\left[F(y_r)\right]\left[F(y_r) - F(y_{r-1})\right]$$

$$= \sum_{r=1}^{n} h(y_r)\left[F(y_r) - F(y_{r-1})\right]$$

Since ϕ is continuous on $[a,b]$

$\Rightarrow \phi$ is uniformly continuous on $[a,b] \Rightarrow \|P\| = 0\ \|Q\| = 0$

Thus if we let $\|P\| = 0$ so $\|Q\| = 0$ then

$$\sum_{r=1}^{n} f(x_r)(x_r - x_{r-1}) \to \int_a^b f(x)\,dx$$

and $\sum h(y_r)\left[F(y_r) - F(y_{r-1})\right] \to \int_{\phi(a)}^{\phi(b)} f\left(F(y)d\left(F(y)\right)\right)$ $\quad [\because h = fF]$.

Hence $\int_a^b f(x)dx = \int_{\phi(a)}^{\phi(b)} f\left[F(y)\right]dF(y)$.

2.12 SUMMARY

Let f be a function integrable over $[a,b]$ then th4e function F on $[a,b]$ given by $F(x) = \int_a^{\pi} f(t)\,dt$, $a \le \pi \le b$ is called the integral function of f.

Let $f \in R[a,b]$. Than the integral function F of f is given by

$F(x)=\int_a^{\pi} f(t)\,dt,\ a\le \pi\le b$ is continuous on $[a,b]$

Let f be a continuous function on $[a,b]$ and let $F(x)=\int_a^x f(t)\,dt,$ for all $x\in[a,b]$ then $F'(x)=f(x)$.

Let f be a continuous function on $[a,b]$ and let & be a differentiable function on $[a,b]$ such that $\phi'(x)=f(x);\ x\in[a,b]$.

Then $\int_a^b f(t)dt=\phi(b)-\phi(a)$.

If $f\in \mathrm{RS}(g)$ on $[a,b]$ there $|f|\in \mathrm{RS}(g)$ on $[a,b]$ and $\left|\int_a^b f\,\mathrm{dg}\right|\le \int_a^b |f|dg$.

2.13 TERMINAL QUESTIONS

Q.1 Explain the Continuity and Differentiability of Integral function.

Q.2 State and prove the fundamental theorem of integral calculus.

Q.3 Show that the relation between Riemann integral and RS integral.

Q.4 If f is continuous on $[a,b]$ and g has a continuous derivative on $[a,b]$ such that $g^1(x)\ne o$ for all $x\in[a,b]$, then. $\int_a^b f\,dg=\int_a^b f(x)g^1(x)\ dx$

Q.5 Let $\overline{f}$ be of bounded variation on $[a,b]$ Then $\overline{f}\,(x+0)$ exists for $a\le x<b$ and $\overline{f}\,(x-0)$ exists for $a<x\le b$, and the set of discontinuous at most countable.

Q.6 Let f and g be monotonically increasing on $[a,b]$. Then $(f-g)$ is of bounded variation on $[a,b]$.

UNIT-3 UNIFORM CONVERGENCE OF SEQUENCE

Structure

3.1 Introduction

3.2 Objectives

3.3 Uniform Bounded Sequence

3.4 Uniform Convergence of Sequence

3.5 Uniform Convergence of a series of functions

3.6 Cauchy’s Criterion for Uniform Convergence

3.7 Tests for Uniform Convergence

3.8 Uniform Convergence and Integration

3.9 Uniform Convergence and Differentiation

3.10 Summary

3.11 Terminal Questions

3.1 INTRODUCTION

Uniform convergence plays a crucial role in the theory of integration, particularly when dealing with sequences of functions. In some cases, the limit of the integrals of a sequence of functions can be expressed as the integral of the limit function. This is known as the theorem on interchanging limits and integrals. However, this interchange is not always valid; uniform convergence is a sufficient condition for this interchange to hold. For series of functions, uniform convergence is essential for ensuring that the series can be integrated term by term.

Uniform convergence is a powerful concept in integration theory, as it allows us to extend properties of individual functions to sequences or series of functions, and it ensures the continuity of the limit operation with respect to integration.

3.2 OBJECTIVES

After studying this unit the learner will be able to understand the :

- Uniform bounded sequence
- Uniform convergence of sequence
- Uniform convergence of a series of functions
- Cauchy's criterion for uniform convergence
- Tests for Uniform Convergence
- Uniform Convergence and Integration
- Uniform Convergence and Differentiation

3.3 UNIFORM BOUNDED SEQUENCE

A sequence of functions $f_1(x), f_2(x), f_3(x), \ldots\ldots, f_n(x)$ defined on a domain D is said to be uniformly bounded if there exists positive number m such that

$|f_n(x)| < M$ for all n and for all $x \in D$.

For example, Let $f_n(x) = \sin nx$, defined on real x is uniformly bounded by I.

3.4 UNIFORM CONVERGENCE OF SEQUENCE

A sequence $\{f_n\}$ of functions defined on $[a,b]$ is said to converge uniformly to a function f if for any $\in > 0$ there exists an integer m such that

$|f_n(x) - f(x)| < \in$, for all $n \geq m$ and for all $x \in [a,b]$.

Here m does not depend upon x .

Note: A uniformly convergent sequence is always a convergent sequence but converge is not necessarily true on $[a,b]$.

3.5 UNIFORM CONVERGENCE OF A SERIES OF FUNCTIONS

A series of functions $\sum f_n$ converge uniformly on $[a,b]$ if the sequence $< S_n >$ of its partial sums, defined by

$S_n(x)=\sum_{i=1}^{n} f_i(x)$. Converges uniformly on $[a,b]$.

Thus a series of functions $\sum f_n$ converges uniformly to f on $[a,b]$ if for any $\in >0$ and for all $x \in [a,b]$, there exists a positive integer m such that

$|S_n - f| < \in$, for all $n \geq m$.

3.6 CAUCHY'S CRITERION FOR UNIFORM CONVERGENCE

Theorem 1. A sequence of functions $< f_n >$ defined on $[a,b]$ converges uniformly on $[a,b]$ if and only if for $\in >0$ and for all $x \in [a,b]$ there exists a positive integer m such that $|f_{n+l}(x) - f_n(x)| < \in$, for all $n \geq m, l \geq 1$

Proof. Let $< f_n >$ be a sequence of functions defined on $[a,b]$ "if part"

Let $< f_n >$ converges uniformly to f then for $\in >0$ and for all $x \in [a,b]$ there exists positive integers m_1, m_2 such that

$|f_n(x) - f(x)| < \frac{\in}{2}$, for all $n \geq m_1$

And $|f_{n+l}(x) - f_n(x)| < \frac{\in}{2}$, for all $n \geq m_2, l \geq 1$

Let $m = \max,(m_1, m_2)$ then

$$|f_{n+l}(x) - f_n(x)| = |f_{n+l}(x) - f(x) + f(x) - f_n(x)|$$

$$\leq |f_{+nl}(x) - f(x)| + |f_n(x) - f(x)|$$

$$< \frac{\in}{2} + \frac{\in}{2}$$

$< \in$, for all $n \geq m, l \geq 1$

Only "if part", suppose $|f_{n+l}(x) - f_n(x)| < \in$, for all $n \geq m, l \geq 1$ to show $< f_n >$ converges to f on $[a,b]$

Since $|f_{n+l}(x) - f_n(x)| < \in$, for all $n \geq m, l \geq 1$

$\Rightarrow < f_n >$ is a Cauchy sequence

$\Rightarrow \lim\limits_{n\to\infty} f_n(x)$ exists for all $x \in [a,b]$

Let $\lim\limits_{n\to\infty} f_n(x) = -f(x)$, for all $x \in [a,b]$

Keeping n fixed and taking $l \to \infty$, we get

$|f_n(x) - f(x)|$, for all $n \geq m$ and for all $x \in [a,b]$

$\Rightarrow < f_n(x) >$ converges to $f(x)$ uniformly on $[a,b]$.

Corollary. A series of functions $\sum f_n$ defined on $[a,b]$ converges uniformly on $[a,b]$ if and only if for every $\in > 0$ and for all $x \in [a,b]$ there exists a positive integer m such that

$|f_{n+1}(x) + f_{n+2}(x) + ... + + f_{n+l}| < \in$, for all $n \geq m$ and all $x \in [a,b]$.

Example.1. Show that the sequence $< x^n >$ is uniformly convergent on $[0,k], k<1$ and only convergent on $[0, 1]$.

Sol. Let $f_n(x) = x^n$

Let $f(x) = \lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} x^n$

$$= \begin{cases} 0, & if \;\; 0 \leq x < 1 \\ 1, & if \;\; x = 1 \end{cases}$$

Remember $\lim\limits_{n\to\infty} x^n = 1, \; if \;\; x = 1$

$= 0 \;\; if \;\; x < 1$

$= \infty \, if \;\; x > 1$

Thus the sequence $< x^n >$ converges to a discontinuous functions on $[0, 1]$.

Further, let $\in < 0$ be given then for $0 < x \leq k < 1$, we get

$$|f_n(x) - f(x)| = |x^n - 0|$$

$$= x^n \text{ as } 0 \leq n < 1$$

$$< \in$$

$$\Rightarrow x^n < \in \Rightarrow \left(\frac{1}{x}\right)^n > \frac{1}{\in} \Rightarrow n \log \frac{1}{x} > \log \frac{1}{\in}$$

$$\Rightarrow \; n.\log\left(\frac{1}{\in}\right) \Big/ \log\frac{1}{x}$$

The number $\left[\dfrac{\log \frac{1}{\in}}{\log \frac{1}{x}}\right]$ increase as x increases.

Its maximum value is $\left[\dfrac{\log\frac{1}{\in}}{\log\frac{1}{k}}\right]$ in $]0,\ k[$

Let m be an integer $\geq \dfrac{\log\frac{1}{\in}}{\log\frac{1}{k}}$

Then. $|f_n(x) - f(x)| < \in$, for all $n \geq m$ and $0 < n < 1$ and at $x = 0$.

$|f_n(x) - f(x)| = 0 < \in$, for all $n \geq 1$

Thus for any $\in > 0$ there exists a positive integer m such that

$|f_n(x) - f(x)| < \in$, for all $n \geq m$ and for all $x \in [0,\ k]$, $k < 1$.

Therefore $< f_n(x) >$ converges uniformly on $[0,\ k]$, $k < 1$

But $\left(\dfrac{\log\frac{1}{\in}}{\log\frac{1}{x}}\right) \to \infty$ as $x \to 1$

Therefore it is not possible to find an positive integer m such that

$|f_n(x) - f(x)| < \in$, for all $n \geq m$ and for all $x \in [0,\ 1]$.

Hence $< f_n(x) >$ is not uniformly convergent on $[0,\ 1]$.

Here the point 1 is a point of non uniform convergence.

Example 2. Show that the sequence $< \dfrac{1}{x+n} >$ is uniformly convergent in any interval $[0,\ k]$, $k > 0$.

Sol. Let $< f_n > = < \dfrac{1}{x+n} >$ be a given sequence

Let $f(x) = \lim\limits_{n\to\infty} f_n = \lim\limits_{n\to\infty} \dfrac{1}{x+n} = 0$, for all $x \in [0,\ k]$

Therefore, for $\in > 0$, we have

$$| f_n(x) - f(x)| = \left|\frac{1}{x+n} - 0\right| < \in$$

$$\frac{1}{x+n} < \in \Rightarrow x + n > \frac{1}{\in}$$

$$\Rightarrow n > \frac{1}{\in} - x$$

Maximum value of $\left(\frac{1}{\in}-x\right)$ is $\frac{1}{\in}$

Let $m \geq \frac{1}{\in}$ then

$|f_n(x)-f(x)|<\in$, for all $n \geq m$

Hence the sequence is uniformly convergent $m[0,\ k]$.

Example 3. Test for uniform convergence, the sequence $<f_n>$ where $f_n(x)=\frac{nx}{1+n^2n^2}$ for all real x.

Sol. Let $f_n(x)=\frac{nx}{1+n^2n^2}$

Now $f(x)=\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty} \frac{n[x]}{n^2\left[\frac{1}{n^2}+x^2\right]}=0$

Now for $\in>0$,

$|f_n(x)-f(x)|=\left|\frac{nx}{1+n^2x^2}-0\right|<\in$ for all $n \geq m$, m is positive integer.

$$\frac{nx}{1+n^2x^2}=\frac{x}{n\left[\frac{1}{n^2}+x^2\right]}<\in$$

Put $x=\frac{1}{p}, n=p$

$$\frac{1}{2}<\in \Rightarrow \in>\frac{1}{2}$$

We can take $\in \to 0$ i.e. if we take $\in=\frac{1}{2}$ then $\frac{1}{2}>\frac{1}{2}$

$\Rightarrow <f_n>$ is not uniformly cost.

Example 4. Show that the series $\sum f_n$ whose sum to n terms $S_n^{(x)}=nxe^{-nx^2}$ is convergent and not uniformly convergent on any interval $[0,\ k], k>0$

Solution:- Let $S_n^{(x)}=nxe^{-nx^2}$

Now $S(x)=\lim_{n\to\infty} S_n(x)=\lim_{n\to\infty} \frac{nx}{e^{nx^2}}$

$$= \lim_{n\to\infty} \frac{nx}{1+\left(nx^2\right)+\frac{\left(nx^2\right)}{\left(^2\right)}+\ldots\ldots.}$$

$$S(x) = \lim_{n\to\infty} \frac{n}{2nx+n^2(\ \ )+\ldots\ldots} = 0$$, for all $x \ge 0$

$\therefore$ Thus $\sum f_n(x)$ converges to 0 on $[0,\ k]$

Checking of uniform convergence, for $\in > 0$, we have

$|f_n(x) - f(x)| < \in,\ \forall\, n \ge m,$ m is positive integer

$$\left|\frac{hx}{e^{nx^2}} - 0\right| < \in,\ \forall\, n \ge m$$

Take $nx^e = 1 \Rightarrow n = \frac{1}{x^2} \Rightarrow x = \frac{1}{\sqrt{n}}$

Take $n = m\,o,\ \ x = \frac{1}{\sqrt{m\,o}}$

$$\therefore mo.\frac{1}{\sqrt{mo}} < \in .e$$

$$\sqrt{m\,o} < e \in$$

$$m\,o < e^2 \in^2$$

i.e. $n < e^2 \in^2$ but n must be $\ge m$

which is a contradiction

$< \sum f_n(n) >$ is not uniformly convergent.

3.7 TESTS FOR UNIFORM CONVERGENCE

Theorem.2. (M_n-Test): Let $< f_n >$ be a sequence of function such that $\lim\limits_{n\to\infty} f_n(x) = f(x)$, for all $x \in [a,\ b]$ and let M_n= sup $|f_n(x) - f(x)|$. Then $f_n \to f$ uniformly on $[a,\ b]$ iff $M_n \to 0$ as $n \to \infty$.

Proof: Necessary condition. Let the sequence $< f_n >$ converge uniformly to f on $[a,\ b]$ than for $\in > 0$ there exists a positive integer m (independent of x) such that

$|f_n(x) - f(x)| < \in$, for all $n \ge m,$ for all $x \in [a,\ b]$

$\Rightarrow$ sup $|f_n(x) - f(x)| \le \in$, for all $n \ge m,$

$\Rightarrow M_n \to 0$ as $n \to \infty$ $\left[\because M_n = \sup\left|f_n(x) - f(x)\right|\right]$

Sufficient condition. Let $M_n \to 0$ as $n \to \infty$

Then for $\in > 0$, there exists a positive integer m such that

$M_n < \in$, for all $n \geq m,$

$\Rightarrow \sup\left|f_n(x) - f(x)\right| < \in$, for all $n \geq m,$

$\Rightarrow \left|f_n(x) - f(x)\right| < \in$, for all $n \geq m,$

$\Rightarrow f_n$ converges to f uniformly on $[a, b]$

Examples

Example.5. Show $< \dfrac{x}{1+nx^2} >, x$ is real, converges uniformly on any closed interval I.

Solution. Let $f_n(x) = \dfrac{x}{1+nx^2}$

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{1+nx^2} = 0, \text{ for all } x \in k$$

Now $\sup\left|f_n(x) - f(x)\right| = \sup\left|\dfrac{x}{1+nx^2} - 0\right|$

$$f(x) = \frac{x}{1+nx^2}$$

$$f'(x) = \frac{\left(1+nx^2\right)1 - x, 2nx}{\left(1+nx^2\right)^2}$$

$$f'(x) = 0 \Rightarrow 2nx^2 = 1 + nx^2$$

$$nx^2 = 1 \Rightarrow x = \pm\frac{1}{\sqrt{n}}$$

$f(x)$ is maximum at $x = \dfrac{1}{\sqrt{n}}$

$$= \frac{1}{2\sqrt{n}}$$

$$\because M_n = \frac{1}{2\sqrt{n}}$$

$$\lim_{n\to\infty} M_n = \lim_{n\to\infty} \frac{1}{2\sqrt{n}} = 0$$

$\Rightarrow M_n \to 0$ as $n \to \infty$

So by M_n test $< \dfrac{x}{1+nx^2} >$

Converges uniformly on any closed interval I.

Example.6. Show $< nx(1-x)^2 >$ does not converge uniformly on $[0, 1]$.

Solution. Let $f_n(x) = nx(1-x)^n$

$$f(x) = \lim_{n\to\infty} f_n(x) = \frac{\lim}{n \to \infty} \frac{nx}{(1-x)^{-n}} \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} \frac{x}{(1-x)^{-n}, \log(1-x)}$$

$$= \lim_{n\to\infty} \frac{x(1-x)^n}{\log(1-x)} = 0 \qquad \left[\because \frac{1}{n \to \infty}(1-x)^n = 0 \, as \, (1-x) < 1\right] \therefore x \in [0, 1]$$

$$\therefore \sup|f_n(x) - f(x)| = \sup\{nx(1-x)^n\}$$

$$= \sup\left\{n.\frac{1}{n}\left(1-\frac{1}{n}\right)^n\right\} \text{ take } x\frac{1}{n}$$

$$= \sup(e^{-1}) = \frac{1}{e} \qquad \left|\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x \;\; x = -1\right.$$

$$\therefore M_n \to \frac{1}{e} \text{ as } n \to \infty$$

Hence the sequence $< f_n >$ does not converge uniformly on $[0, 1]$. Hence 0 is the part of non uniform convergence.

Example.7. Show that the sequence $< f_n >$ where $f_n(x) = nxe^{-nx^2}, x \geq 0$ is not uniformly convergent on $[0, k], \; k > 0$.

Solution. Here we have

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nx\,e^{-nx^2}$$

$$= \lim_{n\to\infty} \frac{nx}{e^{nx^2}} = \lim_{n\to\infty} \frac{nx}{1+(nx^2)+\frac{1}{\sqrt{n}}(nx^2)^2}$$

$$f(x) = \lim_{n\to\infty} \frac{x}{x^2 + (nx^2), x^2 + n^2(\;\;) + \ldots..} = \frac{x}{x^2+\infty} = 0$$

Thus $f(x) = 0, \; \forall \; x \in [0, 1]$

Now $M_n = \sup\{|f_n(x) - f(x)| : x \in [0, 1]\}$

$$= \sup\left\{nx e^{-nx^2}\right\}$$

$$f(x) = nx e^{-nx^2}$$

$$f'(x) = n\left[1.e^{-nx^2} + x.e^{-nx^2}.(-2nx)\right]$$

$$f'(x) = 0 \Rightarrow n - 2n^2n^2 = 0 \qquad 1 - 2x^2 n = 0$$

$$x^2 = \frac{1}{2n} \Rightarrow x = \frac{1}{\sqrt{2n}}$$

$$nxe^{-nx^2} = xn[1 - xn^2 + ...] = (nx)$$

$$x = \frac{1}{\sqrt{2n}} \cdot \frac{1}{\sqrt{e}}$$

$$= n \cdot \frac{1}{\sqrt{2n}} \cdot \frac{1}{\sqrt{e}}$$

$$M_n = \sqrt{\frac{n}{2e}}$$

$\lim_{n \to \infty} M_n \to \infty$ as $n \to \infty$

Therefore, $< f_n >$ is not uniformly convergent on $[0, k]$.

Theorem.3. (Weierstress' s M test): A series $\sum f_n$ of functions will converge uniformly on $[a,b]$ if there exists a convergent series $\sum Mn$ of positive numbers such that

$|f_n(x)| \le Mn$ **for all** $n \ge m$ **and** $x \in [a,b]$**.**

Proof. Let $\sum f_n$ be a series of functions defined on $[a,b]$.

Let $\sum M_n$ be a convergent series of positive numbers such that

$|f_n(x)| \le M_n$, For all $n \ge m, \forall x \in [a,b]$

To prove $\sum f_n$ converges uniformly

It is given $\sum M_n$ is convergent so for $\varepsilon > 0,$ we can obtain a positive number m such that $|M_{n+1} + M_{n+2} + M_{n+3} + ... + M_{n+p}| < \varepsilon, \forall n \ge m, p \ge 1$________(1)

Also $|f_n(x)| \le Mn$______(2)

From equations (1) & (2), we have

$$\left|f_{n+1}(x)+f_{n+2}(x)+....+f_{n+p}(x)\right| \le \left|f_{n+1}(x)\right|+\left|f_{n+2}(x)\right|+.....+\left|f_{n+p^{(x)}}\right.$$

$$\le m_{n+1}+m_{n+2}+.....+m_{n+p} < \varepsilon, \forall n \ge m,\ p \ge 1, \forall x \in [a,b]$$

$\Leftarrow \sum f_n(x)$ converges uniformly on $[a,b]$.

Example.8. Show that the series $\sum 3^x \sin\frac{1}{4^n x}$ converges absolutely and uniformly on $]a,\infty[, a>0$

Sol. Let $f_n(x) = \sum 3^n \sin\frac{1}{4^n x}, x \in]a,\infty[, a>0$

we have

$$U_n = 3^n \sin\frac{1}{4^n x}$$

$$U_{n+1} = 3^{x+1} \sin\frac{1}{4.4^n x}$$

$$\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \left|3, \lim\left\{\frac{\sin\frac{1}{4^{n+1}x}}{\sin\frac{1}{4^n x}}\right\}\right. \quad 4^n x \ge 1 \Leftarrow \frac{1}{4^n x} \le 1$$

$$= \frac{3}{4} < 1 \quad \sin n\theta = \theta \text{ if } \theta \text{ is small}$$

Further, for $n \ge m$ $\left[\begin{array}{l} \sin\frac{1}{4^{n+1}x} = \frac{1}{4.4^n x} \\ \sin\frac{1}{4^n x} = \frac{1}{4^n x} \end{array}\right.$

$$\sin\frac{1}{4^n x} < \frac{1}{4^n x} < \frac{1}{4^{n-m}} \quad \left|3^n \sin\frac{1}{4^n x}\right| < 4^m.\left(\frac{3}{4}\right)^n, \forall n \ge m$$

But $\sum 4^m\left(\frac{3}{4}\right)^n$ is convergent series

$\Leftarrow \sum 3^n \sin\frac{1}{4^n x}$ is convergent as any series les than convergent is convergent $\left[\because \sum 3^n \sin\frac{1}{4^n x} < \sum 4^m\left(\frac{3}{4}\right)^x\right.$ (by weser strass test)

Theorem.4. (Abel's Test): The series $\sum U_n(x) v_n(x)$ will converge. Uniformly in $[a,b]$ if

(i) $\sum U_n(x)$ is uniformly convergent in $[a,b]$

(ii) The sequence $< U_n^{\infty} >$ is monotonic for every $x \leftarrow [a,b]$

(iii) The sequence $< v_n(x) >$ is uniformly bounded in $[a,b]$ i.e. there exists a positive number k independent of x and n such that $|U_n(x)| < k, \forall x \leftarrow N, \forall x \in [a,b]$

Proof. Let

$$ {}_pR_n(x) = U_{n+1}(x) + U_{n+2}(x) + ... + U_{n+p}(x), \forall x \in [a,b] $$

Then

$$ U_{n+1}(x)v_{n+1}(x) + U_{n+2}(x)v_{n+2}(x) + ...U_{n+p}(x)v_{n+p}(x) $$

Take ${}_1R_n(x) = U_{n+1}(x)$

${}_2R_n(x) = U_{n+1}(x) + U_{n+2}(x) \Rightarrow U_{n+2}(x) = {}_2R_n(x) - {}_1R_n(x)$

${}_3R_n(x) = U_{n+1}(x) + U_{n+2}(x) + U_{n+3}(x) \Leftarrow U_{n+3}(x) = {}_3R_n(x) - {}_2R_n(x)$ and so on.

$$ \therefore {}_1R_n(x)v_{n+1}(x) + \left[{}_2R_n(x) - {}_1R_n(x)\right]v_{n+2}(x) $$

$$ + \left[{}_3R_n(x) - {}_2R_n(x)\right]v_{n+3}(x) $$

$$ +.......+\left({}_pR_n^{(x)} - {}_{p-1}R_n^{(x)}\right)v_{n+p}(n) = U_{n+1}(x)v_{n+1}(x) + ... + U_{n+p}(x)v_{n+p}(x) \_\_\_\_\_\_ (1) $$

Since $\sum U_n$ is uniformly converges $\Rightarrow$ for $\varepsilon > 0, \exists m \in N$ such that ______(2)

$$ \left|p^{R_n}(x)\right| < \varepsilon, \forall n \geq m, p \geq 1 \quad , x \in [a,b] $$

Also $< v_n(x) >$ is uniformly bounded $\Rightarrow$ for $k > 0$, we have

$|v_n(x)| < k, \forall n \in N, \forall x \in [a,b]$ _______(3)

$$ \therefore \left(U_{n+1}^{(x)}V_{n+1}^{(x)} + ... + U_{n+p}V_{n+p}^{(x)}\right) $$

$$ < \varepsilon\left|v_{n+1}^{(x)} - v_{n+2}^{(x)} + ...v_{n+p}^{(x)}\right| + \varepsilon\left|v_{n+p}^{(x)}\right| $$

$$ < \varepsilon\left[\left|v_{n+1}^{(x)}\right| + \left|v_{n+p}^{(x)}\right|\right] + \varepsilon\left|v_{n+p}^{(x)}\right| $$

$$ < \varepsilon.2k + \varepsilon.k. $$

$$ < 3\varepsilon k, \forall n \geq m, p \geq 1, x \in [a,b] $$

Hence $\sum U_n^{(x)}V_n^{(x)}$ Converges uniformly in $[a,b]$.

3.8 UNIFORM CONVERGENCE AND INTEGRATION

Theorem.5. If a sequence $< f_n >$ converges uniformly to f on $[a,b]$ and each f_n is inferable on $[a,b]$ and the sequence $< \int_a^x f_n \; at >$ converges uniformly to $\int_a^\pi f \, dt$ on $[a,b]$ i.e. $\int_a^x f \; dt = \lim_{n\to\infty} \int_a^x f_n dt,$ for all $x \in [a,b]$

Proof. **S**ince f_n converges to $f \Rightarrow$ for $\varepsilon > 0$ There exists a positive number m such that $\left| f_n(x) - f(x) \right| < \frac{\varepsilon}{3(b-a)}, n \geq m$ ____(1)

In particular $\left| f_m(x) - f(x) \right| < \frac{\varepsilon}{3(b-a)}$ _____(2)

Since f_m is integrable we choose a partition P of $[a,b]$ such that

$$U[P, f_m] - L[P, f_m] < \frac{\varepsilon}{3} \text{\_\_\_\_\_}(3)$$

From (2),we have

$$f(x) < f_m(x) + \frac{\varepsilon}{3(b-a)}$$

$$\Rightarrow U(P, f) < U(P, f_m) + \frac{\varepsilon}{3} \text{\_\_\_\_\_}(4)$$

Also from (2), we have

$$f(x) > f_m(x) - \frac{\varepsilon}{3(b-a)}$$

$$\Rightarrow L(P, f) > L(P, f_m) - \frac{\varepsilon}{3} \text{\_\_\_\_\_}(5)$$

From (3), (4) and (5), we have

$$U(P, f) - L(P, f) < U(P, f_m) - L(P, f_m) + \frac{2\varepsilon}{3}$$

$$< \frac{\varepsilon}{3} + \frac{2\varepsilon}{3}$$

$$< \varepsilon$$

This implies that the function f is integrable on $[a,b]$. Now since the sequence $< f_n >$ converges uniformly to f, for a given $\varepsilon > 0$ there exists an integer m such that for all $x \in [a,b]$.

$$\left| f_n(x) - f(x) \right| < \frac{\varepsilon}{b-a}, \text{ for all } n \geq m$$

Then for all $x \leftarrow [a,b]$ and for all $n \geq m$, we have

$$\left|\int_a^x f\,dt-\int_a^x f_n\,dt\right|=\left|\int_a^x (f-f_n)\,dt\right|$$

$$\leq \int_a^x |f-f_n|.|dt|$$

$$<\frac{\varepsilon}{b-a}.(x-a)$$

$$<\varepsilon,\ x\in[a,b]$$

This implies that $\int_a^x f_n\,dt$ converges uniformly to $\int_a^x f\,dt$ over $[a,b]$,

i.e. $\int_a^x f\,dt=\lim_{n\to\infty}\int_a^x f_n\,dt,$ for all $x\in[a,b]$.

Theorem.6. (Term by Term integration) If a series $\sum f_n$ converges uniformly to f on $[a,b]$ and f_n is continuous on $[a,b]$ then f is integrable on $[a,b]$ and the series

$\sum\left(\int_a^x f_n dt\right)$ **converges uniformly to** $\int_a^x f\,dt$ **for all values of in** $[a,b]$ **i.e.,**

$$\int_a^x f\,dt=\sum_{r=1}^{\infty}\int_a^x f_n\,dt,\ \textbf{for all}\ x\leftarrow[a,b]$$

Proof. Since the series $\sum f_n$ is uniformly convergent to f on $[a,b]$ and each f_n is continuous on $[a,b]$, the sum function f is continuous and therefore it is integrable on $[a,b]$.

Further since all the functions f_n are continuous, the sum function $\sum_{r=1}^{n} f_r(x)$ of finite number of functions is also continuous and integrable on $[a,b]$ and

$$\sum_{r=1}^{n}\int_a^x f_r\,dt=\int_a^x\sum_{r=1}^{n} f_r dt$$

Since the series $\sum f_n$ is uniformly convergent, for a given $\varepsilon>0,$ we can obtain a positive integer such that for all $x\leftarrow[a,b]$

$$\left|f-\sum_{r=1}^{n} f_r\right|<\frac{\varepsilon}{b-a},\ \text{for all}\ n\geq m$$

and $\left|\int_a^x f\,dt-\sum_{r=1}^{n}\int_a^x f_r dt\right|=\left|\int_a^x\left(f-\sum_{r=1}^{n} f_r\right)dt\right|$

$$<\frac{\varepsilon}{b-a}.\int_a^x dt<\varepsilon$$

This implies that $\sum_{n=1}^{\infty}\left(\int_a^x f_n dt\right)$ converges uniformly to $\int_a^x f\,dt$ on $[a,b]$

i.e.

$$\int_a^x f\,dt = \sum_{n=1}^{\infty}\int_a^x f_n\,dt, \text{ for all } x \in [a,b].$$

3.9 UNIFORM CONVERGENCE AND DIFFERENTIATION

Theorem.7. Let $< f_n >$ be a sequence of real values functions defined on $[a,b]$ such that

(i) f_n is differentiable on $[a,b]$ for n = 1,2,3,…

(ii) $< f_n >$ converges at least one point $x_0 \in [a,b]$

(iii) The sequence $< f_n' >$ converges uniformly on $[a,b]$ Then the given sequence $< f_n >$ converges uniformly to a differentiable limit f and

$\lim_{n\to\infty} f_n'(x) = f'(x)$, **for all** $x \in [a,b]$,

Proof. Let $\varepsilon > 0$ then by the convergence of $< f_n(x_0) >$ and by the uniform convergence of $< f_n' >$ on $[a,b]$ there exists a positive integer m such that for all $x \leftarrow [a,b]$, we have $\left|f_{n+p}(x_0) - f_n(x_0)\right| < \frac{\varepsilon}{2}$, for all $n \geq m, p \geq 1$ ____(1)

And $\left|f_{n+p}'(x) - f_n'(x)\right| < \frac{\varepsilon}{2(b-a)}$, for all $n \geq m, p \geq 1$ ______(2)

Applying Lagrange's mean value theorem to the function $(f_{n+p} - f_n)$ for any two points x and y of $[a,b]$ and for some ε between x and y for all $n \geq m, p \geq 1$, we have

$$\left|f_{n+p}(x) - f_n(x) - f_{n+p}(y) + f_n(y)\right| . \left|f_{n+p}'(\varepsilon) - f_n'(\varepsilon)\right|$$

$$< \frac{|n-y|}{2(b-a)}.\varepsilon$$

$$< \frac{\varepsilon}{2} \quad \left[\because |x-y| \leq (b-a)\right]$$

Now $\left|f_{n+p}(x) - f_n(x)\right| = \left|f_{n+p}(x) - f_n(x) - f_{n+p}(x_0) + f_n(x_0) + f_{n+p}(x_0) - f_n(x_0)\right|$

$$\leq \left|f_{n+p}(x) - f_n(x) - f_{n+p}(x_0) + f_n(x_0)\right| + \left|f_{n+p}(x_0) - f_n(x_0)\right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$< \varepsilon \qquad \text{\_\_\_\_\_(3)}$$

This implies that the sequence $< f_n >$ converges uniformly on $[a,b]$. Let it converges to f say For a fixed

x on $[a,b]$ and $y \in [a,b], \quad y \neq x.$ we define

$$F_n(y) = \frac{f_n(y) - f_n(x)}{y-x}, n = 1, 2, 3 - - - - (4)$$

And $$F(y) = \frac{f(y) - f(x)}{y-x} - - - - - - (5)$$

Since each f_n is differentiable for each n

$$\lim_{y \to x} F_n(y) = f_n'(x) \_\_\_\_\_\_\_\_ (6)$$

Therefore,

$$\left|F_{n+p}(y) - F_n(y)\right| = \frac{1}{|y-x|}\left| f_{n+p}(y) - f_{n+p}(x) - f_n(y) + f_n(x)\right|$$

$$= \frac{1}{|y-x|}\left|\left\{ f_{n+p}(y) - f_n(y)\right\} - \left\{ f_{n+p}(x) - f_n(x)\right\}\right|$$

$$< \frac{\varepsilon}{2(b-a)}, \forall n \geq m, p \geq 1 \qquad \text{(Using (3))}$$

This shows that $< F_n(y) >$ converges Uniformly to $F(y)$ on $[a,b]$ for $y \in [a,b], y \neq x.$

Since the sequence $< f_n >$ converges to f, from (4), we have

$$\lim_{n \to \infty} F_n(y) = \lim_{n \to \infty} \frac{f_n(y) - f_n(x)}{y-x} = \frac{f(y) - f(x)}{y-x} = F(y).$$

Thus the sequence $< f_n(y) >$ converges uniformly to $F(y)$ on $[a,b]$ for $y \in [a,b], \quad y \neq x.$

Using (6) we have

$$\lim_{y \to x} F(y) = \lim_{n \to \infty} f_n'(x) = a(x)$$

This implies that $\lim_{y \to x} F(y)$ exists. Therefore (5) implies that f is

differentiable and $\lim_{y \to x} F(y) = f'(x)$

Hence $f'(x) = G(x) = \lim_{n \to \infty} f_n'(x)$

Theorem.8. (Term by term differentiation): Let a series $\sum f_n$ of differentiable function on $[a,b]$ such that it converges to f on $[a,b]$ and each f_n^1 is continuous on $[a,b]$ and the series $\sum f_n^1$ converges uniformly to G on $[a,b]$ then the given series $\sum f_n$ converges uniformly to f on $[a,b]$ and $f'(x) = G(x)$ **or** $\frac{d}{dx}\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{d}{dx} f_n(x)$, **for all** $x \in [a,b]$.

Proof: Since the series $\sum f_n^1(x)$ of continuous functions converges uniformly to $G(x)$ on $[a,b]$, its sum function G is continuous on $[a,b]$ and hence the function $\int_a^x G(t)\,dt$ is differentiable and $\frac{d}{dx}\int_a^x G(t)dt = G(x)$, for all $x \leftarrow [a,b]$ ____ (1)

For every $x \leftarrow [a,b]$, we have

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

Now each function f_n^1 is continuous, so it is integrable on $[a,b]$ and therefore by the fundamental theorem of calculus, we have

$$\int_a^x f_n^1(t)\,dt = f_n(x) - f_n(a), \text{for all } n \geq 1,\ x \in [a,b].$$

Therefore

$$\sum_{n=1}^{\infty}\int_a^x f_n^1(t)\,dt = f(x) - f(a), \forall x \in [a,b] \qquad ....(2)$$

Further since the series $\sum f_n^1$ of integrable functions converges uniformly to G on $[a,b]$, therefore term by term integrable is valid i.e.

$$\int_a^x G(t)\,dt = \sum_{r=1}^{\infty}\int_a^x f_n^1(t)\,dt, \forall x \in [a,b] \qquad ...(3)$$

Proof (1), (2) & (3) we have $f'(x) = G(x), \forall x \in [a,b]$

i.e. $\frac{d}{dn}\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty}\frac{d}{dn} f_n(x), \forall x \in [a,b]$.

i.e. term by term differentiation is valid.

Examples

Example.9: Show that the series for which

$f_n(x) = \frac{1}{1+nx}$ can be integrated term by term in $[0,1]$ although it is not uniformly convergent.

Solution. Given $f_n(x) = \frac{1}{1+nx}$

Let $f(x) = \lim_{n\to\infty} f_n(x)$

then $f(x) = \lim_{n\to\infty}\frac{1}{1+nx} = 0,$ for $x \in [0,1]$.

And $\lim_{n\to\infty}\int_0^1 f_n(x)\,d_x = \lim_{n\to\infty}\int_0^1 \frac{nx}{r+nx}\,dx$

$$= \lim_{n\to\infty} \frac{1}{n} \log(1+n) = \underset{n\to\infty}{\propto} \frac{\frac{1}{1+b}}{1} = 0$$

Thus $\int_0^1 f(x)dx = \lim_{n\to\infty} \int_0^1 f_n(x)\,dx$

Hence the series can be integrabled term by term. It can be series easly that 0 is a point of non-uniform convergence of the series.

Example.10: Examine for term by term integration the series for which $f_n(x) = nx_e^{-nx^2}$ for $x \in [0,1]$.

Solution. Let $f(x) = \lim_{n\to\infty} f_n(x)$,

given $f_n(x) = nx_e^{-nx^2}$ $= \lim_{n\to\infty} nxe^{-nx^2}$ $= \lim_{n\to\infty} \frac{nx}{e^{nx^2}}$

$$= \lim_{n\to\infty} \frac{nx}{1 + nx^2 + \frac{n^2x^4}{\lfloor 2}} + \ldots\ldots\ldots\ldots \qquad 0 < x < 1$$

$$= \lim_{n\to\infty} \frac{1}{\frac{1}{nx} + x + \frac{nx^3}{\lfloor 2}} + \ldots\ldots\ldots\ldots$$

$f(x) = 0$, for finite values of x

$\therefore \int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0$.

And $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \lim_{n\to\infty} \int_0^1 nx_e^{-nx^2} dx$

$$= \lim_{n\to\infty} \left[-\frac{1}{2} e^{-nx^2} \right]_0^1$$

$$= \lim_{n\to\infty} \frac{1}{2} \left[1 - e^{-n} \right] = \frac{1}{2}$$

Thus $\int_0^1 f(x)a_x \neq \lim_{n\to\infty} \int_0^1 f_n(x)\,dx$

Hence term by term integration over $[0,1]$ is not verified. Also convergence

can not be uniform on $[0,1]$ But if we consider the interval $[c,1], 0 < c < 1$ then

$\underset{n\to\infty}{\propto} \int_0^1 f_n(x) = \underset{n\to\infty}{\propto} \int_c^1 nx_e^{-nx^2} dx$

$$= \lim_{n\to\infty} \left[e^{-nc^2} - e^{-n} \right] = 0$$

Thus $\int_0^1 f(x)dx = \lim_{n\to\infty}\int_0^1 f_n(x)\,dx$

Hence term by term integration is justified.

Example.11: Show that the series for which $f_n(x) = \dfrac{nx}{1+n^2x^2}$, $x \in [0,1]$ can not be differentiated term by term at $x = 0$

Solution: Given $f_n(x) = \dfrac{nx}{1+n^2x^2}$

$$\therefore f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{\left(\frac{1}{n}+n^2\right)} = 0 \text{ for } x \in [0,1]$$

$$\therefore f'(x) = 0$$

Also $$f_0^1(0) = \lim_{h\to 0} \frac{f_n(0+h) - f_n(0)}{h} = \lim_{h\to 0} \frac{\frac{n\cancel{h}}{1+n^2h^2} - 0}{\cancel{h}}$$

$$= \lim_{h\to 0} \frac{n}{1+n^2h^2} = \lim_{h\to 0} \frac{n}{1+n^2h^2} = n$$

$$\therefore \lim_{n\to\infty} f_n^1(0) = \lim_{h\to\infty} n = \infty .$$

Thus $f'(0) \neq \lim_{n\to\infty} f_n^1(0)$

Therefore, the given series can not be differentiated term by term by at $x = 0$.

Example.12: Show that the sequence $< f_n >$ where

$f_n(x) = \dfrac{x}{1+nx^2}$ **converges uniformly**

To a function f on $[0,1]$ and the equation $f'(x) = \lim_{n\to\infty} f_n^1(x)$ is true if $x \neq 0$ and false if $x = 0$.

Solution. It can be show easily that the sequence $< f_n >$ converges uniformly to zero for all real x

Now $f(x) = \lim_{n\to\infty} \dfrac{x}{1+nx^2} \qquad = 0$

Where $x \neq 0 \qquad f_n^1(x) = \dfrac{1-nx^2}{\left(1+nx^2\right)^2}$

$$\therefore \lim_{n\to\infty} f_n^1(x) = \lim_{n\to\infty} \frac{1-nx^2}{\left(1-nx^2\right)^2} = 0$$

Thus $\quad f'(x) = 0 = \lim f_n^1(x)$

Hence equation is true

When $x=0$ $f_n^1(0)=1$

Thus $f'(0)\neq\lim\limits_{n\to\infty} f_n^1(0)$

Hence at $x=0$ the equation

$f'(x)=\lim\limits_{n\to\infty} f_n^1(x)$ is not true

Example.13: Show that the function $\sum\limits_{n=0}^{\infty}\frac{\sin nx}{n^3}$ is differentiable for every x and its derivative is $\sum\limits_{n=0}^{\infty}\frac{\cos nx}{n^2}$

Solution: Let $f(x)=\sum\limits_{n=0}^{\infty}\frac{\sin nx}{n^3}$

And $u_n(x)=\frac{\sin nx}{n^3}$

Then $u_n^1(x)=\frac{\cos nx}{n^2}$

$$\therefore \sum_{n=0}^{\infty}U_n^1(x)=\sum_{n=0}^{\infty}\frac{\cos nx}{n^2}$$

Now $\left|\frac{\cos nx}{n^2}\right|\leq\frac{1}{n^2}$ for all x

and $\sum\frac{1}{n^2}$ is converges.

Hence by weirshass's M test the series $\sum U_n^{'}(x)$ is uniformly convergent for all real values of x and therefore the series $\sum U_n(x)$ can be differentiated term by term.

Hence $f'(x)=\sum\limits_{n=0}^{\infty}U_n^1(x)=\sum\limits_{n=0}^{\infty}\frac{\cos nx}{n^2}$

Example.14. Show that the function

$$f_n(x)=\begin{cases} n^2x, & 0\leq x\leq\frac{1}{n} \\ -n^2x+2x, & \frac{1}{n}\leq x\leq\frac{2}{n} \\ 0, & \frac{2}{n}\leq x\leq 1 \end{cases}$$

is not uniformly convergent on $[0,1]$

Solution. The given sequence converges to f where $f(x)=0, \forall x \in [0,1]$.

Also $\int_0^1 f_n d_x = \int_0^{\frac{1}{x}} n^e x\,dx + \int_{1/x}^{2/x} \left(-n^e x + 2n\right) dx + \int_{2/x}^{1} 0.dx$

And $\int_0^1 f(x)\,dx \neq \lim_{n\to\infty} \int_0^1 f_n\ dx$

Hence the sequence $< f_n >$ cannot converge uniformly on $[0,1]$.

3.10 SUMMARY

A sequence of functions $f_1(x), f_2(x), f_3(x),\ldots\ldots, f_n(x)$ defined on a domain D is said to be uniformly bounded if there exists positive number m such that

$|f_n(x)| < M$ for all n and for all $x \in D$.

A sequence $\{f_n\}$ of functions defined on $[a,b]$ is said to converge uniformly to a function f if for any $\in > 0$ there exists an integer m such that

$|f_n(x) - f(x)| < \in$, for all $n \geq m$ and for all $x \in [a,b]$.

Here m does not depend upon x.

A series of functions $\sum f_n$ converge uniformly on $[a,b]$ if the sequence $< S_n >$ of its partial sums, defined by

$S_n(x) = \sum_{i=1}^{n} f_i(x)$. Converges uniformly on $[a,b]$.

Thus a series of functions $\sum f_n$ converges uniformly to f on $[a,b]$ if for any $\in > 0$ and for all $x \in [a,b]$, there exists a positive integer m such that

$|S_n - f| < \in,$ for all $n \geq m$.

3.11 TERMINAL QUESTIONS

Q.1 Explain the concept of uniform convergence.

Q.2 Examine for term by term integration the series for when $f_n(x) n^2 x[1-x]^n, x \leftarrow [0,1]$

Q.3 Examine for the continuity of the sum function and for term by term integration the series where

x th term is $n^2 x e^{-n^2 x^2} - (n-1)^2_x e^{-(n-1)^2 x^2}$ for all $x \leftarrow [0,1]$.

Q.4 Show $f_n(x) = \dfrac{\log(1+n^2x^2)}{n^2}$ is uniformly convergent on $[0,1]$.

Q.5 Examine for term by term integration the series $\sum x^{n-1}(1-2x^n)$ in $[0,1]$.

Q.6 Let for the series $\sum f_n(x)$, $f_n(x) = \dfrac{1}{2n^2}\log(1+n^4x^2)$. Show that the series $\sum U_n^1(x)$ does not converge uniformly but the given series can be differentiated term by term.

UNIT 4 POWER SERIES

Structure

4.1 Introduction

4.2 Objectives

4.3 Power series

4.4 Cauchy's Theorems on limits

4.5 Radius of Convergence

4.6 Uniform Convergence of Power Series

4.7 Abel's Theorem

4.8 Tauber's Theorem

4.9 Summary

4.10 Terminal Questions

4.1 INTRODUCTION

Power series are commonly used in mathematics, especially in calculus and analysis, for functions that can be expressed as a series expansion around a certain point. They're also used in various other areas, like physics and engineering, for their ability to approximate functions and solve differential equations. In real analysis, power series have several applications, particularly in the study of functions and their properties. Power series can be used to approximate a wide range of functions. By using the Taylor series expansion of a function, one can approximate the function locally around a point. Power series can help analyze the convergence and continuity of functions. Understanding the behavior of a function at its power series expansion point can provide insights into its convergence and continuity properties.

Power series can be differentiated and integrated, term by term, within their interval of convergence. This property is useful for finding derivatives and integrals of functions represented by power series. Power series can be used to solve differential equations. By substituting a power series into a differential equation, one can solve for the coefficients of the series and find a solution to the differential equation. Power series are crucial in the study of analytic functions. A function is said to be analytic at a point if it can be locally represented by a convergent power series. Analytic functions have many nice properties, and power series provide a way to understand and analyze them.

4.2 OBJECTIVES

After reading this unit the learner should be able to understand about the:

- Power series and Cauchy's theorems on limits
- Radius of convergence and uniform convergence of power series
- Abel's theorem and Tabuler's theorem

4.3 POWER SERIES

A series of the type

$\sum_{n=0}^{\infty} a_n (z-z_o)^n = a_o + a_1(z-z_o) + ...+ a_n(z-z_o)^n + ...$ is called a power series.

Theorem: The power series $\sum a_n z^n$ either

(i) Converges for all values of z,

or (ii) Converges only for $z=0$,

or (iii) Converges for z in some region in the complex plane.

Proof. Here we take an example.

(i) The series $\sum \frac{z^n}{n}$ converges absolutely for all values of z.

Let $U_n = \frac{z^n}{n}$. $U_{n+1} = \frac{z^{n+1}}{n+1}$

Then by D'Alembert ratio test, we have

$$\lim_{n\to\infty} \frac{|U_n|}{|U_{n+1}|} = \lim_{n\to\infty} \frac{n+1}{|z|} = \infty$$

The series absolutely convergent for all values of z.

(ii) The series $\sum nz^n$ converges only at $z=0$

$\lim_{n\to\infty} nz^n = \infty$ if $z \neq 0$.

Thus the series does not converge for $z \neq 0$ i.e, it converges only for $z=0$

(iii) The geometric series $\sum_{n=0}^{\infty} z^n$ converges for $|z|<1$ and divergent for $|z| \geq 1$.

Theorem.2: If a power series $\sum_{n=0}^{\infty} a_n z^n$. converges for a particular value Zo of Z then it is converges absolutely for all values of z for which $|z| < |z_o|$.

Proof. Let $\sum a_n z_o^n$ converge.

Then the n^{th} term $a_n z_0^n$ must tend to o as $n \to \infty$ so we can find a number $M > 0$ such that

$$|a_n z_0^n| \le M \;, \qquad \forall n$$

Then we have

$$|a_n z^n| \le M \left| \frac{Z}{Z_o} \right|^n \qquad \ldots(1)$$

Since $|z| < |z_o|$ the geometric series $\sum \left| \frac{Z}{Z_o} \right|^n$ converges.

Then by equation (1) the series $\sum |a_n z^n|$ converges for all values of z for which $|z| < |z_o|$.

4.4 CAUCHY'S THEOREMS ON LIMITS

Theorem.3: If $\{a_n\}$ be a sequence of constant and if $\lim \; an = l$ when $a \to \infty$ then we have

$$\lim_{n \to \infty} \frac{a_1 + a_2 + \ldots\ldots + a_n}{n} = l \,.$$

Theorem.4. If $\{a_n\}$ is a sequence of positive constant then

$$\lim_{n \to \infty} \left(a_n^{1/n} \right) = \lim_{n \to \infty} \left(\frac{a_{n+1}}{a_n} \right) \qquad \ldots(1)$$

Provided the limit on the right hand side of equation (1) exists whethere finite or infinite.

4.5 RADIUS OF CONVERGENCE

The number R is known as the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ if there exist $R > 0$ such that the series converges absolutely when $|z| < R$ and diverges when $|z| > R$.

The circle $|z| = R$ is known as the circle of convergence of the series.

Theorem.5. If the power series $\sum a_n z^n$ is such that $a_n \ne 0$ for all n and for which the number $l = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$

exists finitely or infinitely then the radius of convergence of the series is $R = \frac{1}{l}$ (where $R = 0$ if $l = \infty$ and $R = \infty$ if $l = 0$).

Proof. Using D; alembert's ratio test, we have

$$\sum a_n z^n \text{ is converges absolutely if } \lim\left|\frac{a_{n+1}z^{n+1}}{a_n z^n}\right| < 1$$

i.e., if $|z| < \frac{1}{l}$ $\left\{\because \lim\left|\frac{a_{n+1}}{a_n}\right| = l\right\}$

and if $|z| > \frac{1}{l}$ then $\lim\left|\frac{a_{n+1}z^{n+1}}{a_n z^n}\right| > 1$

So that the n^{th} term $a_n z^n$ does not tend to zero and convergence is impossible. It follows by definition of radius of convergence of the power series is $R = \frac{1}{l}$ if $l = \infty$, $R = 0$ and if $l = 0, R = \infty$.

4.6 UNIFORM CONVERGENCE OF POWER SERIES

Theorem.6. The power series $\sum a_n z^n$ is uniformly convergent for $|z| \le \rho < R$ where R is the radius of convergence.

Proof. Let $\rho < \rho' < R$

Since the series is convergent for $|z| = \rho'$ there is a number k. independent of n so that $|a_n \rho'^n| < k \quad , \forall n$

Hence we have for

$$|z| \le \rho$$

$$|a_n z^n| = \left|a_n \rho'^n \left(\frac{z}{\rho'}\right)^n\right| < k\left(\frac{p}{\rho'}\right)^n$$

Which is independent of z

But the series $k\sum\left(\frac{p}{p'}\right)^n$ is convergent, being a geometric series of common radio $\frac{\rho}{\rho'} < 1$. Hence by weiertxss's M test, the power series is uniformly convergent for $|z| \le \rho < R$. Thus every power series is uniformly convergent within its circle of convergence.

Note. Absore theorem statement can be defined in the following form:

If $f(x)=\sum_{n=0}^{\infty} a_n z^n$ for $|x|<R$ then the series converges uniformly on $[-R+\in, R-\in]$ for each $\in>0$.

4.7 ABEL'S THEOREM

Theorem.7. If the series $\sum_{0}^{\infty} a_n$ is convergent and has the sum s. then the series $\sum_{0}^{\infty} a_n x^n$ is uniformly convergent for $0 \le x \le 1$ and $\lim_{x\to 1} \sum_{0}^{\infty} a_n x^n = s.$

Proof: If is given series $\sum an$ is convergent. We have for $n \ge m$.

$$\left|a_{n+1}+a_{n+2}+........+a_{n+p}\right|<\in$$

For every integral value of $p>0$

Also since the sequence $\{x_n\}$ is monotonic decreasing for all values of x in $0 \le x \le 1$, by abel's inequality $\left|a_n x^n + a_{n+1}x^{n+1}+.........+a_{n+p}x^{n+p}\right| \le \in x^n \le \in (0 \le x \le 1)$.

Thus the series $\sum a_n x^n$ is uniformly convergent for $0 \le x \le 1$.

It follows that $\sum_{0}^{\infty} a_n x^n$ is continuous function of x in $0 \le x \le 1$ and hence

$$\lim_{x\to 1} \sum_{0}^{\infty} a_n x^n = \sum_{0}^{\infty} \lim_{n\to 0} a_n (1-n)^n$$

$$= \sum_{0}^{\infty} a_n$$

$$= s$$

The converge of the above theorem is not true as will be seen by considering the series $\sum_{0}^{\infty} (-1)^n x^n = \frac{1}{1+x}$ when $0 \le x < 1$.

Hence $\lim_{x\to 1} \sum_{0}^{\infty} (-1)^n x^n = \frac{1}{2}$ although $\sum a_n = \sum (-1)^n$ does not converge.

Examples

Example.1. $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} z^n$ determine the radii of convergences of following power series.

Sol. (i) $\frac{1}{R} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ where R is the radius of convergence

Here $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2n+2)!}{((n+1)!)^2} \frac{(n!)^2}{(2n)!}$

$$= \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(n+1)^2}$$

$$= 4$$

Hence $R = \frac{1}{4}$.

Example.2. Solve $\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n$.

Sol. Here $a_n = \frac{n!}{n^n}$. $\quad a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$

Now we have

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$= \frac{1}{e}$$

Hence R = e. $\quad \{2 < e < 3\}$.

4.8 TAUBER'S THEOREM

Theorem.8. If $na_n \to 0$ and $f(x) = \sum a_n x^n \to s$ as $x \to 1$. Then $\sum a_n$ converges to the sum S.

or

If $\lim_{n\to\infty} na_n = 0$ then $\sum a_n$ converges to S.

Proof: We have by cauchy's Ist theorem on limits

$$\lim_{n\to\infty} \frac{|a_1| + 2|a_2| + ... + n|a_n|}{n} = \lim_{n\to\infty} na_n = 0.$$

For given $\in$ we find m such that for $n \geq m$

$$\left| f\left(1-\frac{1}{n}\right) - S \right| < \frac{\in}{3} \qquad ...(1)$$

$$|na_n| < \frac{1}{3} \in$$

i.e., $\quad |a_n| < \frac{\in}{3^n} \qquad ....(2)$

And $\dfrac{|a_1|+2|a_2|+......+n|a_n|}{n}<\dfrac{\in}{3}$ ...(3)

Let $S_n = a_0 + a_1 + ...+ a_n$. then

$$|S_n - S| = |f(x) - S + S_n - f(x)|$$

$$= \left| f(x) - S + a_0 + \sum_1^n a_r - a_0 - \sum_1^\infty a_r x^r \right|$$

$$= \left| f(x) - S + \sum_1^n a_r - \sum_1^\infty a_r x^r \right|$$

$$= \left| f(x) - S + \sum_1^n a_r - \sum_1^n a_r x^r - \sum_{n+1}^\infty a_r x^r \right|$$

$$= \left| f(x) - S + \sum_1^n a_r (1 - x^r) - \sum_{n+1}^\infty a_r x^r \right|$$

$$\le |f(x) - S| + \left| \sum_1^n a_r (1 - x^r) \right| + \left| \sum_{n+1}^\infty a_r x^r \right|.$$

Since $(1-x^r) = (1-x)(1+x+x^2+x^3+.........+x^{r-1})$

$$< (1-x) r \text{ as } (0<x<1)$$

$$< \frac{\in}{3n(1-x)}$$

Now by from equation (4), we have

$$|S_n - S| \le |f(x) - S| + (1-x) \sum_1^n r|a_r| + \frac{\in}{3n(1-x)}$$

Putting $x = 1 - \dfrac{1}{n}$ in this, we get

$$|S_n - S| < \left| f\left(1 - \frac{1}{n}\right) - S \right| + \frac{1}{n} \sum_1^n r|a_r| + \frac{\in}{3}$$

$$< \frac{1}{3}\in + \frac{1}{3}\in + \frac{1}{3}\in \quad \text{(by equations (1) \& (2))}$$

< 3

Hence $S_n \to S$.

Examples

Example.3. Given that

(i) $\sum_{n=0}^{\infty} n\, z^n$ (ii) $\sum_{n=1}^{\infty} (n!)\, z^n$ (iii) $\sum \left(\frac{nz}{n+1}\right)^2$

(iv) $\sum \frac{(n!)^2}{(2n)!} z^{2n}$ (v) $\sum \frac{z^{2n}+1}{2n+1}$ (vi) $\sum \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

Find the radii of convergence of the following power series

Sol.

(i) We have $\frac{1}{R} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ where R is the radius of convergence

We have $a_n = n$ and $a_{n+1} = n+1$

$$\therefore \quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{n}$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)$$

Thus $\frac{1}{R} = 1 \quad \Rightarrow R = 1.$

(ii) Here $a_n = n!, \quad a_{n+1} = (n+1)!$

$$\therefore \quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{n}$$

$$= \lim_{n\to\infty} (n+1)$$

$$= \lim_{n\to\infty} n\left(1 + \frac{1}{n}\right)$$

$$= \infty$$

Thus $\frac{1}{R} = \infty \quad \Rightarrow \quad R = 0.$

(iii) Here $a_n = \frac{n^2}{(n+1)^2}$ $, a_{n+1} = \frac{(n+1)^2}{(n+2)^2}$

$$\therefore \quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2}{(n+2)^2} \times \frac{(n+1)^2}{n^2}$$

$$= 1$$

Thus $\frac{1}{R} = 1 \quad \Rightarrow R = 1$

(iv) Given that $\sum \frac{(n!)^2}{(2n)!} z^{2n}$

Assuming that $2n = m$.

Then

$$a_m = \frac{\left(\left(\frac{m}{2}\right)!\right)^2}{m!}, \quad a_{m+1} = \frac{\left(\left(\frac{m}{2}+1\right)!\right)^2}{(m+1)!}$$

$$\therefore \quad \lim_{m\to\infty} \frac{a_{m+1}}{a_m} = \lim_{m\to\infty} \frac{\left(\left(\frac{m}{2}+1\right)!\right)^2 (m!)}{(m+1)!\left(\left(\frac{m}{2}\right)!\right)^2}$$

$$= \lim_{m\to\infty} \frac{\left(\frac{m}{2}+1\right)^2}{m+1}$$

$$= \infty$$

Thus $\left(\frac{1}{R}\right) = \infty \Rightarrow R = 0.$

(iv) Here $a_m = \frac{1}{2n+1} = \frac{1}{m}$, $a_{m+1} = \frac{1}{m+1}$

Given $\sum \frac{z^{2n+1}}{2n+1}$.

Let $m = 2n+1$ then $\sum \frac{z^m}{m}$

Now we have

$$\therefore \lim_{m\to\infty} \frac{a_{m+1}}{a_m} = \lim_{m\to\infty} \frac{m}{m+1}$$

$$= \lim_{m\to\infty} \left(1 + \frac{1}{m}\right)$$

$$= 1$$

Thus $\frac{1}{R} = 1 \quad \Rightarrow \quad R = 1.$

(vi) Given $\sum \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

Let $m = 2n+1 \quad \Rightarrow \quad n = \dfrac{m-1}{2}$

Then $\quad \sum \dfrac{(-1)^{\frac{m-1}{2}} x^m}{m!}$

Here $a_m = \dfrac{(-1)^{\frac{m-1}{2}}}{m!}, \quad a_{m+1} = \dfrac{(-1)^{\frac{m+1}{2}}}{(m+1)!}$

$$\therefore \lim \frac{a_{m+1}}{a_m} = \lim_{m\to\infty} \frac{(-1)^{(m+1)/2}}{(m+1)!} \times \frac{m!}{(-1)^{(m-1)/2}}$$

$$= \lim_{m\to\infty} \frac{(-1)}{m\left(1+\frac{1}{m}\right)}$$

$$= 0$$

Thus $\quad \dfrac{1}{R} = 0 \Rightarrow R = \infty$.

(vii) $\sum \dfrac{(n!)^2}{(2n)!} z^{2n}$

Sol. Here $a_n = \dfrac{(n!)^2}{(2n)!}, \; a_{n+1} = \dfrac{((n+1)!)^2}{(2n+2)!}$

Now we have

$$\left(\frac{1}{R}\right)^2 = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{((n+1)!)^2 \times (2n)!}{(2n+2)! \times (n!)^2}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)}$$

$$= \lim_{n\to\infty} \frac{n^2\left(1+\frac{1}{n}\right)^2}{4n^2\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right)}$$

Thus $\quad \left(\dfrac{1}{R}\right)^2 = \dfrac{1}{4}$

$\Rightarrow \quad \dfrac{1}{R} = \dfrac{1}{2} \quad \Rightarrow \quad R = 2.$

4.9 SUMMARY

A series of the type

$$\sum_{n=0}^{\infty} a_n (z - z_o)^n = a_o + a_1 (z - z_o) + ... + a_n (z - z_o)^n + ...$$ is called a power series.

The power series $\sum a_n z^n$ either

(i) Converges for all values of z,

or (ii) Converges only for $z = 0$,

or (iii) Converges for z in some region in the complex plane.

If $\{a_n\}$ be a sequence of constant and if $\lim \; an = l$ when $a \to \infty$ then we have

$$\lim_{n\to\infty} \frac{a_1 + a_2 + ... + a_n}{n} = l \;.$$

The number R is known as the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ if there exist $R > 0$ such that the series converges absolutely when $|z| < R$ and diverges when $|z| > R.$

The circle $|z| = R$ is known as the circle of convergence of the series.

If the series $\sum_0^{\infty} a_n$ is convergent and has the sum s. then the series $\sum_0^{\infty} a_n x^n$ is uniformly convergent for $0 \le x \le 1$ and $\lim_{x\to 1} \sum_0^{\infty} a_n x^n = s.$

If $na_n \to 0$ and $f(x) = \sum a_n x^n \to s$ as $x \to 1.$ Then $\sum a_n$ converges to the sum S.

4.10 TERMINAL QUESTIONS

Q.1 Write a short note on power series.

Q.2 What do you mean by radius of convergence.

Q.3 State and prove Abel's theorem.

Q.4 State and prove Tabuler's theorem.

Q.5 If $\{a_n\}$ be a sequence of constant and if $\lim \; an = l$ when $a \to \infty$ then we have

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \dots\dots\dots + a_n}{n} = l\,.$$

Q.6 If $\{a_n\}$ is a sequence of positive constant then

$$\lim_{n\to\infty}\left(a_n^{1/n}\right) = \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)$$

provided the limit on the right hand side of the above equation exists whethere finite or infinite.

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PGMM-101N/
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Advanced Real Analysis
And Integral Equations

BLOCK

2

FUNCTION OF SEVERAL VARIABLES

UNIT-5

Limit and Continuity of Function of Two Variables

UNIT-6

Partial Differentiation

UNIT-7

Euler's Theorem

UNIT-8

Jacobians

BLOCK INTRODUCTION

A function of several variables is a mathematical rule that assigns a unique output to every combination of input values. While functions of one variable take one input and produce one output, functions of several variables take multiple inputs and produce one output. In machine learning and data analysis, functions of several variables are used to model complex relationships in data and make predictions. Many concepts in physics and engineering involve functions of multiple variables. For instance, in fluid dynamics, electromagnetism, or structural analysis, understanding the behavior of functions of two variables is crucial for modeling physical systems accurately. Many real-world phenomena depend on more than one variable. Functions of two variables allow us to model and analyze such complex systems. Limits and continuity provide essential tools for understanding the behavior of these functions, especially as variables approach specific points. In optimization problems, such as maximizing profit or minimizing cost, functions of two variables are often involved. To find optimal solutions, we need to understand the behavior of these functions at critical points, which relies heavily on concepts of limits and continuity.

Partial differentiation is a fundamental concept in multivariable calculus that deals with finding the derivatives of functions of several variables with respect to one of those variables, keeping the others constant. The Jacobian is a concept from multivariable calculus that generalizes the idea of the derivative of a function of several variables. It provides a way to analyze how small changes in the input variables of a multivariable function affect its output values. Economic models often involve functions of several variables representing factors like supply, demand, price, and market conditions.

In the fifth unit, we shall have discussed about limit and continuity of function of two variables and Partial differentiation is in the sixrth unit. Euler’s theorem and Jacobians are discussed in details in unit seventh and eighth respectively.

UNIT 5 LIMIT AND CONTINUITY OF FUNCTION OF TWO VARIABLES

Structure

5.1 Introduction

5.2 Objectives

5.3 Limit of a function of two variables

5.4 Continuity of a function of two variables

5.5 Summary

5.6 Terminal Questions

5.1 INTRODUCTION

Many scientific and engineering problems involve a variable quantity that relies on multiple independent variables to determine its value. This necessitates a solid understanding of partial differentiation methods. Partial derivatives are significant in various fields such as science, management, and engineering. They are essential in optimization techniques, operations research, electricity, computer science, fluid dynamics, probability, statistics, economics, mechanical engineering, electronics, and more. When working with functions of two or more independent variables, for instance, the area of a rectangle dependent on its length and breadth or the volume of a rectangular parallelepiped determined by its length, breadth, and depth, partial derivatives come into play. The area of a rectangle represents a function of two variables, while the volume of a rectangular parallelepiped represents a function of three variables.

Limits and continuity are fundamental concepts in advanced mathematics, serving as building blocks for more complex topics such as differentiation, integration, and topological properties of functions. Therefore, a strong comprehension of partial differentiation is crucial for effectively addressing numerous engineering problems. The importance of understanding limits and continuity of functions of two variables lies in their foundational role in multivariable calculus and their practical applications in various fields.

5.2 OBJECTIVES

After reading this unit the learner should be able to understand about the

- Limit of a function of two variables
- Continuity of a function of two variables

5.3 LIMIT OF A FUNCTION OF TWO VARIABLES

Consider a function $f(x, y)$ of two variables x and y, then $\lim\limits_{\substack{x\to x_0 \\ y\to y_0}} f(x, y)$ exists and equal to l, if for every $\varepsilon > 0,\ \exists\ \delta > 0$ such that $|\ f(x, y) - l\ | < \varepsilon,\quad \forall (x, y)$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is a function and for $|\ x - x_0\ | < \delta$ and $|\ y - y_0\ | < \delta.$

5.4 CONTINUITY OF A FUNCTION OF TWP VARIABLES

Consider a function $f(x, y)$ of two variables x and y then f is said to be continuous at the point (x_0, y_0) if $\lim\limits_{\substack{x\to x_0 \\ y\to y_0}} f(x, y)$ exists and equal to $f(x_0, y_0)$.

or

Consider a function $f(x, y)$ of two variables x and y is said to be continuous at the point (x_0, y_0), if for every $\varepsilon > 0,\ \exists\ \delta > 0$ such that $|\ f(x, y) - f(x_0, y_0)| < \varepsilon,\quad \forall (x, y)$

whenever $|\ x - x_0\ | < \delta$ and $|\ y - y_0\ | < \delta.$

Solved Examples

Example.1. Show that the limit $\lim\limits_{(x,y)\to(0,0)} f(x, y)$, where $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ does not exists.

Solution. The given function is

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

Consider $(x, y) \longrightarrow (0,0)$ along the path $y = mx,$ where $m \in R.$

As $x \to 0$, from $y = mx,$ we have $y \to 0$.

We have

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

Putting $y = mx,$ then we have

$$\lim_{x\to 0} f(x, mx) = \frac{1 - m^2}{1 + m^2},$$ which depend upon m.

Thereforethe $\lim\limits_{x\to 0} f(x, mx)$ is not unique.

Hence the $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exists.

Example.2. If $f(x,y) = y\sin\left(\frac{1}{x}\right) + x\sin\left(\frac{1}{y}\right)$, where $x \neq 0, y \neq 0$, then prove that $f(x,y) \to 0$ as $(x,y) \to (0,0)$.

Solution: The given function is

$$f(x,y) = y\sin\left(\frac{1}{x}\right) + x\sin\left(\frac{1}{y}\right).$$

Consider ε is any given arbitrary small positive number *i.e.*, $\varepsilon > 0$, and assume $\delta = \varepsilon$.

We have

$$|(x,y) - (0,0)| = \sqrt{(x-0)^2 + (y-0)^2}$$

$$\leq |x-0| + |y-0| \qquad \text{(Now if we take } |x-0| < \varepsilon/2,\ |y-0| < \varepsilon/2\text{)}$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Now we have

$$|f(x,y) - 0| = \left|y\sin\left(\frac{1}{x}\right) + x\sin\left(\frac{1}{y}\right)\right|$$

$$\leq \left|y\sin\left(\frac{1}{x}\right)\right| + \left|x\sin\left(\frac{1}{y}\right)\right| \qquad \text{(Using By triangle inequality)}$$

$$\leq |y|\left|\sin\left(\frac{1}{x}\right)\right| + |x|\left|\sin\left(\frac{1}{y}\right)\right|$$

$$\leq |y| + |x| \qquad \left[\because \left|\sin\left(\frac{1}{x}\right)\right| \leq 1 \quad \text{and} \quad \left|\sin\left(\frac{1}{y}\right)\right| \leq 1\right]$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus $\quad |f(x,y) - 0| < \varepsilon.$

Hence $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

Note: If we take the path $y = mx$, then $x \to 0 \Rightarrow y \to 0$.

$$\therefore \quad \lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \left(y\sin\frac{1}{x} + x\sin\frac{1}{y}\right)$$

$$= \lim_{x\to 0}\left(mx\sin\frac{1}{x} + x\sin\frac{1}{mx}\right)$$

$$= 0 \qquad \lim_{x\to 0} x\sin\frac{1}{x} = 0.$$

Hence $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

Example.3. If $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$, then prove that, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exists.

Solution: The given function is

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}.$$

Suppose, if the $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists, then this limit is independent of the path along which we approach the point (x_0, y_0).

Consider $(x,y)\longrightarrow(0,0)$ along the path $y = mx$, where $m\in R$.

As $x\to 0$, from $y = mx$, we have $y\to 0$.

We have

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

Putting $y = mx$, then we have

$$\lim_{(x,mx)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x\,mx}{x^2+m^2x^2}$$

$$= \lim_{x\to 0} \frac{mx^2}{x^2(1+m^2)}$$

$$= \lim_{x\to 0} \frac{m}{1+m^2}$$

$= \dfrac{m}{1+m^2}$,which depend upon m .

Hence the $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Example.4. Show that the function $f(x,y) = \dfrac{xy^3}{x^2+y^6}$, $x\neq 0$, $y\neq 0$ and $f(0,0)=0$ is not continuous at (0, 0) in (x, y).

Solution: The given function is

$$f(x, y) = \frac{xy^3}{x^2 + y^6}, \quad x \neq 0, y \neq 0.$$

Here it is also given $f(0, 0) = 0.$

Let us consider $(x, y) \longrightarrow (0,0)$ through the curve $x = y^3$, so we get

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{y\to 0} \frac{y^6}{y^6 + y^6}$$

$$= \frac{1}{2}$$

Now we have $(x, y) \to (0,0)$ through the line $y = x$, so we get

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{x\to 0} \frac{x.x^3}{x^2 + x^6}$$

$$= \lim_{x\to 0} \frac{x^2}{1 + x^4}$$

$$= 0.$$

Here the limit found by two different approaches are changed thus the $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exists. Hence the given function is not continuous.

OR

If we take the path $x = my^3$ then $x \to 0 \Rightarrow y \to 0$.

$$\therefore \quad \lim_{(x,y)\to(0,0)} f(x, y) = \lim_{y\to 0} \frac{my^3.y^3}{(my^3)^2 + y^6}$$

$$= \lim_{y\to 0} \frac{my^6}{m^2 y^6 + y^6}$$

$$= \frac{m}{m^2 + 1}.$$

which is different value of *m*. So $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist.

Example.5. Show that the function $f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & \text{otherwise} \end{cases}$ is continuous at

$(0, 0)$.

Solution: The given function is

$$f(x,y)=\begin{cases}\dfrac{xy(x^2-y^2)}{x^2+y^2}, & (x, y)\neq(0, 0)\\ 0, & \text{otherwise}\end{cases}.$$

We have

$$|f(x,y)-f(0,0)|=\left|\frac{xy(x^2-y^2)}{x^2+y^2}-0\right|$$

$$=|xy|\left|\frac{x^2-y^2}{x^2+y^2}\right|$$

$$\leq|xy|$$

$$\left[\because |x^2-y^2|\leq|x^2+y^2|\left|\frac{x^2-y^2}{x^2+y^2}\right|\leq 1\right]$$

$$\Rightarrow \qquad |f(x,y)-f(0,0)|\leq|x||y|$$

(We now choose $|x-0|<\sqrt{\varepsilon}$ and $|y-0|<\sqrt{\varepsilon}$)

$$\Rightarrow \qquad |f(x,y)-f(0,0)|<\sqrt{\varepsilon}.\sqrt{\varepsilon}$$

$$\Rightarrow \qquad |f(x,y)-f(0,0)|<\varepsilon$$

Hence the function $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists and equal to $f(0,0)$.

Note: If $\lim\limits_{(x,y)\to(0,0)} f(x,y)=0$ along some path $y=mx$, then this does not means that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ is 0, because there are infinite number of paths passing through a given point.

To confirm that $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)=l.$

We put $x=x_0+r\cos\theta$ and $y=y_0+r\sin\theta$ then

$$|f(x,y)-l|<\in$$

$$\Rightarrow \qquad |f(x_0+r\cos\theta, y_0+r\sin\theta)-l|<\varepsilon$$

must hold for all volume of r less than some number r_0 which is independent of θ and for all value θ such that $|\theta|\leq 2\pi$.

For example: Consider $f\left(x,y\right)=\begin{cases}\dfrac{xy^2}{x^2+y^4}, & x\neq 0\\ 0 & x=0\end{cases}.$

Case-I: If we take the path $y=mx$ then $x\to 0 \Rightarrow y\to 0$.

$$\therefore \quad \lim_{(x,y)\to(0,0)} f\left(x,y\right)=\lim_{x\to 0}\frac{x\left(mx\right)^2}{x^2+\left(mx\right)^4}$$

$$=\lim_{x\to 0}\frac{m^2x^3}{x^2+m^4x^4}$$

$$=\lim_{x\to 0}\frac{xm^2}{1+m^4x^2}$$

$$=0,\quad \text{for all } m.$$

Case-II: But this does not mean that $\lim\limits_{(x,y)\to(0,0)} f\left(x,y\right)=0.$

Because if we take the path $x=my^2$ then $y\to 0 \Rightarrow x\to 0$.

$$\therefore \quad \lim_{(x,y)\to(0,0)} f\left(x,y\right)=\lim_{y\to 0}\frac{my^2.y^2}{\left(my^2\right)^2+y^4}$$

$$=\lim_{y\to 0}\frac{my^4}{m^2y^4+y^4}$$

$$=\lim_{y\to 0}\frac{m}{m^2+1}$$

$$=\frac{m}{m^2+1}.$$

which is different for different value of *m*. Therefore $\lim\limits_{(x,y)\to(0,0)} f\left(x,y\right)$ does not exist.

Example.6. Show that $\lim\limits_{(x,y)\to(0,0)}\dfrac{2x^3-y^3}{x^2+y^2}=0$.

Solution: Let $f\left(x,y\right)=\dfrac{2x^3-y^3}{x^2+y^2}$, then we have

$$\lim_{\substack{x\to 0\\ y\to 0}} f\left(x,y\right)=\lim_{\substack{x\to 0\\ y\to 0}}\frac{2x^3-y^3}{\left(x^2+y^2\right)}$$

Putting $y=mx$, then we have

$$= \lim_{x \to 0} \frac{2x^3 - m^3x^3}{\left(x^2 + m^2x^2\right)}$$

$$= \lim_{x \to 0} \frac{x^3\left(2 - m^3\right)}{x^2\left(1 + m^2\right)}$$

$$= \lim_{x \to 0} \frac{x\left(2 - m^3\right)}{\left(1 + m^2\right)}$$

$$= 0$$

Hence $\lim\limits_{\substack{x \to 0 \\ y \to 0}} f(x, y) = 0.$

Note: Here the limit is zero along the path $y = mx$ but this does not mean that $\lim\limits_{(x,y) \to (0,0)} f(x, y) = 0.$ Because there are infinite number of paths passing through (0, 0). To confirm that limit is zero we proceed as follow:

Here $(x_0, y_0) = (0, 0)$ and $l = 0.$

So we consider

$$\left| f(r\cos\theta, r\sin\theta) - 0 \right| = \left| \frac{2r^3\cos^3\theta - r^3\sin^3\theta}{r^2\left(\cos^2\theta + \sin^2\theta\right)} - 0 \right|$$

$$= \left| 2r\cos^3\theta - r^3\sin^3\theta \right|$$

$$\leq 2|r|\left|\cos^3\theta\right| + |-r|\left|\sin^3\theta\right|$$

or $\quad \left| f(r\cos\theta, r\sin\theta) - 0 \right| \leq 2r + r$

$$= 3r, \quad \text{for all } \theta.$$

If we take $r_0 = \in/3,$ then

$$\left| f(r\cos\theta, r\sin\theta) - 0 \right| < \in, \quad \text{for } r < r_0$$

$\therefore \lim\limits_{(x,y) \to (0,0)} f(x, y) = 0.$

Example.7. Let $f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}}, & \text{when } (x, y) \neq (0, 0) \\ 0, & \text{when } (x, y) = (0, 0) \end{cases}$. Show that $f(x, y)$ is continuous at $(0, 0)$.

Solution. The given function is

$$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & \text{when } (x,y)\neq(0,0)\\ 0, & \text{when } (x,y)=(0,0)\end{cases}.$$

If the simultaneous limit, $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)$ exists and is equal to $f(0,0)$, then $f(x,y)$ will be continuous at $(0,0)$.

Now we have

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y)=\lim_{\substack{x\to 0\\ y\to 0}}\left(\frac{xy}{\sqrt{x^2+y^2}}\right)$$

$$=\lim_{x\to 0}\frac{mx^2}{\sqrt{x^2+m^2x^2}} \qquad \text{[Taking the limit along the line } y=mx\text{]}$$

$$=\lim_{x\to 0}\frac{mx}{\sqrt{1+m^2}}=0.$$

Thus $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)$ exists and is equal to $f(0,0)$.

Hence $f(x,y)$ is continuous at $(0,0)$.

Note: Here the limit is zero along the path $y=mx$. This does not confirm that limit is zero. So we proceed as follow:

Here $\lim\limits_{(x,y)\to(0,0)} f(x,y)=0$ along the path $y=mx$ this does not mean that $\lim\limits_{(x,y)\to(0,0)} f(x,y)=0$ because there are infinite number of paths passing through (0, 0). We consider

$$\left|f(r\cos\theta, r\sin\theta)-0\right|=\left|\frac{r^2\sin\theta\cos\theta}{\sqrt{r^2\cos^2\theta+r^2\sin^2\theta}}-0\right|$$

$$=\left|\frac{r^2\sin\theta\cos\theta}{r.1}\right|$$

$$=\left|\frac{1}{2}r.\sin 2\theta\right|$$

$$\leq\frac{r}{2} \qquad [\because \quad \sin 2\theta\leq 1 \text{ and } r>0]$$

$\therefore$ If we take $r_0=2\in$ then

$$\left|f(r\cos\theta, r\sin\theta)-0\right|<\in, \text{ for } r<r_0 \text{ and for all } \theta.$$

$\therefore \lim\limits_{(x,y)\to(0,0)} f(x,y)=0$ which is equal to the value of $f(x,y)$ at $(0,0)$. Therefore f is continuous at (0, 0).

Example.8. Show that $\lim_{(x,y)\to(0,0)} \frac{2x-y}{x^2+y^2}$ does not exists.

Solution. Let $y = mx,$ then we have

$$\lim_{\substack{x\to 0\\ y\to 0}}\left(\frac{2x-y}{x^2+y^2}\right) = \lim_{x\to 0}\frac{2x-mx}{x^2+m^2x^2}$$

$$= \lim_{x\to 0}\frac{2-m}{x\left(1+m^2\right)}$$

$$= \infty$$

Hence the $\lim_{(x,y)\to(0,0)} \frac{2x-y}{x^2+y^2} = \infty$ does not exists.

Example.9. Let $f : R^2 \to R$ be defined as

$$f(x,y) = \begin{cases} 1, & \text{if } x \text{ is irrational} \\ 0, & \text{if } x \text{ is rational} \end{cases}.$$

Show that for any point (a,b), $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exists.

Solution. Consider *a* be a rational number. Then we have

$$\lim_{\substack{x\to a\\ y\to b}} f(x,y) = \lim_{\substack{x\to a\\ y\to b}}(0) \qquad [\text{i} \because f(x,y) = 0 \text{ f } x \text{ is rational}]$$

$$= 0.$$

Now if *a*is a irrational number, then we have

$$\lim_{\substack{x\to a\\ y\to b}} f(x,y) = \lim_{\substack{x\to a\\ y\to b}}(1) \qquad [\because f(x,y) = 1 \text{ if } x \text{ is irrational}]$$

$$= 1.$$

Therefore $\lim_{(x,y)\to(a,b)} f(x,y)$ is not unique.

Hence the $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exists.

Example.10. Show that the function $f(x,y)$ be defined as $f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2}, & \text{if } x \neq 0, y \neq 0 \\ 0, & \text{if } x = y = 0 \end{cases}$ is discontinuous at the point $(0,\ 0)$.

Solution. The given function is

$$f(x,y)=\begin{cases}\dfrac{x^2-y^2}{x^2+y^2}, & \text{if } \ x\neq 0, y\neq 0\\ 0, & \text{if } \ x=y=0\end{cases}$$

The simultaneous limit of (x,y) at $(0,0)$ is given by

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y)=\lim_{\substack{x\to 0\\ y\to 0}}\frac{x^2-y^2}{x^2+y^2}$$

$$=\lim_{x\to 0}\frac{x^2-m^2x^2}{x^2+m^2x^2} \qquad \text{(taking the limit along the line } y=mx\text{)}$$

$$=\lim_{x\to 0}\frac{1-m^2}{1+m^2}$$

$$=\frac{1-m^2}{1+m^2}$$

Thus the $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)$ depends upon m.

Hence $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exists and hence $f(x,y)$ is discontinuous.

Example.11. Consider

$$f(x,y)=\begin{cases}x\sin\left(\dfrac{1}{y}\right), & \text{if } \ y\neq 0\\ 0, & \text{if } \ (x,y)=(0,0)\end{cases}.$$

Show that $f(x,y)$ is continuous at $(0,\ 0)$.

Solution. The given function is

$$f(x,y)=\begin{cases}x\sin\left(\dfrac{1}{y}\right), & \text{if } \ y\neq 0\\ 0, & \text{if } \ (x,y)=(0,0)\end{cases}.$$

The simultaneous limit of $f(x,y)$ at $(0,0)$ is

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y)=\lim_{\substack{x\to 0\\ y\to 0}} x\sin\frac{1}{y}$$

$$=\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) \qquad [\text{Taking the limit along } y=x]$$

$$=0\times(\text{a finite number}) \qquad \left[\because -1\leq \sin\frac{1}{x}\leq 1\right]$$

$$=0$$

Thus the $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)$ exists and it is equal to $f(0,0)$ *i.e.*, $f(0,0)=0$.

Hence $f(x,y)$ is continuous at $(0,\ 0)$.

5.5 SUMMARY

1. Consider a function $f(x,y)$ of two variables x and y, then $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x,\ y)$ exists and equal to l, if for every $\varepsilon>0,\ \exists\ \delta>0$ such that $|\ f(x,y)-l\ |<\varepsilon,\quad \forall(x,y)$ for $|\ x-x_0\ |<\delta$ and $|\ y-y_0\ |<\delta$.
2. Consider a function $f(x,y)$ of two variables x and y is said to be continuous at the point (x_0,y_0), if $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x,y)$ exists and equal to $f(x_0,y_0)$.

5.6 TERMINAL QUESTIONS

Q.1 Show that $\lim\limits_{(x,y)\to(0,1)} \tan^{-1}\dfrac{y}{x}$ does not exists.

Q.2 Show that $\lim\limits_{(x,y)\to(0,0)} (x^2+y^2)=0.$

Q.3 Show that $\lim\limits_{(x,y)\to(0,0)} f(x,\ y)$ does not exists, where $\dfrac{xy^2}{x^2+y^4},\ (x,\ y)\neq(0,\ 0)$.

Q.4 Give an example to show that the order of iterated limits can be interchanged although the simultaneous limit does not exist.

Q.5 Show that the function $f(x,\ y)=\sqrt{|x\,y|}$ is continuous at $(0,\ 0)$.

Q.6 Show that $\lim\limits_{(x,y)\to(2,3)} xy$ exists and the function $f(x,y)=xy$ is continuous at $(2,3)$.

Q.7 If $f:R^2\to R$ be a function defined by $f(x,y)=\begin{cases}\dfrac{xy^2+x^2y}{x^3+y^3} & if\quad (x,y)\neq(0,0)\\ 0 & if\quad (x,y)=(0,0)\end{cases}$ is not continuous at $(0,0)$.

Q.8 If $f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2} & ,x\neq 0,\ y\neq 0\\ 0 & ,x=0,\ y=0\end{cases}$ then show that $f_x,\ f_y$ exist at $(0,\ 0)$ and examine the continuity of f_x, f_y with respect to x and y and (x, y) together.

Q.9 Let $f(x,y)$ be a function defined by $f(x,y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$

Show that

(i) f, f_x, f_y are continuous in (x, y)

(ii) f_{xy} and f_{yx} exist at every point (x, y) and are continuous except at $(0,0)$

(iii) $f_{xy}(0,0) = 1$ and $f_{yx}(0,0) = -1$.

UNIT 6 PARTIAL DIFFERENTIATION

Structure

6.1 Introduction

6.2 Objectives

6.3 Partial Derivatives

6.4 Higher Order Partial Derivatives

6.5 Summary

6.6 Terminal Questions

6.1 INTRODUCTION

Partial differentiation is a powerful tool that extends the concept of differentiation from functions of one variable to functions of several variables, enabling deeper insights into the behavior of multivariable functions and their applications in diverse fields. Partial derivatives are employed in economic and financial models to analyze how changes in one variable affect others, such as in elasticity calculations or option pricing models. Partial derivatives are used to define the gradient vector, which points in the direction of the steepest ascent of a function. The gradient is crucial in optimization problems and vector calculus.

Partial derivatives help define tangent planes to surfaces and linear approximations to functions. This is essential in understanding the behavior of functions near specific points. In thermodynamics and physics, partial derivatives are used extensively to describe relationships between variables in systems with multiple parameters. In this unit we shall discuss the partial derivatives, higher order partial derivatives and the homogeneous function.

6.2 OBJECTIVES

After reading this unit the learner should be able to understand about:

- the partial derivatives
- the higher order partial derivatives
- the homogeneous function

6.2 PARTIAL DERIVATIVES

Partial differentiation involves determining partial derivatives. When we take the derivative of a function with respect to one independent variable, while keeping all other independent variables constant, it yields the partial derivative of the function with respect to that variable.

Consider $u = f(x, y)$ be the function of two independent variables x and y, and is denoted by $\frac{\partial u}{\partial x}$, or $\frac{\partial f}{\partial x}$, or f_x, or u_x, or $f_x(x, y)$, and $\frac{\partial u}{\partial y}$, or $\frac{\partial f}{\partial y}$, or f_y, or u_y, or $f_y(x, y)$, etc.

Thus we have $\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,\ y) - f(x,\ y)}{h}$ provided that this limit exist and unique.

Similarly, we have $\frac{\partial f}{\partial y} = k \lim_{h \to 0} \frac{f(x, y+k) - f(x, y)}{k}$ provided that this limit exist and unique.

6.4 HIGHER ORDER PARTIAL DERIVATIVES

Suppose $u = f(x, y)$ then $\frac{\partial u}{\partial x}$, or $\frac{\partial f}{\partial x}$, or f_x is the partial derivatives of function of x and y. The partial of $\frac{\partial u}{\partial x}$, or $\frac{\partial f}{\partial x}$, or f_x with respect to x are denoted by $\frac{\partial^2 u}{\partial x^2}$ or $\frac{\partial^2 f}{\partial x^2}$ or f_{xx}.

Similarly, the partial derivation of $\frac{\partial u}{\partial y}$, or $\frac{\partial f}{\partial y}$, or f_y with respect to y are denoted by

$$\frac{\partial^2 u}{\partial y^2} \text{ or } \frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{xy}$$

and

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{yx}$$

If f_{xy} and f_{yx} are continuous, then we have $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = f_{yx}$.

In limit format the derivative of the second order are defined as

$$\frac{\partial^2 f}{\partial x^2} = \lim_{h \to 0} \frac{f(x_0 + 2h, y_0) - 2f(x_0 + h, y_0) + f(x_0, y_0)}{h^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \lim_{k \to 0} \frac{f(x_0, y_0 + 2k) - 2f(x_0, y_0 + k) + f(x_0, y_0)}{k^2}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \lim_{k \to 0} \lim_{h \to 0} \frac{f(x_0+h, y_0+k) - f(x_0, y_0+k) - f(x_0+h, y_0) + f(x_0, y_0)}{hk}$$

and $$\frac{\partial^2 f}{\partial x \partial y} = \lim_{h \to 0} \lim_{k \to 0} \frac{f(x_0+h, y_0+k) - f(x_0+h, y_0) - f(x_0, y_0+k) + f(x_0, y_0)}{hk}$$

Examples

Example.1. Let $f(x,y) = \frac{x^2 y}{x^4 + y^2}$, for $x \neq 0, y \neq 0$ and $f(0,0) = 0$.Show that the partial derivatives f_x, f_y exist everywhere in the region $-1 \leq x \leq 1, -1 \leq y \leq 1$, although $f(x,y)$ is discontinuous in (x,y) at the origin.

Solution. The given function is

$$f(x,y) = \frac{x^2 y}{x^4 + y^2}, \quad (x \neq 0, y \neq 0)$$

We have $f_x = 2xy.\frac{x^2 + y^2 - 2x^4}{(x^4 + y^2)^2}, \quad (x \neq 0, y \neq 0)$

and $f_y = x^2.\frac{x^4 - y^2}{(x^4 + y^2)^2}, (x \neq 0, y \neq 0)$

Also for $x = y = 0$, we get

$$f_x = \lim_{h \to 0} \frac{h.0}{h^4 + 0} = 0 \text{ and } f_y = 0.$$

Similarly, the following results can be prove

$$f_x(x,y) = 0 \text{ for } x = 0, y \neq 0$$

$$f_x(x,y) = 0 \text{ for } x \neq 0, y = 0$$

$$f_y(x,y) = 0 \text{ for } x = 0, y \neq 0$$

$$f_y(x,y) = \frac{1}{x^2} \text{ for } x \neq 0, y = 0$$

Therefore the partial derivative f_x, f_y exists at all points of the given region. Now we shall check the continuity of the given function. The limiting value of $f(x,y)$, along the line $y = 0$, given by

$$\lim_{h \to 0} .0 = 0.$$

And the limiting value of $f(x,y)$ along the line $y = x^2$ is given by

$$\lim_{x\to 0}\frac{x^4}{x^4+x^4}=\frac{1}{2}$$

Here the limit obtained by two different approaches is different. Hence the $f(x,y)$ is discontinuous in (x,y) at the origin.

Example.2. Example.1 Let $f(x,y)=\begin{cases}\frac{1}{4}(x^2+y^2)\log(x^2+y^2), & \text{when}\,(x,y)\neq(0,0)\\ 0 & ,\text{when}\,(x,y)=(0,0)\end{cases}$. Show that $f_{xy}=f_{yx}$ at all points (x,y).

Also, show that none of the derivatives is continuous in (x,y) at the origin.

Solution. The given function is

$$f(x,y)=\begin{cases}\frac{1}{4}(x^2+y^2)\log(x^2+y^2), & \text{when}\,(x,y)\neq(0,0)\\ 0 & ,\text{when}\,(x,y)=(0,0)\end{cases}.$$

For $x\neq 0, y\neq 0$, we have

$$f_x=\frac{1}{2}x\left(1+\log\left(x^2+y^2\right)\right),$$

$$f_y=\frac{1}{2}y\left(1+\log\left(x^2+y^2\right)\right)$$

and $f_{xy}=f_{yx}=\dfrac{xy}{x^2+y^2}$.

For $x=0, y=0$, we have $\quad f_x=y_y=f_{xy}=f_{yx}=0$

Hence, $f_{xy}=f_{yx}$ at every point.

Now, we show that $f_{xy}=f_{yx}$ is not continuous at $(0,0)$.

Since $\lim_{(x,y)\to(0,0)}\dfrac{xy}{x^2+y^2}$ does not exist. If we put $y=mx$, then the limit of the function, depends upon m. Hence the limit does not exists. It follows that $f_{xy}=f_{yx}$ is not continuous at the origin.

Example.3. Let $f(x,y)$ be a function, defined by $f(x,y)=x\sin\frac{1}{x}+y\sin\frac{1}{y},\ x\neq 0, y\neq 0;$

$f(0,y)=y\sin\frac{1}{y}, y\neq 0;$ $f(x,0)=x\sin\frac{1}{x}, x\neq 0;$ $f(0,0)=0.$ Examine the existence of f_x and f_{yx} at $x=0, y=0$.

Solution. The given function is

$$f(x,y) = x\sin\frac{1}{x} + y\sin\frac{1}{y},\ x \neq 0, y \neq 0.$$

We have

$$f_x(0,0) = \lim_{h\to 0}\frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0}\frac{h\sin\frac{1}{h} - 0}{h}$$

$$= \lim_{h\to 0}\sin\frac{1}{h}$$

Since the limit does not exists, therefore $f_x(0,\ 0)$does not exists.

Now $f_{yx} = \lim_{k\to 0}\lim_{h\to 0}\dfrac{f(h,k) - f(0,k) - f(h,0) + f(0,0)}{hk}$

$$= \lim_{k\to 0}\lim_{h\to 0}\frac{h\sin\left(\frac{1}{h}\right) + k\sin\left(\frac{1}{k}\right) - k\sin\left(\frac{1}{k}\right) - h\sin\left(\frac{1}{h}\right)}{hk}$$

$$= \lim_{k\to 0}\lim_{h\to 0}\frac{0}{hk} = 0.$$

In spite of the fact that limit is zero, the derivative $f_{yx}(0, 0)$ cannot be said to exists, since $f_x(0,\ 0)$ does not exists.

Example.4. Find the first order partial derivatives $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ when $u = \tan^{-1}\dfrac{y}{x}$.

Sol. We have $u = \tan^{-1}\dfrac{y}{x}$ …(1)

Differentiating partially equation (1) with respect to x taking y as a constant, we get

$$\frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$= \frac{x^2}{x^2+y^2}.\left(-\frac{y}{x^2}\right)$$

$$= -\frac{y}{x^2+y^2}$$

Now again differentiating partially equation (6.1) with respect to y taking x as a constant, we get

$$\frac{\partial u}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

$$= \frac{x^2}{x^2+y^2}.\left(\frac{1}{x}\right)$$

$$= \frac{x}{x^2+y^2}$$

Example.5. Find the first order partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ when

(i) $u = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$ (ii) $u = \log(x^2 + y^2)$.

Sol. (i) We have $u = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$ …(2)

Differentiating partially equation (1) with respect to x taking y as a constant, we get

$$\frac{\partial u}{\partial x} = \frac{2x}{a^2}$$

Now again differentiating partially equation (1) with respect to y taking x as a constant, we get

$$\frac{\partial u}{\partial x} = \frac{2y}{b^2}$$

(ii) We have $u = \log\left(x^2 + y^2\right)$ … (1)

Differentiating partially equation (1) with respect to x taking y as a constant, we get

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2}.\frac{\partial}{\partial x}\left(x^2+y^2\right)$$

$$= \frac{2x}{x^2+y^2}$$

Now again differentiating partially equation (1) with respect to y taking x as a constant, we get

$$\frac{\partial u}{\partial y} = \frac{1}{x^2+y^2}.\frac{\partial}{\partial y}\left(x^2+y^2\right)$$

$$= \frac{2y}{x^2+y^2}$$

Example.6. Verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$, if $u = ax^2 + 2hxy + by^2$.

Sol. We have $u = ax^2 + 2hxy + by^2$ …(1)

Differentiating partially equation (1) with respect to x we get

$$\frac{\partial u}{\partial x} = 2ax + 2hy \qquad \ldots(2)$$

Differentiating partially equation (2) with respect to y we get

$$\frac{\partial^2 u}{\partial y \partial x} = 2h \qquad \ldots(3)$$

Now differentiating partially equation (1) with respect to y we get

$$\frac{\partial u}{\partial y} = 2hx + 2by \qquad \ldots(4)$$

Again differentiating partially equation (4) with respect to x we get

$$\frac{\partial^2 u}{\partial x \partial y} = 2h \qquad \ldots(5)$$

Using equations (3) and (5), we conclude that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.$$

Example.7. If $u = f\left(\frac{y}{x}\right)$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$

Sol. We have $u = f\left(\frac{y}{x}\right)$ $\qquad \ldots(1)$

Differentiating partially equation (1) with respect to x we get

$$\frac{\partial u}{\partial x} = f'\left(\frac{y}{x}\right).\frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$= f'\left(\frac{y}{x}\right).\left(\frac{-y}{x^2}\right)$$

$$= \frac{-y}{x^2} f'\left(\frac{y}{x}\right)$$

or $$x\frac{\partial u}{\partial x} = \frac{-y}{x} f'\left(\frac{y}{x}\right). \qquad \ldots(2)$$

Now differentiating partially equation (1) with respect to y we get

$$\frac{\partial u}{\partial y} = f'\left(\frac{y}{x}\right).\frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

$$= f'\left(\frac{y}{x}\right).\frac{1}{x}$$

$$= \frac{1}{x} f'\left(\frac{y}{x}\right)$$

or $$y\frac{\partial u}{\partial y} = \frac{y}{x} f'\left(\frac{y}{x}\right). \qquad \dots (3)$$

Adding equations (2) and (3), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

Example.8. If $u(x,\ y,\ z) = \log(x^3 + y^3 + z^3 - 3xyz),$ then show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -3(x+y+z)^{-2}.$$

Sol. Given that $u = \log(x^3 + y^3 + z^3 - 3xyz)$ … (1)

Differentiating partially equation (1) with respect to x, y and z respectively, we get

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz} \qquad \dots (2)$$

$$\frac{\partial u}{\partial y} = \frac{3y^2 - 3zx}{x^3 + y^3 + z^3 - 3xyz} \qquad \dots (3)$$

$$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz} \qquad \dots (4)$$

From equations (2), (3) and (4), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z} \qquad \dots (5)$$

$$\left[\because\ x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx)\right]$$

Now again differentiate partially equations (2), (3) and (4), with respect to x, y and z respectively, we get

$$\frac{\partial^2 u}{\partial x^2} = -3\left(\frac{x^4 - 2xy^3 - 2xz^3 + 3y^2 z^2}{\left(x^3 + y^3 + z^3 - 3xyz\right)^2}\right) \qquad \dots (6)$$

$$\frac{\partial^2 u}{\partial y^2} = 3\left(\frac{-2x^3 y - y^4 + 2yz^3 - 3z^2 x^2}{\left(x^3 + y^3 + z^3 - 3xyz\right)^2}\right) \qquad \dots (7)$$

$$\frac{\partial^2 u}{\partial z^2} = 3\left(\frac{2x^3 z - z^4 - 2y^3 z - 3x^2 y^2}{\left(x^3 + y^3 + z^3 - 3xyz\right)^2}\right) \qquad \dots (8)$$

From equations (6), (7) and (8), we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=-\frac{3}{(x+y+z)^2}.$$

Example.9. If $u=\dfrac{x^2+y^2}{x+y}$, then show that $\left(\dfrac{\partial u}{\partial x}-\dfrac{\partial u}{\partial y}\right)^2=4\left(1-\dfrac{\partial u}{\partial x}-\dfrac{\partial u}{\partial y}\right)$.

Sol. Given that $$u=\frac{x^2+y^2}{x+y} \qquad \ldots (1)$$

Differentiating partially equation (1) with respect to x and y respectively, we get

$$\frac{\partial u}{\partial x}=\frac{(x+y)(2x)-(x^2+y^2).1}{(x+y)^2}$$

$$=\frac{x^2+2xy-y^2}{(x+y)^2}$$

and

$$\frac{\partial u}{\partial y}=\frac{(x+y)(2y)-(x^2+y^2).1}{(x+y)^2}$$

$$=\frac{y^2+2xy-x^2}{(x+y)^2}$$

We have

$$\left(\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}\right)^2=\left[\frac{x^2+2xy-y^2}{(x+y)^2}-\frac{y^2+2xy-x^2}{(x+y)}\right]^2$$

$$=\left[\frac{2x^2-2y^2}{(x+y)^2}\right]^2$$

$$=\frac{4(x+y)^2(x-y)^2}{(x+y)^4}$$

$$=\frac{4(x-y)^2}{(x+y)^2} \qquad \ldots (2)$$

Now we have

$$4\left(1-\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}\right)=4\left[1-\frac{x^2+2xy-y^2}{(x+y)^2}-\frac{y^2+2xy-x^2}{(x+y)^2}\right]$$

$$=4\left[\frac{x^2+y^2+2xy-x^2-2xy+y^2-y^2-2xy+x^2}{(x+y)^2}\right]$$

$$=4.\frac{(x^2+y^2-2xy)}{(x+y)^2}$$

$$= 4.\frac{(x-y)^2}{(x-y)^2} \qquad \ldots (3)$$

Hence by equations (2) and (3), we have

$$\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 = 4\left(1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right).$$

Example.10. If $x^x y^y z^z = c$, then show that $x = y = z,\ \frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$.

Sol. Given that $x^x y^y z^z = c$ … (1)

Taking log both sides of equation (1), we get

$$x \log x + y \log y + z \log z = \log c \qquad \ldots (2)$$

Differentiating partially equation (2) with respect to x (taking z as dependent variable) then we get

$$\left[x\left(\frac{1}{x}\right) + \log x.1\right] + 0 + \left[z.\left(\frac{1}{z}\right) + \log z\right]\frac{\partial z}{\partial x} = 0$$

or $$\frac{\partial z}{\partial x} = -\frac{1+\log x}{1+\log z} \qquad \ldots (3)$$

Similarly, we get $$\frac{\partial z}{\partial y} = -\frac{1+\log y}{1+\log z} \qquad \ldots (4)$$

Differentiating partially equation (4) with respect to x, we get

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(-\frac{1+\log y}{1+\log z}\right)$$

$$= -(1+\log y).\frac{\partial}{\partial x}\left(\frac{1}{1+\log z}\right)$$

$$= -(1+\log y).\left\{(-1)(1+\log z)^{-2}.\frac{1}{z}\frac{\partial z}{\partial x}\right\}$$

$$= \frac{(1+\log y)}{z(1+\log z)^2}.\frac{\partial z}{\partial x}$$

$$= \frac{(1+\log y)}{z(1+\log z)^2}.\left(-\frac{1+\log x}{1+\log z}\right)$$

$$= -\frac{(1+\log y)(1+\log x)}{z(1+\log z)^3} \qquad \ldots (5)$$

At $x = y = z$, from equation (5), we have

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{(1+\log x)^2}{x(1+\log x)^3}$$

$$= -\frac{1}{x(1+\log x)}$$

$$= -\frac{1}{x(\log e + \log x)}$$

$$= -\frac{1}{x \log ex}$$

$$= -[x.\log ex]^{-1}.$$

Example.11. If $u = f(r)$, where $r^2 = x^2 + y^2$, then show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{r} f'(r) + f''(r).$

Sol. Given that $r^2 = x^2 + y^2$ … (1)

Differentiating partially equation (1) with respect to x, we get

$$2r \frac{\partial r}{\partial x} = 2x$$

or $$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly, we get

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

It is also given that

$$u = f(r) \quad \text{… (2)}$$

Differentiating partially equation (2) with respect to x, we get

$$\frac{\partial u}{\partial x} = f'(r) \frac{\partial r}{\partial x}$$

$$= \frac{x}{r} f'(r) \quad \text{… (3)}$$

Again differentiating partially equation (3) with respect to x, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}.\left(\frac{\partial u}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(f'(r).\frac{x}{r}\right)$$

$$= \frac{r\left[f'(r) + x^2 f''(r) \frac{\partial r}{\partial x} \right] - x f'(r) \frac{\partial r}{\partial x}}{r^2}$$

$$= \frac{1}{r^2}\left[r f'(r) + x^2 f''(r) - \frac{x^2}{r} f'(r) \right] \qquad \ldots (4)$$

Similarly, we get

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{r^2}\left[r f'(r) + y^2 f''(r) - \frac{y^2}{r} f'(r) \right] \qquad \ldots (5)$$

Adding equation (4) and (5), we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{r^2}\left[2r f'(r) + (x^2 + y^2) f''(r) - \frac{(x^2 + y^2)}{r} f'(r) \right]$$

$$= \frac{1}{r^2}\left[2r f'(r) + r^2 f''(r) - r f'(r) \right]$$

$$= \frac{1}{r^2}\left[r f'(r) + r^2 f''(r) \right]$$

$$= \frac{1}{r} f'(r) + f''(r).$$

6.5 SUMMARY

1. Partial differentiation is the process of finding the partial derivatives.
2. If $u = f(x, y)$ be the function of two independent variables x and y, then we have $\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,\ y) - f(x,\ y)}{h}$ and $\frac{\partial f}{\partial y} = k \lim_{h \to 0} \frac{f(x, y+k) - f(x, y)}{k}$ provided that these limits are exist and unique.
3. If f_{xy} and f_{yx} are continuous, then we have $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = f_{yx}$.

6.6 TERMINAL QUESTIONS

Q.1 Explain the partial derivatives.

Q.2 Let $f(x, y) = \begin{cases} x^2 y^2 \cos\frac{1}{x}, & \text{for all values of } y \text{ so long as } x \neq 0 \\ 0, & \text{for } x = 0 \end{cases}$. Show that

(i) $f_{xy} = f_{yx}$ at all points (x, y)

(ii) neither f_{xy} nor f_{yx} is continuous in x at $x = 0$, if $y \neq 0$

(iii) both f_{xy} and f_{yx} are continuous in (x, y), together at the origin.

Q.3 If $f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, & x \neq 0, y \neq 0 \\ 0, & x = 0, y = 0 \end{cases}$ then show that f_x, f_y exist at $(0,0)$ and examine the continuity of f_x, f_y with respect to x and y and (x, y) together.

Q.4 Let $f(x,y)$ be a function defined by $f(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2} & \text{for } (x, y) \neq (0,0) \\ 0 & \text{for } (x, y) = (0,0) \end{cases}$

Show that

(i) f, f_x, f_y are continuous in (x, y)

(ii) f_{xy} and f_{yx} exist at every point (x, y) and are continuous except at $(0,0)$

(iii) $f_{xy}(0,0) = 1$ and $f_{yx}(0,0) = -1$.

Q.5 What do you mean by higher order partial derivatives?

Q.6 If $u = \sin^{-1}\dfrac{x}{y} + \tan^{-1}\dfrac{y}{x}$, then show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0.$

Q.7 If $u = \tan^{-1}\dfrac{xy}{\sqrt{1 + x^2 + y^2}}$, then show that $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{1}{(1 + x^2 + y^2)^{3/2}}$.

Q.8 If $u = \log(x^3 + y^3 - x^2 y - xy^2)$, then show that

$(i)\ \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = \dfrac{2}{(x + y)}$ $\qquad (ii)\ \dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = -\dfrac{4}{(x + y)^2}$.

Q.9 If $u = f(x + ay) + g(x - ay)$, then show that $\dfrac{\partial^2 u}{\partial y^2} = a^2\dfrac{\partial^2 u}{\partial x^2}$.

Q.10 If $u = e^{xyz}$, then show that $\dfrac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2 y^2 z^2)e^{xyz}$.

Q.11 If $\dfrac{x^2}{a^2+u}+\dfrac{y^2}{b^2+u}+\dfrac{z^2}{c^2+u}=1,$ then show that

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right).$$

Q.12 Find the value of $\dfrac{1}{a^2}\dfrac{\partial^2 u}{\partial x^2}+\dfrac{1}{b^2}+\dfrac{\partial^2 u}{\partial y^2}$ when $a^2x^2+b^2y^2=c^2z^2.$

Q.13 If $\theta=t^n e^{-(r^2/4t)}$, find the value of n which will make $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial\theta}{\partial r}\right)=\dfrac{\partial\theta}{\partial t}.$

Q.14 If $u=x^2\tan^{-1}\dfrac{y}{x}-y^2\tan^{-1}\dfrac{x}{y}$, then show that $\dfrac{\partial^2 u}{\partial y\partial x}=\dfrac{x^2-y^2}{x^2+y^2}.$

Q.15 If $u=x^2y+y^2z+z^2x,$ then show that $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}=(x+y+z)^2.$

UNIT 7 EULER’S THEOREM

Structure

7.1 Introduction

7.2 Objectives

7.3 Homogeneous Function

7.4 Euler’s Theorem on Homogeneous Functions

7.5 Some deductions from Euler’s Theorem

7.6 Total Differential Coefficient

7.7 Implicit Function

7.8 Summary

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7.1 INTRODUCTION

Euler's Theorem on homogeneous functions is a result from calculus, specifically concerning multivariable calculus and homogeneous functions. It's named after the Swiss mathematician Leonhard Euler, who made significant contributions to various areas of mathematics. This theorem essentially describes a relationship between the partial derivatives of a homogeneous function and the function itself. It provides a useful tool for simplifying expressions involving homogeneous functions, often enabling solutions to problems in physics, economics, engineering, and other fields where such functions arise. Euler's Theorem provides insights into the behavior of these functions, helping to determine critical points, maxima, minima, and saddle points. The concept of the total differential coefficient arises in calculus, particularly in the context of multivariable functions. When you have a function of several variables, its total differential describes how the function changes as each of its variables changes.

The total differential coefficient is crucial for understanding how a function changes as its variables change. It provides a linear approximation of the change in the function near a given point in its domain. In this unit we shall discuss about the Euler’s theorem, Euler’s theorem on homogeneous functions, Some deductions from Euler’s Theorem, Total Differential Coefficient and implicit function.

7.2 OBJECTIVES

After reading this unit the learner should be able to:

- understand the homogeneous function
- understand the Euler's theorem on homogeneous functions
- explain the total differential coefficient
- discuss the implicit function

7.3 HOMOGENEOUS FUNCTION

A function $u(x, y)$ is said to be a homogenous function if all its terms are of the same degree. Let

$$f(x, y) = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + ...+ a_{n-1} x y^{n-1} + a_n y^n \quad \dots (1)$$

be a function of x and y of degree n.

The equation (1) can be written as

$$f(x, y) = x^n \left[a_0 + a_1 \left(\frac{y}{x}\right) + a_2 \left(\frac{y}{x}\right)^2 + ...+ a_{n-1} \left(\frac{y}{x}\right)^{n-1} + a_n \left(\frac{y}{x}\right)^n \right]$$

$$= x^n \phi\left(\frac{y}{x}\right)$$

Examples

Example.1. Find the order of the following homogeneous functions:

(a) $u = \dfrac{x^2 - y^2}{x + y}$ (b) $u = \dfrac{\sqrt{x}\sqrt{y}}{x^2 + y^2}$

Solution: (a) It is given that

$$u = \frac{x^2 - y^2}{x + y}$$

$$= \frac{x^2 \left[1 - \dfrac{y^2}{x^2}\right]}{x \left[1 + \dfrac{y}{x}\right]}$$

$$= x \left(\frac{1 - \dfrac{y^2}{x^2}}{1 + \dfrac{y}{x}} \right)$$

Here u is a homogeneous function of order 1.

(b) It is given that

$$u = \frac{\sqrt{x}\sqrt{y}}{x^2 + y^2}$$

$$= \frac{\sqrt{x}\left[1+\sqrt{\frac{y}{x}}\right]}{x^2\left[1+\left(\frac{y}{x}\right)^2\right]}$$

$$= x^{-3/2}\,\frac{1+\sqrt{\frac{y}{x}}}{1+\left(\frac{y}{x}\right)^2}$$

Here u is a homogeneous function of order $-\frac{3}{2}$.

7.4 EULER'S THEOREM ON HOMOGENEOUS FUNCTIONS

Euler's Theorem on homogeneous functions plays a significant role in various fields of mathematics, physics, and engineering due to its importance and wide range of applications.

If u is a homogenous function of x and y of degree n, then we have $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$

Proof: Since u is a homogenous function of x and y of degree n, therefore it can be written as

$$u = x^n \phi\left(\frac{y}{x}\right) \qquad \ldots (1)$$

Differentiating partially equation (1) with respect to x, we get

$$\frac{\partial u}{\partial x} = nx^{n-1}\phi\left(\frac{y}{x}\right) + x^n \phi'\left(\frac{y}{x}\right).\frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$= nx^{n-1}\phi\left(\frac{y}{x}\right) + x^n \phi'\left(\frac{y}{x}\right).\left(-\frac{y}{x^2}\right)$$

or $$x\frac{\partial u}{\partial x} = nx^n\phi\left(\frac{y}{x}\right) - yx^{n-1}\phi'\left(\frac{y}{x}\right) \qquad \ldots (2)$$

Again differentiating partially (1) with respect to y, we get

$$\frac{\partial u}{\partial y} = x^n \phi'\left(\frac{y}{x}\right) \frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$= x^n \phi'\left(\frac{y}{x}\right).\frac{1}{x}$$

or $$y\frac{\partial u}{\partial y} = y\, x^{n-1} \phi'\left(\frac{y}{x}\right) \qquad \ldots (3)$$

Adding equations (2) and (3), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\, x^n \phi'\left(\frac{y}{x}\right) = nu$$

Hence if u is a homogenous function of x and y of degree n, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

7.5 SOME DEDUCTIONS FORM EULER'S THEOREM

If $u = x^n \phi\left(\frac{y}{x}\right)$ then $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.$

Solution. If $u = x^n \phi\left(\frac{y}{x}\right)$, then we know that by Euler's theorem, we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \qquad \ldots (1)$$

Differentiating partially equation (1) with respect to x and y respectively, we have

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = n\frac{\partial u}{\partial x} \qquad \ldots (2)$$

and $$x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = n\frac{\partial u}{\partial y} \qquad \ldots (3)$$

Multiply equation (2) by x and equation (3) by y and adding them, we get

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = n\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

Using equation (1), we have

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + nu = n^2 u.$$

or $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u.$$

Note: 1. If u is a homogeneous function of $x_1, x_2, \ldots\ldots, x_n$ of degree n, then we have $$x_1\frac{\partial u}{\partial x_1}+x_2\frac{\partial u}{\partial x_2}+x_3\frac{\partial u}{\partial x_3}+\ldots\ldots.+x_m\frac{\partial u}{\partial x_m}=nu.$$

2. If $u=f\left(x_1, x_2, \ldots\ldots, x_n\right)$ is a homogeneous function of $x_1, x_2, \ldots\ldots, x_m$ of degree n, then we have $f\left(tx_1, tx_2, \ldots\ldots, tx_n\right)=t^n f\left(x_1, x_2, \ldots\ldots, x_n\right)$.

Examples

Example.1. If $u=\cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ then prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{2}\cot u.$

Sol. Given that $u=\cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ ….. (1)

The above equation (1) can be written as

$$\cos u=\frac{x+y}{\sqrt{x}+\sqrt{y}} \qquad \ldots..(2)$$

Now consider $V=\frac{x+y}{\sqrt{x}+\sqrt{y}}$ ….. (3)

Here V is a homogenous function of x and y of degree (1/2). Therefore, by Euler's theorem, we have

$$x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}=\frac{1}{2}V \qquad \ldots.(4)$$

From equations (2) and (3), we have

$$V=\cos u \qquad \ldots.(5)$$

Now differentiating partially equation (5) with respect to x and y respectively, we get

$$\frac{\partial V}{\partial x}=-\sin u\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial V}{\partial y}=-\sin u\frac{\partial u}{\partial y}$$

Putting these values of $\frac{\partial V}{\partial x}$ and $\frac{\partial V}{\partial y}$ in equation (4), we get

$$-x\sin u\frac{\partial u}{\partial x}-y\sin u\frac{\partial u}{\partial y}=\frac{1}{2}\cos u$$

or $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{2}\cot u.$$

Example.2. If $u = \sin^{-1}\left(\dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}\right)$ then prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{20}\tan u.$

Sol. Given that $$u = \sin^{-1}\left(\frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}\right) \quad \ldots\ldots (1)$$

The above equation (1) can be written as

$$\sin u = \frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}} \quad \ldots\ldots (2)$$

Now consider $V = \dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$ ….. (3)

Here V is a homogenous function of x and y of degree (1/20). Therefore, by Euler's theorem, we have

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} = \frac{1}{20}V \quad \ldots (4)$$

From equations (2) and (3), we have

$$V = \sin u \quad \ldots(5)$$

Now differentiating partially equation (5) with respect to x and y respectively, we get

$$\frac{\partial V}{\partial x} = \cos u\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial V}{\partial y} = \cos u\frac{\partial u}{\partial y}$$

Putting these values of $\dfrac{\partial V}{\partial x}$ and $\dfrac{\partial V}{\partial y}$ in equation (4), we get

$$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \frac{1}{20}\sin u$$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{20}\tan u.$$

Example.3. If $u = \sin^{-1}\left\{\dfrac{x^2 + y^2}{x + y}\right\}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u.$

Sol. Given that $$u = \sin^{-1}\left\{\frac{x^2 + y^2}{x + y}\right\} \quad \ldots(1)$$

We have $$\sin u = \frac{x^2 + y^2}{x + y} \quad \ldots(2)$$

Now consider V= $\dfrac{x^2+y^2}{x+y}$ …(3)

Here V is a homogenous function of x and y of degree 1. Therefore, by Euler's theorem, we have

$$x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}=V \qquad \text{…(4)}$$

From equations (6.51) and (6.52), we have

$$V=\sin u \qquad \text{…(5)}$$

Now differentiating partially equation (5) with respect to x and y respectively, we get

$$\frac{\partial V}{\partial x}=\cos u\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial V}{\partial y}=\cos u\frac{\partial u}{\partial y}$$

Putting these values of $\dfrac{\partial V}{\partial x}$ and $\dfrac{\partial V}{\partial y}$ in equation (4), we get

$$x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=\sin u$$

or
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u.$$

Example.4. Verify Euler's theorem when $u=\dfrac{x\left(x^3-y^3\right)}{x^3+y^3}$.

Sol. Given that
$$u=\frac{x\left(x^3-y^3\right)}{x^3+y^3}. \qquad \text{…(1)}$$

Here u is a homogenous function of x and y of degree 1. Therefore, by Euler's theorem, we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=u \qquad \text{….(2)}$$

Taking log in equation (1) both sides, we have

$$\log u=\log x+\log(x^3-y^3)-\log(x^3-y^3) \qquad \text{….(3)}$$

Differentiating partially equation (3) with respect to x and y, we get

$$\frac{1}{u}\frac{\partial u}{\partial x}=\frac{1}{x}+\frac{3x^2}{x^3-y^3}-\frac{3x^2}{x^3-y^3}$$

or
$$\frac{x}{u}\frac{\partial u}{\partial x}=1+\frac{3x^3}{x^3-y^3}-\frac{3x^3}{x^3-y^3} \qquad \text{….(4)}$$

and $$\frac{1}{u}\frac{\partial u}{\partial y}=\frac{1}{x}+\frac{3y^2}{x^3-y^3}-\frac{3y^2}{x^3-y^3}$$

or $$\frac{y}{u}\frac{\partial u}{\partial y}=-\frac{3y^3}{x^3-y^3}-\frac{3y^3}{x^3-y^3} \qquad \text{....(5)}$$

Adding equations (4) and (5), we get

$$\frac{x}{u}\frac{\partial u}{\partial x}+\frac{y}{u}\frac{\partial u}{\partial y}=1+\frac{3\left(x^3-y^3\right)}{x^3-y^3}-\frac{3\left(x^3+y^3\right)}{x^3-y^3}$$

or $$\frac{x}{u}\frac{\partial u}{\partial x}+\frac{y}{u}\frac{\partial u}{\partial y}=1+3-3=1$$

Hence $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=u.$

Example.5. If $u=\tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$, then show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u.$

Sol. Given that $$u=\tan^{-1}\left(\frac{x^3+y^3}{x+y}\right) \qquad \text{...(1)}$$

We have $$\tan u=\frac{x^3+y^3}{x+y} \qquad \text{...(2)}$$

Now consider $$V=\frac{x^3+y^3}{x+y} \qquad \text{...(3)}$$

Here V is a homogenous function of x and y of degree 2. Therefore, by Euler's theorem, we have

$$x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}=2V \qquad \text{...(4)}$$

From equations (2) and (3), we have

$$V=\tan u \qquad \text{...(5)}$$

Now differentiating partially equation (5) with respect to x and y respectively, we get

$$\frac{\partial V}{\partial x}=\sec^2 u\,\frac{\partial u}{\partial x} \qquad \text{and} \qquad \frac{\partial V}{\partial y}=\sec^2 u\,\frac{\partial u}{\partial y}$$

Putting these values of $\frac{\partial V}{\partial x}$ and $\frac{\partial V}{\partial y}$ in equation (4), we get

$$x\sec^2 u\,\frac{\partial u}{\partial x}+y\sec^2 u\,\frac{\partial u}{\partial y}=2\tan u$$

or $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\frac{\tan u}{\sec^2 u}$$

or $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\sin u\cos u$$

or $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u$$

Hence $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u.$$

Example.6. If V is a homogeneous function of x and y of degree n and $V=f(u)$ then $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}$.

Sol. Given that $$V=f(u) \qquad \ldots(1)$$

It is given that V is a homogeneous function of x and y of degree n, then by then by Euler's theorem, we have

$$x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}=nV \qquad \ldots(2)$$

Differentiating partially equation (1) with respect to x and y respectively, we have

$$\frac{\partial V}{\partial x}=f'(u)\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial V}{\partial y}=f'(u)\frac{\partial u}{\partial y}$$

Putting these values of $\frac{\partial V}{\partial x}$ and $\frac{\partial V}{\partial y}$ in equation (2), we get

$$xf'(u)\frac{\partial u}{\partial x}+yf'(u)\frac{\partial u}{\partial y}=nf(u)$$

or $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}$$

Hence $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}.$$

Example.7. Verify Euler's theorem for the function $f(x,y)=\frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}$.

Sol. Given that $$f(x,y)=\frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}} \qquad \ldots(1)$$

Here $f(x,y)$ is a homogenous function of x and y of degree (1/20). Therefore, by Euler's theorem, we have

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=\frac{1}{20}f \qquad \dots (2)$$

Now verify to Euler theorem, we have

$$\frac{\partial f}{\partial x}=\frac{\left(x^{1/5}+y^{1/5}\right)\left(\frac{1}{4}x^{-3/4}\right)-\left(x^{1/4}+y^{1/4}\right)\left(\frac{1}{5}x^{-4/5}\right)}{\left(x^{1/5}+y^{1/5}\right)^2} \qquad \dots (3)$$

and $$\frac{\partial f}{\partial y}=\frac{\left(x^{1/5}+y^{1/5}\right)\left(\frac{1}{4}y^{-3/4}\right)-\left(x^{1/4}+y^{1/4}\right)\left(\frac{1}{5}y^{-4/5}\right)}{\left(x^{1/5}+y^{1/5}\right)^2} \qquad \dots (4)$$

Multiply equation (3) by x and equation (4) by y and adding them, we get

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=\frac{1}{20}f$$

Hence Euler's theorem is verified.

Example.8. If $u=x\sin^{-1}\left(\frac{y}{x}\right)$ then prove that $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0.$

Sol. Given that $$u=x\sin^{-1}\left(\frac{y}{x}\right) \qquad \dots(1)$$

Differentiating partially equation (1) with respect to x and y respectively, we have

$$\frac{\partial u}{\partial x}=x\frac{1}{\sqrt{1-\frac{y^2}{x^2}}}\left(-\frac{y}{x^2}\right)+\sin^{-1}\left(\frac{y}{x}\right)$$

or $$x\frac{\partial u}{\partial x}=\frac{-xy}{\sqrt{x^2-y^2}}+x\sin^{-1}\left(\frac{y}{x}\right) \qquad \dots(2)$$

Similarly, we get

$$y\frac{\partial u}{\partial y}=\frac{xy}{\sqrt{x^2-y^2}} \qquad \dots(3)$$

From equations (2) and (3), we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=x\sin^{-1}\left(\frac{y}{x}\right) \qquad \dots(4)$$

Differentiating partially equation (4) with respect to x and y respectively, we have

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=\frac{-y}{\sqrt{x^2-y^2}}+\sin^{-1}\left(\frac{y}{x}\right) \qquad \dots (5)$$

and $$x\frac{\partial^2 u}{\partial y \partial x}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=\frac{x}{\sqrt{x^2-y^2}} \quad \ldots (6)$$

Multiply equation (5) by x and equation (6) by y and adding them, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x \partial y}+y^2\frac{\partial^2 u}{\partial y^2}+\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=x\sin^{-1}\left(\frac{y}{x}\right)$$

Using equation (4), we have

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x \partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0.$$

7.6 TOTAL DIFFERENTIAL COEFFICIENT

Suppose $$u=f(x,y) \quad \ldots (1)$$

where $$x=\phi(t), y=\psi(t) \quad \ldots (2)$$

then u can be expressed as a function of a single variables t, if we put for x and y and in $u=f(x,y)$ and then the derivative of u with respect to t, is the ordinary differential coefficient $\frac{du}{dt}$. This $\frac{du}{dt}$ is called the total derivative of u with respect to t is known as the total derivative of u is given by

$$\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}$$

If $u=f(x,y,z)$ and $x=x(t), y=y(t), z=z(t)$ then the total derivative of u is

$$\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}+\frac{\partial u}{\partial z}\frac{dz}{dt}$$

Suppose $u=f(x,y,z)$ and let y and z are the function of x, u is a function of one independent variable x, we have

$$\frac{du}{dx}=\frac{\partial u}{\partial x}\frac{dx}{dx}+\frac{\partial u}{\partial y}\frac{dy}{dx}+\frac{\partial u}{\partial z}\frac{dz}{dx}$$

or $$\frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{dy}{dx}+\frac{\partial u}{\partial z}\frac{dz}{dx}.$$

Examples

Example.9. If $u=x^3+y^3$ where $x=a\cos t$ and $y=b\sin t$ then find $\frac{du}{dt}$.

Sol. Given that $$u=x^3+y^3 \quad \ldots(1)$$

Also $x = a\cos t$ …(2)

and $y = b\sin t$ …(3)

Differentiating partially equation (1) with respect to x and y respectively, we get

$$\frac{\partial u}{\partial x} = 3x^2 \quad \text{and} \quad \frac{\partial u}{\partial y} = 3y^2 \qquad \text{….(4)}$$

Differentiating equations (2) and (3) with respect to x and y respectively, we have

$$\frac{dx}{dt} = -a\sin t \quad \text{and} \quad \frac{dy}{dt} = b\cos t$$

We know that the total derivative of u with respect to t is

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

or $$\frac{du}{dt} = 3x^2(-a\sin t) + 3y^2(b\cos t)$$

or $$\frac{du}{dt} = 3\left(-ax^2\sin t + by^2\cos t\right).$$

7.1 IMPLICIT FUNCTION

Let us consider the implicit function $f(x, y) = 0.$

Here y is some function of x.

Consider $u = f(x, y)$, where $u = 0$, then we have

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{du}{dy} + \frac{dy}{dx} = 0$$

or $$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{dy}{dx} = 0$$

or $$\frac{dy}{dx} = \frac{-\partial u/\partial x}{\partial u/\partial y} \qquad \text{… (1)}$$

Now if $z = f(x, y)$ and $x = x(u, v),\ y = y(u, v),\ i.e.,\ x, y$ are function of u and v, more than one variable. Then we have

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

and $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \qquad \text{… (2)}$$

Note 1: If $f(x, y, z) = 0$ i.e., if z is an implicit function of x and y then by equation (2), we have

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}.\frac{\partial z}{\partial x}=0$$

and
$$\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}.\frac{\partial z}{\partial y}=0$$

Hence $\frac{\partial z}{\partial x}=-\frac{\partial f/\partial x}{\partial f/\partial z}$ and $\frac{\partial z}{\partial y}=-\frac{\partial f/\partial y}{\partial f/\partial z}$.

Note 2: If $f(x,y)=0$ then we have $\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}$.

Differentiating again with respect to x, we get

$$\frac{d^2y}{dx^2}=-\frac{\frac{\partial f}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y\partial x}.\frac{dy}{dx}\right)-\frac{\partial f}{\partial x}\left(\frac{\partial^2 f}{\partial x\partial y}+\frac{\partial^2 f}{\partial y^2}.\frac{dy}{dx}\right)}{\left(\frac{\partial f}{\partial y}\right)^2}$$

$$=-\frac{\frac{\partial^2 f}{\partial x^2}\left(\frac{\partial f}{\partial y}\right)-2\frac{\partial^2 f}{\partial x\partial y}.\frac{\partial f}{\partial x}.\frac{\partial f}{\partial y}+\frac{\partial^2 f}{\partial y^2}\left(\frac{\partial f}{\partial x}\right)^2}{\left(\frac{\partial f}{\partial y}\right)^3}.$$

Examples

Example.10. If the curve $f(x,y)=0$ and $\phi(x,y)=0$ touch each other then show that $\frac{\partial f}{\partial x}\frac{\partial \phi}{\partial y}-\frac{\partial f}{\partial y}\frac{\partial \phi}{\partial x}=0.$

Sol. Given that $f(x,y)=0$ …(1)

and $\phi(x,y)=0$ …(2)

From equation (1), we get

$$\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}. \qquad ….(3)$$

From equation (2), we get

$$\frac{dy}{dx}=-\frac{\partial \phi/\partial x}{\partial \phi/\partial y}. \qquad ….(4)$$

At the point of contact the two values of $\frac{dy}{dx}$ will be the same i.e., from equation (3) and (4), we have

$$-\frac{\partial f/\partial x}{\partial f/\partial y}=-\frac{\partial \phi/\partial x}{\partial \phi/\partial y}$$

or
$$\frac{\partial f}{\partial x}\frac{\partial \phi}{\partial y}=\frac{\partial f}{\partial y}\frac{\partial \phi}{\partial x}$$

or $$\frac{\partial f}{\partial x}\frac{\partial \phi}{\partial y}-\frac{\partial f}{\partial y}\frac{\partial \phi}{\partial x}=0.$$

Example.11. Prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial \xi^2}+\frac{\partial^2 u}{\partial \eta^2}$, where $x=\xi\cos\alpha-\eta\sin\alpha$, $y=\xi\sin\alpha+\eta\cos\alpha$.

Sol. Let $u=f(x,y)$ … (1)

Given that $x=\xi\cos\alpha-\eta\sin\alpha,\ \ y=\xi\sin\alpha+\eta\cos\alpha.$

From equation (1), we have

$$\frac{\partial u}{\partial \xi}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \xi}$$

or $$=\frac{\partial u}{\partial x}(\cos\alpha)+\frac{\partial u}{\partial y}(\sin\alpha) \qquad \dots (2)$$

and $$\frac{\partial u}{\partial \eta}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \eta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \eta}$$

or $$=\frac{\partial u}{\partial x}(-\sin\alpha)+\frac{\partial u}{\partial y}(\cos\alpha) \qquad \dots (3)$$

Differentiating partially equations (2) and (3) with respect to x and y respectively, we get

$$\frac{\partial^2 u}{\partial \xi^2}=cos\,\alpha\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial \xi}+sin\,\alpha\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial \xi}$$

$$=cos^2\alpha\frac{\partial^2 u}{\partial x^2}+sin^2\alpha\frac{\partial^2 u}{\partial y^2} \qquad \dots (4)$$

and $$\frac{\partial^2 u}{\partial \eta^2}=-sin\,\alpha\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial \eta}+cos\,\alpha\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial \eta}$$

$$=sin^2\alpha\frac{\partial^2 u}{\partial x^2}+cos^2\alpha\frac{\partial^2 u}{\partial y^2} \qquad \dots (5)$$

From equations (4) and (5), we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial \xi^2}+\frac{\partial^2 u}{\partial \eta^2}.$$

7.8 SUMMARY

1. A function $u(x,y)$ is said to be a homogenous function if all its terms are of the same degree.

2. If u is a homogenous function of x and y of degree n, then we have $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu.$

3. If $u=f(x,y,z)$ and $x=x(t),\ y=y(t),\ z=z(t)$ then the total derivative of u is

$$\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}+\frac{\partial u}{\partial z}\frac{dz}{dt}.$$

4. Suppose $u=f(x,y,z)$ and let y and z are the function of x, u is a function of one independent variable x, we have $\frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{dy}{dx}+\frac{\partial u}{\partial z}\frac{dz}{dx}.$

7.9 TERMINAL QUESTIONS

Q.1 State and prove Euler's theorem.

Q.2 Verify the Euler's theorem in the following cases:

$(i)\ u=axy+byz+czx$

$(ii)\ u=x^n\log(y/x)$

$(iii)\ u=ax^2+by^2+2hxy$

$(iv)\ u=x^n\sin(y/x)$

$(v)\ u=3x^2yz+5xy^2z+4z^4.$

Q.3 If $u=\sin^{-1}\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$, then show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0.$

Q.4 If $u=xyf(y/x)$, then show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u.$

Q.5 If $u=\sin^{-1}\frac{x-y}{\sqrt{x}+\sqrt{y}}$, then show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\tan u.$

Q.6 If $u=\tan^{-1}\frac{\sqrt{x^2+y^2}}{x-y}$, then show that $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0.$

Q.7 If $\sin u=\frac{x^2y^2}{x+y}$, then show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3\tan u.$

Q.8 If $u=\log\frac{x^4+y^4}{x+y}$, then show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3.$

Q.9 If $u = \log \frac{x^3 + y^3}{x + y}$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2.$

Q.10 If $u = \sec^{-1}\left(\frac{x^3 + y^3}{x + y}\right)$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cot u.$

Q.11 If u be a homogeneous function of x and y of degree n, then show that

$(i)\ x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = (n-1)\frac{\partial u}{\partial x}.$

$(ii)\ x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y}.$

$(iii)\ x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.$

Q.12 If $u = f(y - z, z - x, x - y)$, then show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$

Q.13 If $x^y + y^x = a^b$, then show that $\frac{dy}{dx} = 0.$

Q.14 If $\sqrt{1 - x^2} + \sqrt{1 - y^2} = a(x - y)$, then show that $\frac{dy}{dx} = \frac{\sqrt{1 - y^2}}{\sqrt{1 - x^2}}.$

Q.15 If $u(x, y, z) = \frac{ax^2 + by^2 + cz^2}{ax + by + cz}$ then show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = u(x, y, z).$$

UNIT 8 JACOBIANS

Structure

8.1 Introduction

8.2 Objectives

8.3 Jacobian

8.4 Jacobian of Functions of Functions

8.5 Jacobian of Implicit Functions

8.6 Necessary and Sufficient Condition for a Jacobian to Vanish

8.7 Summary

8.8 Terminal Questions

8.1 INTRODUCTION

The Jacobian is a mathematical concept pivotal in calculus and linear algebra. It's a matrix composed of first-order partial derivatives. In essence, it elucidates how changes in one set of variables influence another set within a function. For a function with n variables, the Jacobian matrix comprises n rows and m columns (where m denotes the number of functions in the output). Each entry signifies the partial derivative of one function concerning one variable. Jacobian matrices find extensive applications in physics, engineering, economics, and other disciplines. They facilitate solving systems of equations, stability analysis, function optimization, and more.

The Jacobian matrix plays a central role in the Implicit Function Theorem, which provides conditions under which implicit functions can be differentiated. This theorem is essential in various areas of mathematics, including differential geometry, optimization, and differential equations. When dealing with functions of several variables, understanding how small changes in the input variables affect the output is crucial. The Jacobian provides this information through its partial derivatives, aiding in optimization, integration, and curve/surface fitting.

8.2 OBJECTIVES

After reading this unit the learner should be able to understand about:

- the Jacobian
- the Jacobian of Functions of Functions
- the Jacobian of Implicit Functions
- the Necessary and Sufficient Condition for a Jacobian to Vanish

8.3 JACOBIAN

If u and v are functions of two independent variables x and y then the jacobian of u and v with respect to x and y is

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= J(u,v).$$

Similarly, if u, v and w are the functions of three independent variables x, y and z, then the jocobian of u, v and w with respect to the independent variables x, y and z is

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= J(u,v,w).$$

If $u_1, u_2, u_3, \ldots\ldots\ldots\ldots, u_{n-1}, u_n$ are the function of n independent variables $x_1, x_2, x_3, \ldots\ldots\ldots\ldots, x_{n-1}, x_n$ then then the jocobian of $u_1, u_2, u_3, \ldots\ldots\ldots\ldots, u_{n-1}, u_n$ with respect to the independent variables $x_1, x_2, x_3, \ldots\ldots\ldots\ldots, x_{n-1}, x_n$ is

$$\frac{\partial(u_1,u_2,u_3,.........,u_{n-1},u_n)}{\partial(x_1,x_2,x_3,.........,x_{n-1},x_n)}=\begin{vmatrix}\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} .... \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} .... \frac{\partial u_2}{\partial x_n} \\ ..... & ..... \quad ..... \quad ..... \quad .... \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} .... \frac{\partial u_n}{\partial x_n}\end{vmatrix}$$

Note: If the functions $u_1,u_2,u_3,..........,u_{n-1},u_n$ of n independent variables $x_1,x_2,x_3,.........,x_{n-1},x_n$ are in the following forms:

$u_1=f_1(x_1),\ u_2=f_2(x_1,x_2),........,u_n=f_n(x_1,x_2,.......,x_n)$, then we have

$$\frac{\partial(u_1,u_2,u_3,.........,u_{n-1},u_n)}{\partial(x_1,x_2,x_3,.........,x_{n-1},x_n)}=\begin{vmatrix}\frac{\partial u_1}{\partial x_1} & 0 & 0 & ...... & 0 \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & 0 & ...... & 0 \\ ...... & .... & .... & .... & ..... \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial x_3} & .... & \frac{\partial u_n}{\partial x_n}\end{vmatrix}$$

$$=\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2}\frac{\partial u_3}{\partial x_3}...... ....\frac{\partial u_n}{\partial x_n}$$

Examples

Example.1. If $x=r\cos\theta,\ y=r\sin\theta$ show that $\frac{\partial(x,y)}{\partial(r,\theta)}=r.$

Sol. Given that

$$x=r\cos\theta,\ y=r\sin\theta$$

If x and y are functions of two independent variables r and θ then the jacobian of x and y with respect to r and θ is

$$\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta}\end{vmatrix}$$

$$=\begin{vmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{vmatrix}$$

$$= r\left(\cos^2\theta + \sin^2\theta\right)$$

$$= r..$$

Example.2. If $x = r\sin\theta\cos\varphi,\ y = r\sin\theta\sin\varphi,\ z = r\cos\theta,$ show that $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = r^2\sin\theta.$

Sol. Given that

$$x = r\sin\theta\cos\varphi,\ y = r\sin\theta\sin\varphi,\ z = r\cos\theta.$$

The jacobian of x, y and z with respect to r,θ and ϕ is

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial\theta} & \dfrac{\partial x}{\partial\varphi} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial\theta} & \dfrac{\partial y}{\partial\varphi} \\ \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial\theta} & \dfrac{\partial z}{\partial\varphi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \cos\theta\left(r^2\sin\theta\cos\theta\cos^2\varphi + r^2\sin\theta\cos\theta\sin^2\varphi\right)$$
$$+ r\sin\theta\left(r\sin^2\theta\cos^2\varphi + r\sin^2\theta\sin^2\varphi\right)$$

$$= r^2\sin\theta\cos^2\theta + r^2\sin^3\theta$$

$$= r^2\sin\theta\left(\cos^2\theta + \sin^2\theta\right)$$

$$= r^2\sin\theta\,.$$

Example.4. To show that the Jacobian of x, y and z with respect to r,θ and ϕ is

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \frac{r^2\left(m^2\cos^2\varphi + n^2\cos^2\theta\right)}{\sqrt{\left[\left(1-n^2\sin^2\theta\right)\left(1-m^2\sin^2\varphi\right)\right]}}$$ **, given that**

$x = r\cos\theta\cos\phi, y = r\sin\theta\sqrt{(1-m^2\sin^2\phi)},\ z = r\sin\phi\sqrt{(1-n^2\sin^2\theta)},$ **where** $m^2 + n^2 = 1$**.**

Sol. Given that

$$x = r\cos\theta\cos\phi, y = r\sin\theta\left(1-m^2\sin^2\phi\right),\ z = r\sin\phi\left(1-n^2\sin^2\theta\right),$$

We have

$$x^2+y^2+z^2=r^2\cos^2\theta\cos^2\varphi+r^2\sin^2\theta-r^2m^2\sin^2\theta\sin^2\varphi$$
$$+r^2\sin^2\varphi-r^2n^2\sin^2\varphi\sin^2\theta$$

$$x^2+y^2+z^2=r^2\left(\cos^2\theta\cos^2\varphi+\sin^2\theta+\sin^2\varphi-\sin^2\theta\sin^2\varphi\right) \qquad \left[\because m^2+n^2=1\right]$$

$$=r^2\left(\cos^2\theta\cos^2\varphi+\sin^2\theta+\sin^2\varphi\cos^2\theta\right)$$

$$=r^2\left(\sin^2\theta+\cos^2\theta\right).$$

$$=r^2.$$

and
$$\left.\begin{aligned}\therefore x\frac{\partial x}{\partial r}+y\frac{\partial y}{\partial r}+z\frac{\partial z}{\partial r}&=r\\ x\frac{\partial x}{\partial\theta}+y\frac{\partial y}{\partial\theta}+z\frac{\partial z}{\partial\theta}&=0\\ x\frac{\partial x}{\partial\varphi}+y\frac{\partial y}{\partial\varphi}+z\frac{\partial z}{\partial\varphi}&=0\end{aligned}\right\} \qquad \ldots..(1)$$

The jacobian of x, y and z with respect to r,θ and ϕ is

$$J(x,y,z)=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\varphi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\varphi}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\varphi}\end{vmatrix}$$

$$=\frac{1}{x}\begin{vmatrix}x\frac{\partial x}{\partial r} & x\frac{\partial x}{\partial\theta} & x\frac{\partial x}{\partial\varphi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\varphi}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\varphi}\end{vmatrix}$$

$$=\frac{1}{x}\begin{vmatrix}r & 0 & 0\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\varphi}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\varphi}\end{vmatrix}$$, by adding yR_2+zR_2 to R_1 and using the equations (1)

$$=\frac{r}{x}\begin{vmatrix}\frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\varphi}\\ \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\varphi}\end{vmatrix}$$

$$=\frac{r}{x}\left\{\frac{\partial y}{\partial\theta}\frac{\partial z}{\partial\varphi}-\frac{\partial z}{\partial\theta}\frac{\partial y}{\partial\varphi}\right\}$$

$$=\frac{r}{x}\left\{\begin{array}{l} r\cos\theta\sqrt{\left(1-m^2\sin^2\varphi\right)}.r\cos\varphi\sqrt{\left(1-n^2\sin^2\theta\right)}\\ -\frac{r\sin\varphi.n^2\sin\theta\cos\theta}{\sqrt{\left(1-n^2\sin^2\theta\right)}}.\frac{r\sin\theta.m^2\sin\varphi\cos\varphi}{\sqrt{\left(1-m^2\sin^2\varphi\right)}}\end{array}\right\}$$

$$=\frac{r^3\cos\theta\cos\varphi}{x}\left[\frac{\left(1-m^2\sin^2\varphi\right)\left(1-n^2\sin^2\theta\right)-n^2m^2\sin^2\theta\sin^2\varphi}{\sqrt{\left[\left(1-n^2\sin^2\theta\right)\left(1-m^2\sin^2\varphi\right)\right]}}\right]$$

$$=\frac{r^3\cos\theta\cos\varphi}{r\cos\theta\cos\varphi}\left[\frac{\begin{array}{l}1-m^2\sin^2\varphi-n^2\sin^2\theta+m^2n^2\sin^2\varphi\sin^2\theta\\ -m^2n^2\sin^2\theta\sin^2\varphi\end{array}}{\sqrt{\left[\left(1-n^2\sin^2\theta\right)\left(1-m^2\sin^2\varphi\right)\right]}}\right]$$

$$=\frac{r^2\left(m^2\cos^2\varphi+n^2\cos^2\theta\right)}{\sqrt{\left[\left(1-n^2\sin^2\theta\right)\left(1-m^2\sin^2\varphi\right)\right]}}. \qquad \left[\because m^2+n^2=1\right]$$

Example.4. If $y_1=r\sin\theta_1\sin\theta_2,\ y_2=r\sin\theta_1\cos\theta_2,\ y_3=r\cos\theta_1\sin\theta_3,$ $y_4=r\cos\theta_1\cos\theta_3$ then find show that $\frac{\partial\left(y_1,y_2,y_3,y_4\right)}{\partial\left(r,\theta_1,\theta_2,\theta_3\right)}=r^3\sin\theta_1\cos\theta_1.$

Sol. Given that

$$y_1=r\sin\theta_1\sin\theta_2,\ y_2=r\sin\theta_1\cos\theta_2,\ y_3=r\cos\theta_1\sin\theta_3,$$
$$y_4=r\cos\theta_1\cos\theta_3.$$

Squaring and adding the above given relations, we have

$$y_1^{\ 2}+y_2^{\ 2}+y_3^{\ 2}+y_4^{\ 2}=r^2.$$

and
$$\left.\begin{array}{l}\therefore y_1\frac{\partial y_1}{\partial r}+y_2\frac{\partial y_2}{\partial r}+y_3\frac{\partial y_3}{\partial r}+y_4\frac{\partial y_4}{\partial r}=r\\ y_1\frac{\partial y_1}{\partial\theta_n}+y_2\frac{\partial y_2}{\partial\theta_n}+y_3\frac{\partial y_3}{\partial\theta_n}+y_4\frac{\partial y_4}{\partial\theta_n}=0;\ n=1,2,3\end{array}\right\} \qquad .....(1)$$

Also $y_3{}^2 + y_4{}^2 = r^2\cos^2\theta_1$, so that

$$\left.\begin{aligned} y_3\frac{\partial y_3}{\partial\theta_1} + y_4\frac{\partial y_4}{\partial\theta_1} &= -r^2\cos\theta_1\sin\theta_1; \\ y_3\frac{\partial y_3}{\partial\theta_n} + y_4\frac{\partial y_4}{\partial\theta_n} &= 0,\ \ n = 2, 3. \end{aligned}\right\} \qquad \ldots..(2)$$

The Jacobian of $y_1,\ y_2, y_3,\ y_4$ with respect to $r,\ \theta_1,\ \theta_2,\ \theta_3$ is

$$J = \begin{vmatrix} \frac{\partial y_1}{\partial r} & \frac{\partial y_1}{\partial\theta_1} & \frac{\partial y_1}{\partial\theta_2} & \frac{\partial y_1}{\partial\theta_3} \\ \frac{\partial y_2}{\partial r} & \frac{\partial y_2}{\partial\theta_1} & \frac{\partial y_2}{\partial\theta_2} & \frac{\partial y_2}{\partial\theta_3} \\ \frac{\partial y_3}{\partial r} & \frac{\partial y_3}{\partial\theta_1} & \frac{\partial y_3}{\partial\theta_2} & \frac{\partial y_3}{\partial\theta_3} \\ \frac{\partial y_4}{\partial r} & \frac{\partial y_4}{\partial\theta_1} & \frac{\partial y_4}{\partial\theta_2} & \frac{\partial y_4}{\partial\theta_3} \end{vmatrix}$$

Applying $y_1R_1 + \left(y_2R_2 + y_3R_3 + y_4R_4\right)$. and using the equation (1), we get

$$= \frac{1}{y_1}\begin{vmatrix} r & 0 & 0 & 0 \\ \frac{\partial y_2}{\partial r} & \frac{\partial y_2}{\partial\theta_1} & \frac{\partial y_2}{\partial\theta_2} & \frac{\partial y_2}{\partial\theta_3} \\ \frac{\partial y_3}{\partial r} & \frac{\partial y_3}{\partial\theta_1} & \frac{\partial y_3}{\partial\theta_2} & \frac{\partial y_3}{\partial\theta_3} \\ \frac{\partial y_4}{\partial r} & \frac{\partial y_4}{\partial\theta_1} & \frac{\partial y_4}{\partial\theta_2} & \frac{\partial y_4}{\partial\theta_3} \end{vmatrix}$$

$$= \frac{r}{y_1}\begin{vmatrix} \frac{\partial y_2}{\partial\theta_1} & \frac{\partial y_2}{\partial\theta_2} & \frac{\partial y_2}{\partial\theta_3} \\ \frac{\partial y_3}{\partial\theta_1} & \frac{\partial y_3}{\partial\theta_2} & \frac{\partial y_3}{\partial\theta_3} \\ \frac{\partial y_4}{\partial\theta_1} & \frac{\partial y_4}{\partial\theta_2} & \frac{\partial y_4}{\partial\theta_3} \end{vmatrix}$$

$$= \frac{r}{y_1y_3}\begin{vmatrix} \frac{\partial y_2}{\partial\theta_1} & \frac{\partial y_2}{\partial\theta_2} & \frac{\partial y_2}{\partial\theta_3} \\ -r^2\cos\theta_1\sin\theta_1 & 0 & 0 \\ \frac{\partial y_4}{\partial\theta_1} & \frac{\partial y_4}{\partial\theta_2} & \frac{\partial y_4}{\partial\theta_3} \end{vmatrix}$$

Now adding $y_4 R_3$ to $y_3 R_2$ and using the equation (2), we get

$$= \frac{r}{y_1 y_3} . r^2 \cos\theta_1 \sin\theta_1 \left[\frac{\partial y_2}{\partial \theta_2} . \frac{\partial y_4}{\partial \theta_3} - \frac{\partial y_4}{\partial \theta_2} . \frac{\partial y_2}{\partial \theta_3} \right]$$

$$= \frac{r^3 \cos\theta_1 \sin\theta_1}{y_1 y_3} \left[\left(-r \sin\theta_1 \sin\theta_2 \right) \left(-r \cos\theta_1 \sin\theta_3 \right) - 0 \right]$$

$$= \frac{r^5 \sin^2\theta_1 \cos^2\theta_1 \sin\theta_2 \sin\theta_3}{r^2 \sin\theta_1 \cos\theta_1 \sin\theta_2 \sin\theta_3}$$

$$= r^3 \sin\theta_1 \cos\theta_1 .$$

8.4 JACOBIAN OF FUNCTIONS OF FUNCTIONS

If u_1, u_2 are the functions of y_1, y_2 and y_1, y_2 are the functions x_1, x_2 then

$$\frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} = \frac{\partial(u_1, u_2)}{\partial(y_1, y_2)} . \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} .$$

Proof. We have

$$\left. \begin{aligned} \frac{\partial u_1}{\partial x_1} &= \frac{\partial u_1}{\partial y_1} \frac{\partial y_1}{\partial x_1} + \frac{\partial u_1}{\partial y_2} \frac{\partial y_2}{\partial x_1} , \\ \frac{\partial u_1}{\partial x_2} &= \frac{\partial u_1}{\partial y_1} \frac{\partial y_1}{\partial x_2} + \frac{\partial u_1}{\partial y_2} \frac{\partial y_2}{\partial x_2} , \\ \frac{\partial u_2}{\partial x_1} &= \frac{\partial u_2}{\partial y_1} \frac{\partial y_1}{\partial x_1} + \frac{\partial u_2}{\partial y_2} \frac{\partial y_2}{\partial x_1} , \\ \frac{\partial u_2}{\partial x_2} &= \frac{\partial u_2}{\partial y_1} \frac{\partial y_1}{\partial x_2} + \frac{\partial u_2}{\partial y_2} \frac{\partial y_2}{\partial x_2} \end{aligned} \right\} \qquad \ldots..(1)$$

Now we have

$$\frac{\partial(u_1, u_2)}{\partial(y_1, y_2)} . \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} = \begin{vmatrix} \frac{\partial u_1}{\partial y_1} & \frac{\partial u_1}{\partial y_2} \\ \frac{\partial u_2}{\partial y_1} & \frac{\partial u_2}{\partial y_2} \end{vmatrix} \times \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u_1}{\partial y_1} \frac{\partial y_1}{\partial x_1} + \frac{\partial u_1}{\partial y_2} \frac{\partial y_2}{\partial x_1} & \frac{\partial u_1}{\partial y_1} \frac{\partial y_1}{\partial x_2} + \frac{\partial u_1}{\partial y_2} \frac{\partial y_2}{\partial x_2} \\ \frac{\partial u_2}{\partial y_1} \frac{\partial y_1}{\partial x_1} + \frac{\partial u_2}{\partial y_2} \frac{\partial y_2}{\partial x_1} & \frac{\partial u_2}{\partial y_1} \frac{\partial y_1}{\partial x_2} + \frac{\partial u_2}{\partial y_2} \frac{\partial y_2}{\partial x_2} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} \end{vmatrix}, \text{ using the equation (1)}$$

$$= \frac{\partial(u_1, u_2)}{\partial(x_1, x_2)}.$$

Note. If $u_1, u_2, \ldots\ldots\ldots, u_{n-1}, u_n$ are functions of $y_1, y_2, \ldots\ldots\ldots, y_{n-1}, y_n$ and $y_1, y_2, \ldots\ldots\ldots, y_{n-1}, y_n$ are functions of $x_1, x_2, \ldots\ldots\ldots, x_{n-1}, x_n$, then we have

$$\frac{\partial(u_1, u_2, \ldots\ldots\ldots, u_{n-1}, u_n)}{\partial(x_1, x_2, \ldots\ldots, x_{n-1}, x_n)} = \frac{\partial(u_1, u_2, \ldots\ldots\ldots, u_{n-1}, u_n)}{\partial(y_1, y_2, \ldots\ldots, y_{n-1}, y_n)} \cdot \frac{\partial(y_1, y_2, \ldots\ldots, y_{n-1}, y_n)}{\partial(x_1, x_2, \ldots\ldots, x_{n-1}, x_n)}.$$

8.5 JACOBIAN OF IMPLICIT FUNCTIONS

Suppose $u_1, u_2, u_3, \ldots\ldots\ldots\ldots, u_{n-1}, u_n$ instead of being given explicitly in terms of $x_1, x_2, x_3, \ldots\ldots\ldots\ldots, x_{n-1}, x_n$ are connected with them by equations such as

$$F_1(u_1, u_2, u_3, \ldots\ldots, u_{n-1}, u_n, x_1, x_2, x_3, \ldots\ldots, x_{n-1}, x_n) = 0$$

$$F_2(u_1, u_2, u_3, \ldots\ldots, u_{n-1}, u_n, x_1, x_2, x_3, \ldots\ldots, x_{n-1}, x_n) = 0$$

….. …… ……. …….

$$F_n(u_1, u_2, u_3, \ldots\ldots, u_{n-1}, u_n, x_1, x_2, x_3, \ldots\ldots, x_{n-1}, x_n) = 0$$

Then, we have

$$\frac{\partial(u_1, u_2, u_3, \ldots\ldots\ldots\ldots, u_{n-1}, u_n)}{\partial(x_1, x_2, x_3, \ldots\ldots, x_{n-1}, x_n)} = (-1)^n \frac{\dfrac{\partial(F_1, F_2, F_3, \ldots\ldots\ldots, F_{n-1}, F_n)}{\partial(x_1, x_2, x_3, \ldots\ldots\ldots, x_{n-1}, x_n)}}{\dfrac{\partial(F_1, F_2, F_3, \ldots\ldots\ldots\ldots, F_{n-1}, F_n)}{\partial(u_1, u_2, u_3, \ldots\ldots\ldots\ldots, u_{n-1}, u_n)}}$$

Proof. We will now derive the outcome for two variables, with the proof readily extendable to n variables. Students are encouraged to craft their own proof for n variables using the framework presented below for two variables. In the case of two variables, the connecting relations are

$$\left.\begin{aligned} F_1(u_1, u_2, x_1, x_2) = 0 \\ F_2(u_1, u_2, x_1, x_2) = 0 \end{aligned}\right\} \qquad \ldots\ldots(1)$$

From equation (1), we have

$$\left.\begin{aligned}
\frac{\partial F_1}{\partial x_1}+\frac{\partial F_1}{\partial u_1}\frac{\partial u_1}{\partial x_1}+\frac{\partial F_1}{\partial u_2}\frac{\partial u_2}{\partial x_1}=0\\
\frac{\partial F_1}{\partial x_2}+\frac{\partial F_1}{\partial u_1}\frac{\partial u_1}{\partial x_2}+\frac{\partial F_1}{\partial u_2}\frac{\partial u_2}{\partial x_2}=0\\
\frac{\partial F_2}{\partial x_1}+\frac{\partial F_2}{\partial u_1}\frac{\partial u_1}{\partial x_1}+\frac{\partial F_2}{\partial u_2}\frac{\partial u_2}{\partial x_1}=0\\
\frac{\partial F_1}{\partial x_2}+\frac{\partial F_2}{\partial u_1}\frac{\partial u_1}{\partial x_2}+\frac{\partial F_2}{\partial u_2}\frac{\partial u_2}{\partial x_2}=0
\end{aligned}\right\} \qquad \ldots\ldots(2)$$

Now we have

$$\frac{\partial(F_1,F_2)}{\partial(u_1,u_2)}.\frac{\partial(u_1,u_2)}{\partial(x_1,x_2)}=\begin{vmatrix}\frac{\partial F_1}{\partial u_1} & \frac{\partial F_1}{\partial u_2}\\ \frac{\partial F_2}{\partial u_1} & \frac{\partial F_2}{\partial u_2}\end{vmatrix}\times\begin{vmatrix}\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2}\\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2}\end{vmatrix}$$

$$=\begin{vmatrix}\frac{\partial F_1}{\partial u_1}\frac{\partial u_1}{\partial x_1}+\frac{\partial F_1}{\partial u_1}\frac{\partial u_2}{\partial x_1} & \frac{\partial F_1}{\partial u_1}\frac{\partial u_1}{\partial x_2}+\frac{\partial F_1}{\partial u_2}\frac{\partial u_1}{\partial x_2}\\ \frac{\partial F_2}{\partial u_1}\frac{\partial u_1}{\partial x_1}+\frac{\partial F_2}{\partial u_2}\frac{\partial u_2}{\partial x_1} & \frac{\partial F_2}{\partial u_1}\frac{\partial u_1}{\partial x_2}+\frac{\partial F_2}{\partial u_2}\frac{\partial u_2}{\partial x_2}\end{vmatrix}$$

or $\dfrac{\partial(F_1,F_2)}{\partial(u_1,u_2)}.\dfrac{\partial(u_1,u_2)}{\partial(x_1,x_2)}=\begin{vmatrix}\frac{-\partial F_1}{\partial x_1} & \frac{-\partial F_1}{\partial x_2}\\ \frac{-\partial F_2}{\partial x_1} & \frac{-\partial F_2}{\partial x_2}\end{vmatrix}$, using the equation (2)

$$=(-1)^2\,\frac{\partial(F_1,F_2)}{\partial(x_1,x_2)}$$

Thus we have

$$\frac{\partial(u_1,u_2)}{\partial(x_1,x_2)}=(-1)^2\,\frac{\dfrac{\partial(F_1,F_2)}{\partial(x_1,x_2)}}{\dfrac{\partial(F_1,F_2)}{\partial(u_1,u_2)}}$$

Examples

Example.5. Prove that $\dfrac{\partial(u,v)}{\partial(x,y)}\times\dfrac{\partial(x,y)}{\partial(u,v)}=1$.

Sol. Consider $u=f_1(x,y),\ v=f_2(x,y)$. …..(1)

Differentiate partially equation (1) with respect to u and v, we get

$$\left.\begin{aligned}
1 &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u}, \\
0 &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v}, \\
0 &= \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial u}, \\
1 &= \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v}.
\end{aligned}\right\} \qquad \ldots.(2)$$

Now we have

$$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \text{ using the equation (2)}$$

$$= 1.$$

Example.6. If $u^3 + v + w = x + y^2 + z^2,\ u + v^3 + w = x^2 + y + z^2,\ u + v + w^3 = x^2 + y^2 + z$

then show that $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \dfrac{1 - 4(xy + yz + zx) + 16xyz}{2 - 3(u^2 + v^2 + w^2) + 27u^2v^2w^2}$.

Sol. The given relations can be written as

$$F_1 \equiv u^3 + v + w - x - y^2 - z^2 = 0,$$

$$F_2 \equiv u + v^3 + w - x^2 - y - z^2 = 0,$$

$$F_3 \equiv u + v + w^3 - x^2 - y^2 - z = 0.$$

We know that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\dfrac{\partial(F_1,F_2,F_3)}{\partial(x,y,z)}}{\dfrac{\partial(F_1,F_2,F_3)}{\partial(u,v,w)}} \qquad \ldots\ldots(1)$$

Here $\dfrac{\partial(F_1,F_2,F_3)}{\partial(x,y,z)}=\begin{vmatrix}-1 & -2y & -2z\\ -2x & -1 & -2z\\ -2x & -2y & -1\end{vmatrix}$

$$=-1(1-4yz)+2x(2y-4yz)-2x(4yz-2z)$$

$$=-1+4(yz+zx+xy)-16xyz.$$

and $\dfrac{\partial(F_1,F_2,F_3)}{\partial(u,v,w)}=\begin{vmatrix}3u^2 & 1 & 1\\ 1 & 3v^2 & 1\\ 1 & 1 & 3w^2\end{vmatrix}$

$$=3u^2(9v^2w^2-1)-1(3w^2-1)+.(1-3v^2)$$

$$=2-3(u^2+v^2+w^2)+27u^2v^2w^2.$$

From equation (1), we have

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}=\frac{1-4(yz+zx+xy)+16xyz}{2-3(u^2+v^2+w^2)+27u^2v^2w^2}.$$

8.6 NECESSARY AND SUFFICIENT CONDITION FOR A JACOBIAN TO VANISH

Let us consider $u_1,u_2,u_3,........,u_{n-1},u_n$ be functions of n independent variables $x_1,x_2,x_3,........,x_{n-1},x_n$. In order that these n functions may not be independent, *i.e,* there may exist between these n functions a relation

$$F\left(u_1,u_2,u_3,........,u_{n-1},u_n\right)=0 \qquad .....(1)$$

It is necessary and sufficient that the Jacobian $\dfrac{\partial\left(u_1,u_2,u_3,........,u_{n-1},u_n\right)}{\partial\left(x_1,x_2,x_3,........,x_{n-1},x_n\right)}$ should vanish identically.

Proof. The condition is necessary i.e., if there exists between $u_1,u_2,u_3,........,u_{n-1},u_n$ a relation

$$F\left(u_1,u_2,u_3,........,u_{n-1},u_n\right)=0 \qquad .....(1)$$

Differentiate partially equation (1) with respect to $x_1,x_2,x_3,........,x_{n-1},x_n$ we get

$$\frac{\partial F}{\partial u_1}\frac{\partial u_1}{\partial x_1}+\frac{\partial F}{\partial u_2}\frac{\partial u_2}{\partial x_1}+..........+\frac{\partial F}{\partial u_n}\frac{\partial u_n}{\partial x_1}=0,$$

$$\frac{\partial F}{\partial u_1}\frac{\partial u_1}{\partial x_2}+\frac{\partial F}{\partial u_2}\frac{\partial u_2}{\partial x_2}+..........+\frac{\partial F}{\partial u_n}\frac{\partial u_n}{\partial x_2}=0,$$

....

$$\frac{\partial F}{\partial u_1}\frac{\partial u_1}{\partial x_n}+\frac{\partial F}{\partial u_2}\frac{\partial u_2}{\partial x_n}+..........+\frac{\partial F}{\partial u_n}\frac{\partial u_n}{\partial x_n}=0.$$

Eliminating $\frac{\partial F}{\partial u_1},\frac{\partial F}{\partial u_2},\frac{\partial F}{\partial u_3},..........,\frac{\partial F}{\partial u_n}$ from these equations, we get

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & .... & \frac{\partial u_n}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & .... & \frac{\partial u_n}{\partial x_2} \\ .... & .... & .... & ...... \\ \frac{\partial u_1}{\partial x_n} & \frac{\partial u_2}{\partial x_n} & ...... & \frac{\partial u_n}{\partial x_n} \end{vmatrix}=0$$

or $$\frac{\partial\left(u_1,u_2,u_3,........,u_{n-1},u_n\right)}{\partial\left(x_1,x_2,x_3,........,x_{n-1},x_n\right)}=0$$

The condition is sufficient, *i.e.*, if the Jacobian $J\left(u_1,u_2,u_3,........,u_{n-1},u_n\right)$ is zero, then there must exist a relation between $u_1,u_2,u_3,........,u_{n-1},u_n$.

The equations connecting the functions $u_1,u_2,u_3,........,u_{n-1},u_n$ and the variables $x_1,x_2,x_3,........,x_{n-1},x_n$ are always capable of being put into the following form:

$$\begin{aligned}
&\varphi_1\left(x_1,x_2,....,x_n,u_1\right)=0\\
&\varphi_2\left(x_2,x_3,.....,x_n,u_1,u_2\right)=0\\
&...\\
&\varphi_r\left(x_r,x_{r+1},......,x_n,u_1,u_2,......,u_r\right)=0\\
&...\\
&\varphi_n\left(x_n,u_1,u_2,......,u_n\right)=0.
\end{aligned}$$

Then, we have

$$J=\frac{\partial(u_1,u_2,....,u_n)}{\partial(x_1,x_2,....,x_n)}=(-1)^n\,\frac{\dfrac{\partial(\phi_1,\phi_2,....,\phi_n)}{\partial(x_1,x_2,....,x_n)}}{\dfrac{\partial(\phi_1,\phi_2,....,\phi_n)}{\partial(u_1,u_2,....,u_n)}}$$

$$=(-1)^n\,\frac{\dfrac{\partial\phi_1}{\partial x_1}\dfrac{\partial\phi_2}{\partial x_2}........\dfrac{\partial\phi_n}{\partial x_n}}{\dfrac{\partial\phi_1}{\partial u_1}\dfrac{\partial\phi_2}{\partial u_2}........\dfrac{\partial\phi_n}{\partial u_n}}.$$

Now, if $J=0$, we have

$$\frac{\partial \phi_1}{\partial x_1}\cdot\frac{\partial \phi_2}{\partial x_2}\ldots\ldots\ldots\frac{\partial \phi_r}{\partial x_r}\ldots\ldots\ldots\frac{\partial \phi_n}{\partial x_n}=0$$

i.e., $\frac{\partial \phi_r}{\partial x_r}=0$ for some value of r between 1 and n.

Hence, for that particular value of r the function ϕ_r must not contain x_r; and accordingly the corresponding equation is of the form

$$\varphi_r\left(x_{r+1}, x_{r+2}, \ldots\ldots, x_{n-1}, x_n, u_1, u_2, u_3, \ldots\ldots, u_{n-1}, u_n\right)=0.$$

Therefore, as a result of this and the remaining equations $\varphi_{r+1}=0,\ \varphi_{r+2}=0, \ldots\ldots, \varphi_n=0$ the variables $x_{r+1}, x_{r+2}, \ldots\ldots, x_n$ can be eliminated so as to give a final equation between $u_1, u_2, u_3, \ldots\ldots\ldots, u_{n-1}, u_n$ alone.

Examples

Example.7. Show that the functions $u=x+y-z,\quad v=x-y.+z,\quad w=x^2+y^2+z^2-2yz$ are not independent of one another. Also find the relation between them.

Sol. We know that the jocobian of u, v and w with respect to x, y and z is

$$\frac{\partial(u, v, w)}{\partial(x, y, z)}=\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$=\begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2x & 2(y-z) & 2(z-y) \end{vmatrix}$$

$$=\begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2x & 2(y-z) & 0 \end{vmatrix}, \text{ adding } C_2 \text{ to } C_3$$

$$=0.$$

Since the Jacobian is zero, the functions are not independent.

Now we have

$$u+v=2x \text{ and } u-v=2(y-z).$$

Therefore we have

$$(u+v)^2+(u-v)^2=4\left(x^2+y^2+z^2-2yz\right)$$
$$=4w.$$

Which is the required relation between $u, v, w.$

8.4 SUMMARY

1. If u and v are functions of two independent variables x and y then the jacobian of u and v with respect to x and y is

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}$$

$$=J(u,v).$$

2. If u,v and w are the functions of three independent variables x, y and z, then the jocobian of u,v and w with respect to the independent variables x, y and z is

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{vmatrix}$$

$$=J(u,v,w).$$

3. If $u_1, u_2, u_3, \ldots\ldots\ldots\ldots, u_{n-1}, u_n$ are the function of n independent variables $x_1, x_2, x_3, \ldots\ldots\ldots\ldots, x_{n-1}, x_n$ then then the jocobian of $u_1, u_2, u_3, \ldots\ldots\ldots\ldots, u_{n-1}, u_n$ with respect to the independent variables $x_1, x_2, x_3, \ldots\ldots\ldots, x_{n-1}, x_n$ is

$$\frac{\partial(u_1,u_2,u_3,\ldots\ldots\ldots,u_{n-1},u_n)}{\partial(x_1,x_2,x_3,\ldots\ldots\ldots,x_{n-1},x_n)}=\begin{vmatrix}\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2}\ldots\ldots\ldots\ldots\ldots & \frac{\partial u_1}{\partial x_n}\\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2}\ldots\ldots\ldots\ldots\ldots & \frac{\partial u_2}{\partial x_n}\\ \ldots\ldots & \ldots\ldots\quad\ldots\ldots\quad\ldots\ldots & \ldots\ldots\\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2}\ldots\ldots\ldots\ldots\ldots & \frac{\partial u_n}{\partial x_n}\end{vmatrix}$$

4. If the functions $u_1, u_2, u_3, \ldots\ldots\ldots\ldots, u_{n-1}, u_n$ of n independent variables $x_1, x_2, x_3, \ldots\ldots\ldots\ldots, x_{n-1}, x_n$ are in the following forms:

$u_1 = f_1(x_1),\ u_2 = f_2(x_1, x_2), \ldots\ldots\ldots, u_n = f_n(x_1, x_2, \ldots\ldots\ldots, x_n)$, then we have

$$\frac{\partial(u_1, u_2, u_3, \ldots\ldots\ldots\ldots, u_{n-1}, u_n)}{\partial(x_1, x_2, x_3, \ldots\ldots\ldots\ldots, x_{n-1}, x_n)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & 0 & 0 & \ldots\ldots & 0 \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & 0 & \ldots\ldots & 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial x_3} & \ldots\ldots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

$$= \frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2}\frac{\partial u_3}{\partial x_3}\ldots\ldots\ \ldots\ldots\ \ldots\ldots\frac{\partial u_n}{\partial x_n}$$

5. If $u_1,\ u_2$ are the functions of $y_1,\ y_2$ and $y_1,\ y_2$ are the functions $x_1,\ x_2$ then

$$\frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} = \frac{\partial(u_1, u_2)}{\partial(y_1, y_2)}.\frac{\partial(y_1, y_2)}{\partial(x_1, x_2)}.$$

6. Consider $u_1, u_2, u_3, \ldots\ldots\ldots, u_{n-1}, u_n$ be functions of n independent variables $x_1, x_2, x_3, \ldots\ldots\ldots, x_{n-1}, x_n$. In order that these n functions may not be independent, *i.e,* there may exist between these n functions a relation $F(u_1, u_2, u_3, \ldots\ldots\ldots, u_{n-1}, u_n) = 0$. It is necessary and sufficient that the Jacobian $\frac{\partial(u_1, u_2, u_3, \ldots\ldots\ldots, u_{n-1}, u_n)}{\partial(x_1, x_2, x_3, \ldots\ldots\ldots, x_{n-1}, x_n)}$ should vanish identically.

8.5 TERMINAL QUESTIONS

Q.1 What do you mean by Jacobian?

Q.2 If $x = r\cos\theta,\ y = r\sin\theta$ show that $\frac{\partial(r, \theta)}{\partial(x, y)} = \frac{1}{r}$.

Q.3 If $x = c\cos u\cosh v,\ y = c\sin u\sinh v$ show that

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{2}c^2(\cos 2u - \cosh 2v).$$

Q.4 If $x+y+z=u,\ y+z=uv,\ z=uvw$ show that $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}=u^2v.$

Q.5 Prove that $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}\times\dfrac{\partial(x,y,z)}{\partial(u,v,w)}=1.$

Q.6 Show that $ax^2+2hxy+by^2$ and $Ax^2+2Hxy+By^2$ are independent unless

$$\frac{a}{A}=\frac{h}{H}=\frac{b}{B}.$$

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Advanced Real Analysis And Integral Equations

BLOCK

3

FOURIER SERIES

UNIT-9

Introduction to Fourier Series

UNIT-10

Half Range Fourier Series

BLOCK INTRODUCTION

The Fourier series holds immense significance across diverse fields such as mathematics, physics, engineering, and signal processing. It offers a method to express periodic functions as a summation of sinusoidal functions (sine and cosine waves), simplifying the analysis and manipulation of many real-world phenomena that exhibit inherent periodicity. Fourier series serves as a fundamental tool for analyzing and processing signals. Fourier series are employed to analyze seismic signals and investigate the Earth's subsurface structure. By decomposing seismic signals into frequency components, reserchers can identify seismic waves, infer properties of geological formations, and locate subsurface features such as oil and gas reservoirs. Fourier series plays a central role in harmonic analysis, which explores the representation of functions or signals as combinations of basic oscillatory components. This analysis is pivotal in understanding the behavior of complex systems found in areas like quantum mechanics, acoustics, and optics.

Fourier series determine the crucial applications in medical imaging techniques such as MRI (Magnetic Resonance Imaging) and CT (Computed Tomography) scans. In MRI, Fourier transforms are utilized to reconstruct images from acquired data in k-space, enabling visualization of internal body structures with high resolution. Similarly, Fourier techniques are integral to CT scans for image reconstruction and analysis. In the ninth unit, we shall have discussed about the introduction about fourier series, periodic function, even and odd functions with their properties, and the Euler formulas for the Fourier coefficients and in the tenth unit we deal with half rage series, change of interval and Parseval’s identity for Fourier series.

UNIT 9 INTRODUCTION OF FOURIER SERIES

Structure

9.1 Introduction

9.2 Objectives

9.3 Trigonometric Fourier Series

9.4 Periodic Function

9.5 Even and Odd Function

9.6 Some Important Identities

9.7 Euler Formulas for the Fourier Coefficients

9.8 Some assumptions and definition for expansion of f(x) by Fourier Series

- 9.8.1 Assumption for Expression of f(x) by Fourier Series
- 9.8.2 Fourier Series of Even and Odd functions
- 9.8.3 Dirichlet's Conditions
- 9.8.4 Smooth and Piecewise Smooth Function
- 9.8.5 Jump Discontinuities
- 9.8.6 A Criterion for the Convergence of Fourier Series

9.9 Summary

9.10 Terminal Questions

9.1 INTRODUCTION

The Fourier series holds immense significance across a spectrum of disciplines, including heat conduction, rotating machinery, sound waves, planetary dynamics, cardiac physiology, chemical kinetics, and acoustics. This mathematical concept was pioneered by the French mathematician Fourier in 1807. Essentially, the Fourier series provides an infinite series representation of periodic functions using trigonometric sine and cosine functions. Its versatility lies in its application as a powerful tool for solving both ordinary and partial differential equations involving periodic functions.

Engineers and scientists extensively utilize the Fourier series to tackle various physical and engineering challenges. Its utility stems from its capability to accurately represent functions that may not be differentiable. Moreover, it transcends beyond continuous functions, extending its application to periodic functions as well as functions exhibiting discontinuities in their values and derivatives. This broad applicability enables the Fourier series to serve as a fundamental analytical tool across diverse domains, aiding in the understanding and solution of complex problems encountered in scientific and engineering contexts.

9.2 OBJECTIVES

After reading this unit the learner should be able to understand about:

- understand the trigonometric Fourier series
- comprehend the Periodic Functions
- explain the even and odd functions with their properties
- discuss the Euler formulas for the Fourier coefficients

9.3 TRIGONOMETRIC FOURIER SERIES

A series $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin x)$

Where a_0, a_n, b_n are Fourier constants independent of x, and f is bounded and integrable on $(-\pi. \pi)$, $(0, \pi)$, is called the trigonometric Fourier series.

9.4 PERIODIC FUNCTION

A function $f(x)$ is said to be periodic function if

$$f(x + T) = f(x), \ \forall x, \ T \neq 0.$$

smallest such T is called the period of $f(x)$.

For example $\sin(x + 2\pi) = \sin x$, i.e., 2π is period

$\cos(x + 2\pi) = \cos x$, i.e., 2π is period

$\sec(x + 2\pi) = \sec x$, i.e., 2π is period

$\operatorname{cosec}(x + 2\pi) = \operatorname{cosec}$, i.e., 2π is period

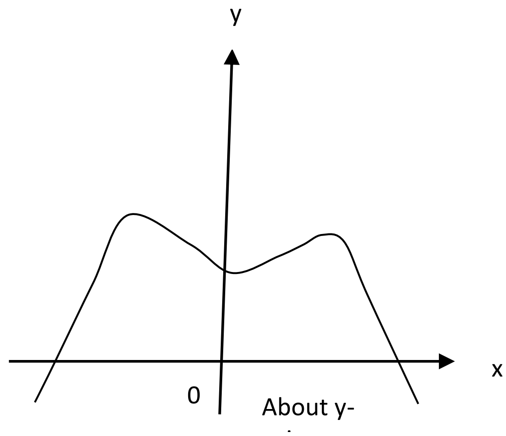
$\cot(x + \pi) = \cot(x)$, i.e., π is period

$\tan(x + \pi) = \tan(x)$, i.e., π is period

$$sin\ (nx + 2\pi) = sin\ n \left(x + \frac{2\pi}{n} \right), \text{ i.e., } \frac{2\pi}{n} \text{ is period.}$$

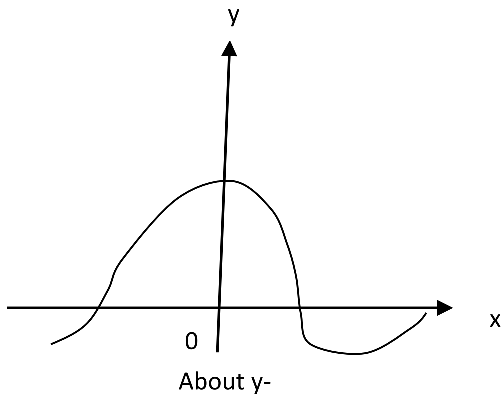
9.5 EVEN AND ODD FUNCTION

Let the function *f(x)* defined in an interval I which is symmetric to origin, we say that *f(x)* is an even function if $f(-x) = f(x)\ \forall x$



This implies the graph of any even function y = *f(x)* is symmetric with respect to the *y* axis. It follows from the interpretation of the integral as on area that for even function we have

$$\int_{-l}^{l} f(x)dx = 2\int_{0}^{l} f(x)dx \quad for\ \ any\ l \qquad \ldots.(1)$$



Provide that *f(x)* is defined and integrable as interval [-l, l].

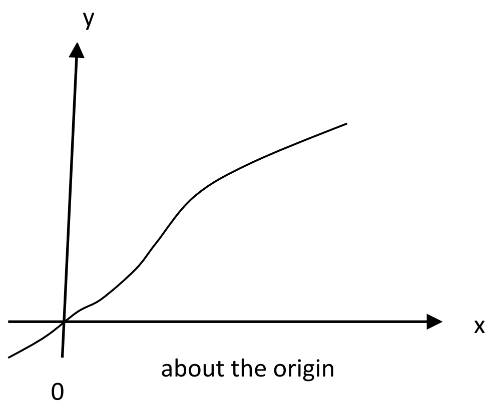
The function *f(x)* is odd if

$$f(-x) = -f(x)\ \forall x$$

In particular we have for an odd function

$$f(-x) = -f(x) \forall x$$

The graph of any odd function y = $f(x)$ is symmetric with respect to the origin.



For odd function $\int_{-l}^{l} f(x)dx = 0$ … (2)

for any l provided that $f(x)$ is defined and integrable on the interval $[-l, l]$.

Properties of even and odd functions

(a) The product for two even function or odd function is again an even function.

Suppose $\varphi(x)$, and $\Psi(x)$, are even function, then we have

Let $f(x) = \varphi(x)\ \Psi(x)$

Now we have

$$f(-x) = \varphi(-x)\ \Psi(-x)$$
$$= \varphi(x)\ \Psi(x)$$
$$= f(x)$$

while if $\varphi(x)$ and $\Psi(x)$ are odd functions then we have

$$f(-x) = \varphi(-x)\ \Psi(-x)$$
$$= [-\varphi(x)]\ [-\Psi(x)]$$
$$= f(x)$$

(b) The product of an even and an odd function is an odd function.

Suppose $\varphi(x)$ is even and $\Psi(x)$ is odd function then

$$f(-x) = \varphi(-x)\ \Psi(-x)$$
$$= \varphi(x)\ [-\Psi(x)]$$
$$= -\varphi(x)\ \Psi(x)$$
$$= -f(x)$$

Note. In case of even function the graph of the curve is symmetrical about y-axis whereas in case of odd function the graph of the curve is symmetrical about origin.

9.6 SOME IMPORTANT IDENTITIES

1. $\int_{-\pi}^{\pi} \sin n\pi \, dx = 0$ (even function)

2. $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \cos bx - b \sin bx)$

3. $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx + b \cos bx)$

4. $\int_{-\pi}^{\pi} \cos nx \, dx = 2 \int_{0}^{\pi} \cos nx \, dx$ (even function)

5. $\int_{0}^{2\pi} \sin nx \, dx = 0$

6. $\int_{0}^{2\pi} \cos nx \, dx = 0$

7. $\int_{-\pi}^{\pi} \cos nx \, dx = 2 \int_{0}^{\pi} \cos nx \, dx$ (even function)

$= 2\left[\frac{\sin nx}{n}\right]_0^{\pi}$

$= \frac{2}{n} [\sin n\pi - \sin n.0]$

$= \frac{2}{n} [0 - 0]$

$= 0$

8. $\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 2 \int_{0}^{\pi} \cos mx \cos nx \, dx$ (even function)

$= \int_{0}^{\pi} 2 \cos mx \operatorname{cox} nx \, dx$

$= \int_{0}^{\pi} [\cos (m - n)x - \cos(m + n)x] \, dx$

$= \int_{0}^{\pi} \cos(m - n) x \, dx - \int_{0}^{\pi} \cos (m + n) x \, dx$

$$= \left[\frac{\sin(m-n)x}{m-n}\right]_0^{\pi} - \left[\frac{\sin(m+n)x}{m+n}\right]_0^{\pi}$$

$= 0 - 0$

$= 0.$

9. $\int_{-\pi}^{\pi} \sin mx \sin nx \, dx \quad = 2\int_0^{\pi} \sin mx \sin nx \, dx$

$$= \int_0^{\pi} [\cos(m-n)x - \cos(m+n)] \, dx$$

$$= \int_0^{\pi} \cos(m-n)x \, dx - \int_0^{\pi} \cos(m+n)x \, dx$$

$$= \left[\frac{\sin(m-n)x}{m-n}\right]_0^{\pi} - \left[\frac{\sin(m+n)x}{m+n}\right]_0^{\pi}$$

$= 0 - 0$

$= 0.$

10. $\int_{-\pi}^{\pi} \cos^2 nx \, dx \ = 2\int_0^{\pi} \cos^2 nx \, dx$ (even function)

$$= \int_0^{\pi} 2\cos^2 nx \, dx$$

$$= \int_0^{\pi} (1 + \cos 2nx) \, dx$$

$$= \int_0^{\pi} 1.\,dx + \int_0^{\pi} \cos 2nx \, dx$$

$$= [x]_0^{x} + \left[\frac{\sin 2nx}{2n}\right]_0^{\pi}$$

$= \pi - 0 + 0 - 0$

$= \pi.$

11. $\int_{-\pi}^{\pi} \sin^2 nx \, dx \ = 2\int_0^{\pi} \sin^2 nx \, dx$ (even function)

$$= \int_0^{\pi} 2\sin^2 nx \, dx$$

$$= \int_0^{\pi} (1 - \cos 2nx)\, dx$$

$$= \int_0^{\pi} 1.\, dx - \int_0^{\pi} \cos 2nx\, dx$$

$$= [x]_0^{\pi} + \left[\frac{\sin 2nx}{2n}\right]_0^{\pi}$$

$$= \pi - 0 - 0 + 0 = \pi.$$

9.7 EULER FORMULAS FOR THE FOURIER COEFFICIENTS

Let $f(x)$ can be expanded in the term of Fourier series

$$f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \ldots\ldots + b_1 \sin x + b_2 \sin 2x + \ldots\ldots \qquad \ldots(3)$$

where $a_0, a_1, a_2, \ldots a_n, \ldots\ldots, b_1, b_2, \ldots\ldots$ are Fourier constants

The above equation (3) can be written as in forms

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad \ldots(4)$$

To find a_0, integrating equation (4) both sides on the interval $-\pi$ to π.

$$\int_{-\pi}^{\pi} f(x)\, dx = \int_{-\pi}^{\pi} a_0\, dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} a_n \cos nx\, dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} b_n \sin nx\, dx$$

$$= a_0 \int_{-\pi}^{\pi} 1.\, dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx\, dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} b_n \sin nx\, dx$$

$$= a_0 .\, 2\pi + \sum_{n=1}^{\infty} a_n .\, 0 + \sum_{n=1}^{\infty} b_n .\, 0$$

$$= 2\pi a_0$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

To find a_n multiplying by $\cos nx$ both sides of the equation (3) and integrating in interval $-\pi$ *to* π.

$$\int_{-\pi}^{\pi} f(x) \cos nx\, dx = a_0 \int_{-\pi}^{\pi} \cos nx\, dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos^2 nx\, dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \cos nx\, dx$$

$$= a_0 \int_{-\pi}^{\pi} \cos nx\, dx + \sum_{n=1}^{\infty} a_n \int_0^{\pi} 2\cos^2 nx\, dx + \sum_{n=1}^{\infty} \frac{b_n}{2} \int_{-\pi}^{\pi} \sin 2nx\, dx$$

$$= a_0 .\, 0 + a_n .\, \pi + \frac{b_n}{2} .\, 0$$

$$= \pi a_n$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \; \mathrm{d}x$$

To find b_n, multiplying by sin nx both sides of equation (3) and integrating $-\pi$ *to* π

$$\int_{-\pi}^{\pi} f(x)\sin nx\, dx = a_0\int_{-\pi}^{\pi} \sin nx\, dx + \sum_{n=1}^{\infty} a_n\int_{-\pi}^{\pi} \sin nx \cos nx\, dx + \sum_{n=1}^{\infty} b_n\int_{-\pi}^{\pi} \sin^2 nx\, dx$$

$$= a_0 \,.\, 0 + \sum_{n+1}^{\infty} \frac{a_n}{2}\int_{-\pi}^{\pi} \sin 2nx\, dx + \sum_{n+1}^{\infty} b_n\int_{0}^{\pi} 2\sin^2 nx\, dx$$

$$= a_0 .\, 0 + \sum_{n=1}^{\infty} \frac{a_n}{2} \,.\, 0 + b_n\pi$$

$$= \pi b_n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \; \mathrm{d}x$$

Hence the Fourier series is defined.

9.8 SOME ASSUMPTIONS AND DEFINITION FOR EXPANSION OF *f*(*x*) BY FOURIER SERIES

9.8.1 ASSUMPTION FOR EXPANSION OF f(x) BY FOURIER SERIES

1. The Fourier function is assumed to be single valued continuous and integrable in the given internals.
2. The series $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ be integrating term by term, the series should be uniformly convergent.
3. The series converges to $f(x)$ at every point where $f(x)$ is continuous. At the point of discontinuity in the interval $(-\pi, \pi)$ the series converges to

$$\frac{1}{2}[f(x+0) + f(x-0)].$$

If $x = c$ is the point of discontinuity then

$$f(x) = \frac{1}{2}\left[f(c+0) + f(c-0)\right].$$

At $x = \pm\pi$, the series converges to $\frac{1}{2}[f(-\pi+0) + f(\pi-0)]$ limit exist.

9.8.2 FOURIER SERIES OF EVAN AND ODD FUNCTIONS

Case-1: If $f(x)$ is an even function then $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{0}^{\pi} f(x)\,dx$

and $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ \mathrm{d}x = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\ \mathrm{d}x \qquad \left(\because\ f(x).\cos nx \text{ is even function}\right)$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ \mathrm{d}x = 0$

$\left(\because\ f(x) \text{ is even and } \sin x \text{ is odd and so } f(x).\sin nx \text{ is odd function}\right)$

Case-2: If $f(x)$ is an odd function then $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$

and $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ \mathrm{d}x = 0 \qquad \left(\because\ f(x).\cos nx \text{ is odd function}\right)$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ \mathrm{d}x = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\ \mathrm{d}x$

$\left(\because\ f(x).\sin nx \text{ is even function}\right)$

Note: Thus a series of even function will contain only cosine terms and series of odd function will contain only sine terms.

9.8.3 DIRICHLET'S CONDITIONS

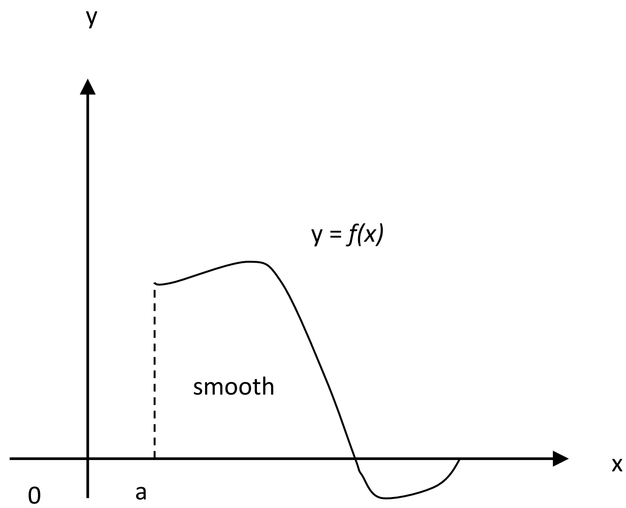
If a function *f* is bounded Riemann integrable in the interval [-π, π] and if it is possible to divide the interval [-π, π] into a finite number of subinterval in each of which *f(x)* is monotonic, then the Fourier series corresponding to *f(x)* converges for every *x* and if *S(x)* (sum function) of the series is given as follows:

$$S(x) = \frac{1}{2}\left[f(x+0) + f(x-0)\right] \qquad (-\pi < x < \pi)$$

Also $$S(x) = \frac{1}{2}\left[f(-\pi+0) + f(\pi-0)\right] \qquad \text{when } x = \pm\pi.$$

9.8.4 SMOOTH AND PIECEWISE SMOOTH FUNCTION

The function *f(x)* is said to be piecewise continuous on [*a*, *b*] if it has a continuous derivative on [*a*, *b*]. The graph of a smooth function is a smooth curve without any corners (a point at which the curve has two distinct tangents).A continuous or a discontinuous function *f(x)* which is defined on the whole *x*-axis is said to be piecewise smooth if it is smooth on every interval on finite length. The concept applies to periodic function every piecewise smooth function is bounded and has a bounded derivative every pair except its corner and points of discontinuity.



9.8.5 JUMP DISCONTINUITIES

In this case Right Hand Limit (R.H.L.) and Left Hand Limit (L.H.L.) both exist but not are equal to each other. We know that when R.H.L. and L.H.L. of a given function at a point exist but are not equal then the function is said to have the discontinuity of the first kind or a point of jumps discontinuities.

Also x_0 is the point then $f(x_0 + 0) - f(x_0 - 0) = \delta$ is called the jumps of the function $x = x_0$.

For example, $f(x) = \begin{cases} -x^3 & for \quad x<1 \\ 0 & \quad x=1 \\ \sqrt{x} & \quad x>1 \end{cases}$

At $x = 1$, L.H.L. $f(1 - 0) = -1$

R.H.L. $f(1 + 0) = 1$

The function has 2 jumps at $x = 1$ which is the point of discontinuity.

Note. In removal L.H.L. and R.H.L. exist and equal but not equal to function.

9.8.6 A CRITERION FOR THE CONVERGENCE OF FOURIER SERIES

This Fourier of a piecewise smooth (continuous of discontinuous) function *f(x)* of period 2π converge for all value of *x* the sum of the series equals *f(x)* at every point of continuity and is

$$\frac{1}{2}[f(x + 0) + f(x - 0)] \text{ at every point of discontinuity}$$

{arithmetic mean of R. H.L. and L.H.L.}

If *f(x)* is continuous at everywhere then the series converges absolutely and uniformly at end points of the interval (-π, π).

1. If $f(-\pi) = f(\pi)$ then the function is continuous at the point $\pm\pi$ and the Fourier series converge to *f(x)* at the end points of the closed interval -π to π.

2. If $f(-\pi) \neq f(\pi)$ then the function is discontinuous at the point $\pm\pi$ and its sum function $\left[\frac{f(-\pi)+f(\pi)}{2}\right]$.

Examples

Example.1. *Find the Fourier series of the function* $f(x) = x^2$ *in interval* $(-\pi, \pi)$ *and deduce that*

$$(\text{i}) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\ldots = \frac{\pi^2}{6} \quad (\text{ii}) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\ldots = \frac{\pi^2}{12}$$

$$(\text{iii}) = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\ldots\ldots = \frac{\pi^2}{8}$$

Expand $f(x)= x^2$ $(-\pi \leq x \leq \pi)$ *in Fourier series.*

Sol. We know that the Fourier series for $f(x)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\ dx$$

$$= \frac{2}{2\pi}\int_{0}^{\pi} x^2\ dx \ \ \text{(even function)}$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\,\frac{\pi^3}{3}$$

$$= \frac{\pi^2}{3}$$

and
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\ dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 cosnx\, dx \qquad \text{(even function)}$$

$$= \frac{2}{\pi}\left[\left(x^2\frac{\sin nx}{n}\right)_0^{\pi} - \left\{2x\left(-\frac{\cos nx}{n^2}\right)\right\}_0^{\pi} + \left(\frac{-2\sin nx}{n^2}\right)_0^{\pi}\right]$$

$$= \frac{2}{\pi}\left[0-0+\frac{2\pi}{n^2}\cos \mathrm{n}\pi - 0+0\right]$$

$$= \frac{4}{n^2}\cos\, n\pi$$

$$= \frac{4}{n^2}(-1)^n.$$

Also $\qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\, dx \qquad \text{(odd function)}$$

Since the given function x^2 is even function and $\sin nx$ is odd function, therefore all b_n will be zero. Now we have

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

or $\qquad x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx$

$$x^2 = \frac{\pi^2}{3} - \frac{4}{1^2}\cos x + \frac{4}{2^2}\cos 2x - \frac{4}{3^2}\cos 3x + \ldots.. \qquad \ldots(5)$$

(i) Put $x = \pi$ in the equation (5), we get

$$\pi^2 = \frac{\pi^2}{3} + 4 + \frac{4}{2^2} + \frac{4}{3^2} + \frac{4}{4^2}\ldots\ldots$$

or $\qquad \frac{2\pi^2}{3} = 4\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots\ldots\ldots\ldots\right)$

or $\qquad \frac{\pi^2}{6} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots.. \qquad \ldots(6)$

(ii) Putting $x = 0$ in the equation (5), we get

$$0 = \frac{\pi^2}{3} - \frac{4}{1^2} + \frac{4}{2^2} + \frac{4}{3^2} + \frac{4}{4^2}\ldots\ldots$$

or $\quad \dfrac{\pi^2}{3} = 4\left[\dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + ...\right]$

or $\quad \dfrac{\pi^2}{12} = \dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + ...$ …(7)

(iii) Putting $x = \dfrac{\pi}{2}$ in the equation (5), we get

$$\frac{\pi^2}{4} = \frac{\pi^2}{3} + 4\left[(-1).\,0 - \frac{1}{2^2} - \frac{1}{3^2}.0 + \frac{1}{4^2}.1 + ...\right]$$

or $-\dfrac{\pi^2}{12} = 4\left[-\dfrac{1}{2^2} + \dfrac{1}{4^2} - \dfrac{1}{6^2} + ...\right]$

or $\quad \dfrac{\pi^2}{48} = \left[\dfrac{1}{2^2} - \dfrac{1}{4^2} + \dfrac{1}{6^2} - .....\right]$ …(8)

Adding equations (6) and (7), we get

$$\frac{\pi^2}{6} + \frac{\pi^2}{12} = \frac{2}{1^2} + \frac{2}{3^3} + \frac{2}{5^2} + ...$$

or $\quad \dfrac{\pi^2}{4} = 2\left[\dfrac{1}{1^2} + \dfrac{1}{3^2} - \dfrac{1}{5^2} + ...\right.$

or $\quad \dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^3} + \dfrac{1}{5^2} + \dfrac{1}{7^2} + ...$

Note that $f(x)$ is is an even function so its fourier series will contain only cosine terms.

Note: The fourier series of $f(x) = x^3; -\pi < x < \pi$ will contain only sin terms because $f(x) = x^3$ is odd function.

Example.2: *Expand f(x) by Fourier series where f(x) is defined*

$$f(x) = \begin{cases} -1 & if \ -\pi < x < 0 \\ 0 & if \ x = 0 \\ 1 & if \ 0 < x < \pi \end{cases}$$

Sol. We know that the Fourier series for $f(x)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad …(9)$$

where $\quad a_0 = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} (-1)\mathrm{dx} + \frac{1}{2\pi}\int_{0}^{\pi} 1.\mathrm{dx}$$

$$= \frac{1}{2\pi}[-x]_{-\pi}^{0} + \frac{1}{2\pi}[x]_{0}^{\pi}$$

$$= \frac{1}{2\pi}[-\pi + \pi]$$

$$= 0$$

and $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\cos nx\,dx + \int_{0}^{\pi} 1.\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{-\sin nx}{n}\right)_{-\pi}^{0} + \left(\frac{\sin nx}{n}\right)_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}[-0 - 0 + 0 + 0]$$

$$= 0$$

Also $\quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\sin nx\,dx + \int_{0}^{\pi} 1.\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\cos nx}{n}\right)_{-\pi}^{0} + \left(-\frac{\cos nx}{n}\right)_{0}^{\pi}\right]$$

$$= \frac{1}{n\pi}[\cos n.0 - \cos n(-\pi) - \cos n\pi + \cos n.0]$$

$$= \frac{1}{n\pi}[2\cos n.0 - 2\cos n\pi]$$

$$= \frac{2}{n\pi}[1 - \cos n\pi]$$

If n is even, $\cos n\pi = 1$, then $b_2 = b_4 = b_6 = \ldots.0$

If n is odd, $\cos n\pi = -1$, then, $b_1 = \frac{4}{\pi}$, $b_3 = \frac{4}{3\pi}$, $b_5 = \frac{4}{5\pi}$, ...

Putting these values of a_0, a_n, b_n in the equation (9), we get

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$= 0 + 0 + [b_1\sin x + b_2\sin 2x + \ldots]$$

$$= \frac{4}{\pi} \sin x + 0.\ \sin 2x + \frac{4}{3\pi} \sin 3x + 0.\ \sin 4x + \frac{4}{5\pi} \sin 5\pi x + \ldots$$

$$= \frac{4}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots..\right] \qquad \ldots(10)$$

At $x = \pi/2$, $0 < x < \pi$, it is given that $f(x) = 1$, so that form equation (10), we get

$$1 = \frac{4}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots..\right]$$

or
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots..$$

Example.3. *Expand $f(x) = |x|$ by the Fourier series in the interval $(-\pi, \pi)$. Also show that* $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \ldots\ldots$.

Sol. We know that the Fourier series for $f(x)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad \ldots(11)$$

where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx$ *for* $x \geq 0$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|\, dx$$

$$= \frac{2}{2\pi}\int_{0}^{\pi} |x|\, dx \qquad (\because \ x > 0 \ \therefore \ |x| = x)$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{1}{\pi} \cdot \frac{\pi^2}{2}$$

$$= \frac{\pi}{2}$$

and $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx\, dx$$

$$= \frac{2}{2\pi}\int_{0}^{\pi} x \cos nx\, dx$$

$$= \frac{2}{\pi}\left[\left(\frac{x\sin nx}{n}\right)_0^{\pi} - \left(1\left(-\frac{\cos nx}{n^2}\right)\right)_0^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{n}.0 - 0 + \frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{2}{\pi n^2}\left[\cos \pi n - 1\right]$$

$$= \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

If n is given, $n = 2, 4, 6, ...,$ we get $a_2 = a_4 = a_6 = ... = 0$

$$a_1 = \frac{2}{\pi.1^2}[-1-1] = -\frac{4}{\pi.1^2}$$

$$a_3 = \frac{2}{\pi 3^2}[-1-1] = -\frac{4}{\pi 3^2}$$

$$a_5 = -\frac{4}{\pi 5^2},$$

$$a_7 = -\frac{4}{\pi 7^2}, ...$$

Since all b_n's will be zero because the given function $f(x) = |x|$ is even function. Putting these values of a_0, a_n, b_n in the equation (11), we get

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + ...\right] \qquad ...(12)$$

Now put $x = 0$ in the equation (12), we get

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos 0}{1^2} + \frac{\cos 0}{3^2} + \frac{\cos 0}{5^2} + ...\right]$$

or
$$\frac{\pi}{2} = \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} ...\right]$$

or
$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} ....$$

Example.4. *Show that an even function can have no sine term in its Fourier expansion* $f(x) = x\sin x;\ [-\pi, \pi]$. *Also show that* $\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + ...$

Sol. We know that the Fourier series for $f(x)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad ...(13)$$

where $$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{f}(\mathrm{x})\,\mathrm{dx}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x \sin x \, dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} x \sin x \, dx$$

$$= \frac{1}{\pi}\left[(-x \cos x)_0^{\pi} - (-\sin x)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\pi \cos\pi + 0 + \sin\pi - \sin 0\right] \qquad [\because \quad \cos\pi = -1]$$

$$= -\cos\pi$$

$$= 1.$$

and $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin x \cos nx \, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x \sin x \cos nx \, dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} x \; 2 \sin x \cos nx \, dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} x (2 \cos nx \sin x) \, dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} x \{\sin (n+1)x - \sin (n-1)x\} \, dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} x \sin (n+1)x \; dx - \int_{0}^{\pi} x \sin (n-1)x \; dx$$

$$= \frac{1}{\pi}\left[\left(\frac{-x \cos(n+1)\,x}{n+1}\right)_0^{\pi} - \left(\frac{-1.\,\sin(n+1)x}{(n+1)^2}\right)_0^{\pi}\right]$$

$$- \frac{1}{\pi}\left[\left(\frac{-x \cos(n-1)\,x}{n-1}\right)_0^{\pi} - \left(\frac{-1.\,\sin(n-1)x}{(n-1)^2}\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(-\frac{\pi\ \cos(n+1)\ \pi}{n+1}\right)_0^{\pi} - 0 + 0 + \left(\frac{\pi\cos(n-1)\pi}{(n-1)^2}\right)_0^{\pi} - 0 - 0\right]$$

$$= -\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}$$

$$= -\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}$$

$$= \frac{(-1)^{n+2}}{n+1} + \frac{(-1)^{n-1}}{n-1}\text{, } where\ n \geq 2$$

Putting n = 2, 3, 4,……., we get

$$a_2 = \frac{(-1)^4}{3} + \frac{(-1)}{1} = \frac{1}{3} - 1,$$

$$a_3 = \frac{(-1)^5}{4} + \frac{(-1)^2}{2} = -\frac{1}{4} + \frac{1}{2}$$

and $$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\cos x\,dx$$

$$= \frac{2}{2\pi}\int_{-\pi}^{\pi} x\sin x\cos x\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x(2\sin x\cos x)dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x\sin 2x\,dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} x\sin 2x\,dx \qquad \text{(even function)}$$

$$= \frac{1}{\pi}\left[\left(\frac{-x\cos 2x}{2}\right)_0^{\pi} - \left(\frac{-1.\sin 2x}{4}\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{-\pi\cos 2\pi}{2} - 0 + 0 - 0\right)\right]$$

$$= -\frac{\cos 2\pi}{2}$$

$$= (-1)\frac{(-1)^2}{2}$$

$$= -\frac{1}{2}$$

Since the given function *f(x)* is even therefore all b_n's are zero. Now putting these values of a_0, a_n, b_n in the equation (13), we get

$$x\sin x = 1 - \frac{1}{2}\cos x + \left(\frac{1}{3}-1\right)\cos 2x + \left(-\frac{1}{4}+\frac{1}{2}\right)\cos 3x + \left(\frac{1}{5}-\frac{1}{3}\right)\cos 4x + ... \qquad ...(14)$$

Put $x = \pi/2$ in the equation (14), we get

$$\frac{\pi}{2}\sin\frac{\pi}{2} = 1 - \frac{1}{2}\cos x + \left(\frac{1}{3}-1\right)\cos 2x + \left(-\frac{1}{4}+\frac{1}{2}\right)\cos 3x + \left(\frac{1}{5}-\frac{1}{3}\right)\cos 4x + ...$$

or $$\frac{\pi}{2} = 1 - \left(\frac{1}{3}-1\right) + \left(\frac{1}{5}-\frac{1}{3}\right) - \left(\frac{1}{7}-\frac{1}{5}\right) + ...$$

or $$\frac{\pi}{2} = 1 + \frac{2}{3} - \frac{2}{3.5} + \frac{2}{5.7} - ....$$

or $$\frac{\pi}{2} = 2\left[\frac{1}{2} + \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7}........\right]$$

or $$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + ...$$

Example.5. *Find the Fourier series for f(x)= x + x*^{2} *in (-π, π).*

Sol. We know that the Fourier series for $f(x)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad ...(15)$$

where $$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}(x + x^2)dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx + \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\, dx$$

$$= \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{\pi} + \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\pi^2 - \pi^2\right] + \frac{1}{2\pi}\left[\frac{\pi^3}{3} + \frac{\pi^3}{3}\right]$$

$$= 0 + \frac{2\pi^2}{6}$$

$$= \frac{\pi^2}{3}.$$

and $\quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x + x^2\right)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\left(\frac{x\sin nx}{n}\right)_{-\pi}^{\pi} - \left(\frac{-1.\cos nx}{n^2}\right)_{-\pi}^{\pi}\right] + \frac{2}{\pi}\int_{0}^{\pi} x^2\cos nx\,dx$$

$$= \frac{1}{\pi}\left[0 + \frac{\cos n\pi}{n^2} - \frac{\cos n\pi}{n^2}\right]$$

$$+ \frac{2}{\pi}\left[\left(\frac{x^2\sin nx}{n}\right)_{-\pi}^{\pi} - \left(\frac{-2x\cos nx}{n^2}\right)_{0}^{\pi} + \left(\frac{-2\sin nx}{n^3}\right)_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}[0] + \frac{2}{\pi}\left[0 + \frac{2\pi\cos n\pi}{n^2} - 0 - 0 + 0\right]$$

$$= \frac{2}{\pi}\left[\frac{2\pi\cos n\pi}{n^2}\right]$$

$$= \frac{4\cos n\pi}{n^2}$$

$$= \frac{4}{n^2}(-1)^n$$

Also $\quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x + x^2\right)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin nx\,dx$$

$$= \frac{2}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin nx\,dx$$

$$= \frac{2}{\pi}\left[\left(\frac{x\cos nx}{n}\right)_0^{\pi} - \left(\frac{-1.\sin nx}{n^2}\right)_0^{\pi}\right]$$

$$+\frac{1}{\pi}\left[\left(\frac{-x^2\cos nx}{n}\right)_{-\pi}^{\pi} - \left(\frac{-2x.\sin nx}{n^2}\right)_{-\pi}^{\pi} + \left(\frac{2\cos nx}{n^3}\right)_{-\pi}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{-\pi\cos n\pi}{n} - 0 + 0\right)\right] + \frac{1}{\pi}\left[\frac{-\pi^2\cos n\pi}{n} + \frac{\pi^2\cos n\pi}{n^2} + 0 + \frac{2\cos n\pi}{n^3} - \frac{2\cos n\pi}{n^3}\right]$$

$$= \frac{-2\cos n\pi}{n} + 0 + 0$$

$$= \frac{-2\,(-1)^n}{n}$$

Now putting these values of a_0, a_n, b_n in the equation (15), we get

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx - \sum_{n=1}^{\infty} \frac{2}{n}(-1)^n \sin nx$$

or $$x + x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \left[\frac{\cos nx}{n^2} - \frac{\sin nx}{2n}\right].$$

Example.6. *Find the Fourier series for* $f(x) = \dfrac{\pi - x}{2}$ *in* $[0, 2\pi]$.

Sol. We know that the Fourier series for $f(x)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad \ldots(16)$$

where $a_0 = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(x)\,dx$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\,dx$$

$$= \frac{1}{2\pi}\left[\frac{1}{2}\left(\pi x - \frac{x^2}{2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{1}{2}\left(\pi.2\pi - \frac{4\pi^2}{2}\right) - 0\right]$$

$$= \frac{1}{2\pi}\left[\pi^2 - \pi^2\right]$$

$$= 0$$

and $$a_n = \frac{1}{\pi}\left[\int_0^{2\pi} \frac{1}{2}(\pi - x)\cos nx \, dx\right]$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} \pi \cos nx dx - \int_0^{2\pi} x \cos x nx \, dx\right]$$

$$= \frac{1}{2\pi}\left[\pi\left(\frac{\sin nx}{n}\right)_0^{2\pi} - \left(\frac{x \sin nx}{n}\right)_0^{2\pi} + \left(\frac{-1.\cos nx}{n^2}\right)_0^{2\pi}\right]$$

$$= \frac{1}{2\pi}\left[-0 - 0 - \frac{\cos 2n\pi}{n^2} + \frac{1}{n^2}\right]$$

$$= \frac{1}{2\pi}\left[-\frac{(-1)^{2n}}{n^2} + \frac{1}{n^2}\right]$$

$$= \frac{1}{2\pi}\left[-\frac{1}{n^2} + \frac{1}{n^2}\right]$$

$$= \frac{1}{2\pi}[0]$$

$$= 0$$

Also $$b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\sin nx dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \pi \sin nx dx - \int_0^{2\pi} x \cos nx dx$$

$$= \frac{1}{2\pi}\left[\pi\left(\frac{-\cos nx}{n}\right)_0^{2\pi} - \left(\frac{-x \cos nx}{n}\right)_0^{2\pi} + \left(\frac{-1.\sin nx}{n^2}\right)_0^{2\pi}\right]$$

$$= \frac{1}{2\pi}\left[-\frac{\pi}{n}(\cos 2n\pi - \cos 0) + \frac{1}{\pi}2\pi(\cos 2n\pi - 0) + 0\right]$$

$$= \frac{1}{2\pi}\left[-\frac{\pi}{n}\left[(-1)^{2n} - 1\right] + \frac{2\pi}{n}(-1)^{2n}\right]$$

$$= \frac{1}{2\pi}\left[0 + \frac{2\pi}{n}\right]$$

$$= \frac{1}{n}$$

Now putting these values of a_0, a_n, b_n in the equation (16), we get

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)$$

$$\frac{1}{2}(\pi - x) = 0 + \sum_{n=1}^{\infty} 0.\ \cos nx + \sum_{n=1}^{\infty} \frac{1}{n} \sin\ nx$$

$$\frac{1}{2}(\pi - x) = 0 - 0 + \frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \ldots\ldots.. \qquad \ldots(17)$$

Put $x = \pi/2$ in the equation (17), we get

$$\frac{\pi}{4} = \frac{1}{1} + 0 - \frac{1}{3} + 0 + \frac{1}{5} + 0 - \frac{1}{7} + \ldots\ldots$$

or $$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots .$$

Example.7. *Find the Fourier series which converge to f (x) in (-π< x <π) where f(x)= x+ x²and f(x)= π² when x = ±π.*

Sol. For the given *f(x)*= *x* + *x*² in [-π, π] (from example 5), we have

$$x + x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\left[\frac{\cos nx}{n^2} - \frac{\sin nx}{2n}\right]$$

It is given that when $x = \pm\pi$ then $f(x) = \pi^2$.

Now we have

$$S(x) = \frac{1}{2}\left[f(\pi - 0) + f(-\pi + 0)\right]$$

$$= \frac{1}{2}\left[\pi - 0 + (\pi - 0)^2 + -\pi + (-\pi)^2\right]$$

$$= \frac{1}{2}\left[\pi + \pi^2 - \pi + \pi^2\right]$$

$$= \pi^2$$

At $x = \pm\pi$, we have

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\left[\frac{\cos n\pi}{n^2} - \frac{\sin nx}{2n}\right]$$

$$= \frac{\pi^2}{3} + 4\sum(-1)^n\left[\frac{(-1)^n}{n^2} - 0\right]$$

or $$\pi^2 - \frac{\pi^2}{3} = 4\sum \frac{(-1)^n}{n^2}$$

or $$\frac{2\pi^2}{3} = 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots\dots\dots\right]$$

or $$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots\dots\dots .$$

Example.8. *Find the Fourier series of a function* $f(x) = \begin{cases} x - \pi & \text{when } -\pi < x < 0 \\ \pi - x & \text{when } 0 < x < \pi \end{cases}$.

Sol. We know that the Fourier series for $f(x)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad \dots(18)$$

where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx$

$$= \frac{1}{2\pi}\int_{-\pi}^{0}(x-\pi)dx + \frac{1}{2\pi}\int_{0}^{\pi}(\pi - x)dx$$

$$= \frac{1}{2\pi}\left[\frac{x^2}{2} - \pi x\right]_{-\pi}^{0} + \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_{0}^{\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{\pi^2}{2} - \pi^2\right] + \frac{1}{2\pi}\left[\pi^2 - \frac{\pi^2}{2}\right]_{0}^{\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{\pi^2}{2} - \pi^2 + \pi^2 - \frac{\pi^2}{2}\right]$$

$$= -\frac{\pi^2}{2\pi}$$

$$= -\frac{\pi}{2}$$

and $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(x-\pi)\cos nx\, dx + \frac{1}{\pi}\int_{0}^{\pi}(\pi - x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} x\cos nx\, dx - \frac{1}{\pi}\int_{-\pi}^{0}\pi\cos nx\, dx + \frac{1}{\pi}\int_{0}^{\pi}(\pi - x)\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\left(\frac{x\sin nx}{n}\right)_{-\pi}^{0} - \left(-(1).\frac{\cos nx}{n^2}\right)_{-\pi}^{0} + \left(\frac{\pi.\sin nx}{n}\right)_{-\pi}^{0}\right]$$

$$+\frac{1}{\pi}\left[\left(\pi\frac{\sin nx}{n}\right)_0^\pi-\left(x\,\frac{\sin nx}{n}\right)_0^\pi+\left(-1\,\frac{\cos nx}{n^2}\right)_0^\pi\right]$$

$$=\frac{1}{\pi}\left[0+\frac{1}{n^2}-\frac{\cos n\pi}{n^2}-0\right]+\frac{1}{\pi}\left[0-0-\frac{\cos n\pi}{n^2}+\frac{1}{n^2}\right]$$

$$=\frac{1}{\pi}\left[\frac{1}{n^2}-\frac{(-1)^n}{n^2}-\frac{(-1)^n}{n^2}+\frac{1}{n^2}\right]$$

$$=\frac{2}{\pi n^2}\left[1-(-1)^n\right]$$

$$=\frac{2}{\pi n^2}\left[1+(-1)^{n+1}\right]$$

Also $\quad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$$=\frac{1}{\pi}\int_{-\pi}^{0}(x-\pi)\sin nx\,dx+\frac{1}{\pi}\int_0^\pi(\pi-x)\sin nx\,dx$$

$$=\frac{1}{\pi}\left[\left(-x\frac{\cos nx}{n}\right)_{-\pi}^0-\left(-1\,\frac{\sin nx}{n^2}\right)_{-\pi}^0-\left(-\pi\frac{\cos nx}{n}\right)_{-\pi}^0\right]$$

$$+\frac{1}{\pi}\left[\left(-\pi\frac{\cos nx}{n}\right)_0^\pi-\left(-x\,\frac{\cos nx}{n}\right)_0^\pi+\left(-1.\,\frac{\sin nx}{n^2}\right)_0^\pi\right]$$

$$=\frac{1}{\pi}\left[-\frac{\pi\cos n\pi}{n}+0+\frac{\pi\cos 0}{n}-\frac{\pi\cos n\pi}{n}\right]$$

$$+\frac{1}{\pi}\left[-\frac{\pi\cos n\pi}{n}+\frac{\pi\cos 0}{n}-\frac{\pi\cos n\pi}{n}-0+0\right]$$

$$=\frac{1}{\pi}\left[-2\pi\,\frac{(-1)^n}{n}+\frac{\pi}{n}+\frac{\pi}{n}\right]$$

$$=\frac{1}{\pi}\left[-2\pi\,\frac{(-1)^n}{n}+\frac{2\pi}{n}\right]$$

$$=\frac{2\pi}{\pi}\left[\frac{(-1)^{n+1}}{n}+\frac{1}{n}\right]$$

$$=\frac{2}{n}\left[(-1)^{n+1}+1\right]$$

Now putting these values of a_0, a_n, b_n in the equation (18), we get

$$f(x)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$$

$$=-\frac{\pi}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

$$=-\frac{\pi}{2}+\frac{2}{\pi}+\sum_{n=1}^{\infty}\left[\frac{1+(-1)^{n+1}}{n^2}\right]_n\cos nx+2\sum_{n=1}^{\infty}\frac{1}{n}\left[(-1)^{n+1}+1\right]\sin nx$$

$$=-\frac{\pi}{2}+\frac{2}{\pi}\left[\frac{2\cos x}{1^2}+\frac{2\cos 3x}{3^2}-\frac{2\cos 5x}{5^2}+\ldots\ldots\right]$$

$$+2\left[\frac{2\sin x}{1}+\frac{2\sin 3x}{3}-\frac{\sin 5x}{5}+\ldots\ldots\right]$$

$$=-\frac{\pi}{2}+\frac{4}{\pi}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}-\frac{\cos 5x}{5^2}+\ldots\ldots\right]$$

$$+4\left[\frac{\sin x}{1}+\frac{\sin 3x}{3}-\frac{\sin 5x}{5}+\ldots\ldots\right]$$

Example.9. *Expand f(x) by Fourier series where f(x) is defined by f(x)= e^x in [-π, π].*

Sol. We know that the Fourier series for $f(x)$ is

$$f(x)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right) \qquad \ldots(19)$$

where $a_0=\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}f(x)\,dx$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^x\,dx$$

$$=\frac{1}{2\pi}\left[e^x\right]_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left[\frac{e^{\pi}-e^{-\pi}}{2}\right]$$

$$=\frac{1}{\pi}\sin h\pi$$

and $a_n=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}f(x)\cos nx\,dx$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}e^x\cos nx\,dx$$

$$=\frac{1}{\pi}\left[\frac{e^x}{1+n^2}\left(\cos nx+n\sin nx\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(1+n^2)}\left[e^x(\cos nx + n\sin nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(1+n^2)}\left[e^\pi(\cos n\pi + 0) - e^{-\pi}(\cos n\pi + 0)\right]$$

$$= \frac{1}{\pi(1+n^2)}\left[e^\pi - e^{-\pi}\right]\cos n\pi$$

$$= \frac{2.(-1)^n}{\pi(1+n^2)}\left(\frac{e^\pi - e^{-\pi}}{2}\right)$$

$$= \frac{2(-1)^n \sin h\pi}{\pi(1+n^2)} \ or \ \frac{2\sin h\pi \cos n\pi}{\pi(1+n^2)}$$

Note: $$\left\{\begin{array}{l} \int e^{ax}\cos bx\, dx = \dfrac{e^{ax}}{a^2+b^2}\ (a\cos bx + b\sin bx) \\ \int e^{ax}\sin bx\, dx = \dfrac{e^{ax}}{a^2+b^2}\ (a\sin bx - b\cos bx) \end{array}\right\}$$

Also $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin nx\, dx$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}(\sin nx - n\cos nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(1+n^2)}\left[e^x(\sin nx - n\cos nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(1+n^2)}\left[e^\pi(0 - n\cos n\pi) - e^{-\pi}(0 - n\cos n\pi)\right]$$

$$= \frac{1}{\pi(1+n^2)}\left[-e^\pi\ n\cos n\pi + e^{-\pi}\ n\cos n\pi\right]$$

$$= -\frac{n\cos n\pi}{\pi(1+n^2)}\left(e^\pi - e^{-\pi}\right)$$

$$= -\frac{2n(-1)^\pi}{\pi(1+n)^2}\left(\frac{e^\pi - e^{-\pi}}{2}\right)$$

$$= -\frac{2n(-1)^\pi}{\pi(1+n)^2} \ or \ -\frac{2n\cos n\pi \sin h\,\pi}{\pi(n+1)^2}$$

Now putting these values of a_0, a_n, b_n in the equation (19), we get

$$f(x) = \frac{1}{\pi} \sinh \pi + \sum_{n=1}^{\infty} \left[\frac{2(-1)^{\pi} \sin h\pi \cos nx}{\pi(1+n^2)} - \frac{2n(-1)^2 \sin h\pi \sin nx}{\pi(1+n^2)} \right].$$

9.9 SUMMARY

1. A French Mathematician Fourier, in 1807 introduced the Fourier series.
2. A series $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin x)$, Where a_0, a_n, b_n are Fourier constants and interval (-π. π), (0, π) independent of x, is called the trigonometric Fourier series.
3. A function $f(x)$ is said to be periodic if $f(x+T) = f(x)$; for all real x and some positive number T.
4. A function $f(x)$ is said to be even if $f(-x) = f(x)$.
5. A function $f(x)$ is said to be odd if $f(-x) = -f(x)$.
6. The Fourier series for $f(x)$ in [-π, π] is

   $$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

   where $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{f}(\mathrm{x})\, \mathrm{dx}$, $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$ and $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$
7. Let $f(x)$ be an odd function in $-\pi < x < \pi$ then the graph $y = f(x)$ will be symmetrical about the origin, then we get.

   $a_0 = a_n = 0$, since $f(x)$ is odd.

   $$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx\, dx$$

   The Fourier sine series in $0 < x < \pi$ is given by $f(x) = \sum_{n=1}^{\infty} b_n \sin nx\, dx$
8. Let $f(x)$ be an even function in $-\pi < x < \pi$ then the graph $y = f(x)$ will be symmetrical about the y-axis, then we get

   $b_n = 0$, *since $f(x)$* is even

   $$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos nx\, dx$$

and $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos dx$

The Fourier cosines series in $0 < x < \pi$ is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$

9.10 TERMINAL QUESTIONS

Q.1 Explain the Periodic function.

Q.2 Write the Euler formulas for the Fourier coefficients.

Q.3 Find the Fourier series of the function $f(x) = x - x^2$ in interval $[-\pi, \pi]$

Q.4 Find the Fourier series for $f(x) = e^{-x}$ in the interval $[0, 2\pi]$.

Q.5 Find the Fourier series of

$$f(x) = \begin{cases} 0, & when \quad -\pi \le x \le 0 \\ x^2, & when \quad 0 < x < \pi \end{cases}$$

which is assumed to be periodic with period 2π.

Q.6 Find the Fourier series of

$$f(x) = \begin{cases} 0, & when \quad -\pi < x < 0 \\ \sin x, & when \quad 0 < x < \pi \end{cases}$$

and deduce the

$$\frac{\pi - 2}{4} = \frac{1}{1.3} - \frac{1}{1.5} + \frac{1}{5.7} + \ldots..$$

Q.7 Find the Fourier series of

$$f(x) = \begin{cases} 0, & for \quad 0 < x < \pi \\ 2\pi - x, & for \quad \pi \le x \le 2\pi \end{cases}$$

Q.8 Find a Fourier series of the periodic function $f(x)$ with period 2π which is defined as follows:

$$f(x) = \begin{cases} -1 & for \quad -\pi \le x \le 0 \\ 1, & for \quad 0 \le x \le 2\pi \end{cases}$$

Q.9 Find the Fourier series of the function $f(x) = |\cos x|$ in the interval $(-\pi, \pi)$.

Q.10 Find the Fourier of the function defined as

$$f(x)=\begin{cases}0 & for \quad -\pi\le x\le 0\\ \dfrac{\pi x}{4} & for \quad 0<x\le\pi\end{cases}.$$

Q.11 If f is bounded and integrable on (-π, π) and a_n, b_n are the fourier coefficient then show that $\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)$ converges.

Answers

3. $$-\frac{2\pi^2}{3}+4\left[\frac{\cos x}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\frac{\cos 4x}{4^2}......\right]+2\left[\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\frac{\sin 4x}{4}+...\right]$$

4. $$\frac{\left(1-e^{-2\pi}\right)}{\pi}\left[1+\left(\frac{1}{2}\cos x+\frac{1}{5}\cos 2x+\frac{1}{10}\cos 3x+....\right)+\left(\frac{1}{2}\sin x+\frac{2}{5}\sin 2x+\frac{3}{10}\sin 3x+......\right)\right]$$

5. $$\frac{\pi^3}{3}+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx+\sum_{n=1}^{\infty}\left[\frac{\pi}{n}(-1)^{n+1}+\frac{2}{\pi n^3}[(-1)^n-1]\right]\sin nx.$$

6. $$\frac{2}{\pi}+\frac{\sin x}{2}-\frac{2}{\pi}\left\{\frac{\cos 2x}{3}+\frac{\cos 4x}{15}+\frac{\cos 6x}{35}+\frac{\cos 8x}{63}+.......\right\}$$

7. $$\pi-\frac{4}{\pi}\left\{\cos x+\frac{\cos 3x}{3}+\frac{\cos 5x}{5^2}+......\right\}$$

8. $$\frac{4}{\pi}\left\{\sin x+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+.......+\frac{\sin(2n+1)x}{2n+1}+..........\right\}.$$

9. $$\frac{4}{\pi}+\frac{4}{\pi}\left[\frac{\cos 3x}{3}-\frac{\cos 4x}{15}+......\right]$$

10. $$\frac{\pi^2}{8}-\frac{1}{2}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+.......\right]+\frac{\pi}{4}\left[\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-....\right].$$

UNIT-10 HALF RANGE FOURIER SERIES

Structure

10.1 Introduction

10.2 Objectives

10.3 Derivatives Using Forward Difference Formula

10.4 Derivatives Using Backward Difference Formula

10.5 Summary

10.6 Terminal Questions

10.1 INTRODUCTION

The Fourier series offers a robust mathematical framework essential for comprehending and controlling periodic phenomena, with broad applications spanning various scientific and engineering domains. In data compression, Fourier series assume a pivotal role by enabling representation of data in the frequency domain, facilitating the elimination of redundant or less critical information. This leads to enhanced efficiency in data storage and transmission. Moreover, in image processing, Fourier series find extensive use in tasks like image enhancement, compression, and feature extraction. Through techniques such as the Discrete Fourier Transform (DFT), images can be efficiently transformed into the frequency domain, facilitating their manipulation and analysis.

The Half Range Fourier Series is a specialized form of the Fourier series that deals with functions defined on a specific interval, typically from 0 to L, rather than over the entire real line. This series is particularly useful for analyzing and representing functions that are defined only on half of the interval, such as symmetric functions. In electrical engineering, Fourier series play a vital role in the analysis and design of electrical circuits, encompassing filters, amplifiers, and communication systems. They aid engineers in comprehending the frequency response of circuits, thus optimizing their performance.

10.2 OBJECTIVES

After reading this unit the learner should be able to understand about:

- understand the half range expansions in Fourier series
- comprehend the change of interval in Fourier series
- discuss the Parseval's identity for Fourier series

10.3 HALF-RANGE EXPANSIONS

In Fourier series, periodic function defined in an interval 0 to $2T$, (or C to $C + 2T$ or $-\pi$ to π). Now we suppose that a Fourier series for a function $f(x)$ which is defined only in half-period say ($0 < x < \pi$). There are two cases arise:

(i) Fourier Sine series on $0 < x < \pi$

Half-range Fourier sine series containing only sine terms. Let $f(x)$ be an odd function in

$-\pi < x < \pi$ then the graph $y = f(x)$ will be symmetrical about the origin, we get

$a_0 = a_n = 0$, since $f(x)$ is odd.

and $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\, dx$$

Hence the Fourier sine series in $0 < x < \pi$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx\, dx$$

(ii) Fourier Cosine Series in $0 < x < \pi$

Half-range Fourier cosine series containing only cosine terms. Let $f(x)$ be an even function in $-\pi < x < \pi$ then the graph $y = f(x)$ will be symmetrical about the y-axis, we get

$b_n = 0$, since $f(x)$ is even.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\, dx$$

and $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\, dx$$

Hence the Fourier cosine series in $0 < x < \pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Note: Here in next articles 5.9 to 5.12, we assume the Fourier series is in form $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_0 \cos nx + b_n \sin nx)$.

10.4 CHANGE OF INTERVAL

Generally we have considered only intervals of length of π or 2π, but in many engineering problems the period of functions are different say: T or 2l. In such cases, this interval must be converted to the length 2π. Let $f(x)$ be a periodic function defined in the interval $-l < x < l$. We introduce a new variable z. We have

$$z = \frac{\pi}{l} x \text{ or } x = \frac{l}{\pi} z$$

Then $f(x) = f\left(\frac{l}{\pi} z\right) = F(z)$

At $x = -l,\ z = -\pi$, and at $x = l,\ z = \pi$

Hence *F(z)* is defined in $(-\pi, \pi)$

Now let the Fourier series of *F(z)* defined in the interval $(-\pi, \pi)$ with period 2πbe

$$F(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nz + b_n \sin nz) \qquad \ldots(20)$$

where

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} F(z)\, dz \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} F(z) \cos nz\, dz \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} F(z) \sin nz\, dz \end{aligned}\right\} \qquad \ldots(21)$$

Now to find the Fourier series for *f(x)* in $-l < x < l$. Changing the variable *z* to *x*, we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} F(z) dz \left\{\because z = \frac{\pi}{l} x\right\}$

$$= \frac{1}{\pi}\int_{-l}^{l} f(x) \frac{\pi}{l} dx$$

$$= \frac{1}{l}\int_{-l}^{l} f(x) dx$$

and $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(z) \cos nz\, dz$

$$= \frac{1}{\pi}\int_{-l}^{l} f(x) \cos\frac{n\pi x}{l} . \frac{\pi}{l} dx$$

$$= \frac{1}{l}\int_{-l}^{l} f(x) \cos\frac{n\pi x}{l} dx$$

Similarly, $b_n = \frac{1}{\pi}\int_{-l}^{l} f(x) \sin\frac{n\pi x}{l} dx.$

10.5 PARSEVAL'S IDENTITY FOR FOURIER SERIES

Theorem: Let the Fourier series $\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ converges uniformly to *f(x)* at every points of (0, 2π) then show that $\frac{1}{\pi}\int_0^{2\pi}[f(x)]^2\,dx = \frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2)$.

Proof. We know that Fourier expansion of a function is given

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \qquad \ldots(22)$$

Since the series uniformly converges term by term integration is justified.

where $$a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \qquad \ldots(23)$$

and $$b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \qquad \ldots(24)$$

Equation (22) multiplying both sides by *f(x)* and integrating between limit 0 to 2π, we get

$$\int_0^{2\pi}[f(x)]^2\,dx = \int_0^{2\pi}\frac{a_0}{2}f(x)dx + \int_0^{2\pi}\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\, f(x)dx$$

or $$\int_0^{2\pi}[f(x)]^2\,dx = \frac{a_0}{2}\int_0^{2\pi} f(x)dx + \sum_{n=1}^{\infty}a_n\int_0^{2\pi} f(x)\cos nx\,dx + \sum_{n=1}^{\infty}b_n\int_0^{2\pi} f(x)\sin nx\,dx \qquad \ldots(25)$$

Put n=0 in the equation (23), we get

$$a_0=\frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \quad \Rightarrow \quad \pi a_0=\int_0^{2\pi} f(x)\,dx.$$

Also from the equations (23) and (24), we get

$$\pi a_n=\int_0^{2\pi} f(x)\cos nx\,dx$$

And $$\pi\, b_n=\int_0^{2\pi} f(x)\sin nx\,dx$$

From equation (25), we get

$$\int_0^{2\pi}[f(x)]^2\,dx = \frac{a_0}{2}\pi a_0 + \sum_{n=1}^{\infty}a_n\,.\pi a_n + \sum_{n=1}^{\infty}b_n\,.\pi b_n$$

$$\int_0^{2\pi}[f(x)]^2\,dx = \pi\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_0^2 + b_0^2\right)\right]$$

Hence

$$\frac{1}{\pi}\int_0^{2\pi}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

Examples

Example.1. Find the half-range sine series for x in the interval (0, 2).

Sol. It is given

$$f(x) = x \text{ in interval } (0, 2) \qquad \ldots(26)$$

The Fourier series for $f(x)$ over [-*l*, *l*,] will contain only sine terms given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad \ldots(27)$$

Here l = 2, so we have

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx$$

$$= 2\int_0^2 x.\sin\frac{n\pi x}{2}\,dx$$

$$= \left[x\left\{-\frac{\cos(n\pi x/2)}{(n\pi/2)}\right\} - 1.\left\{-\frac{\cos(n\pi x/2)}{(n^2\pi^2/4)}\right\}\right]_0^2$$

$$= \left[\frac{2.(-\cos n\pi)}{(n\pi/2)} + \frac{\sin n\pi}{(n^2\pi^2/4)}\right]$$

$$= -\frac{4}{n\pi}\cos n\pi$$

$$= -\frac{4}{n\pi}(-1)^n$$

Now putting these values of b_n in equation (27), we get

$$f(x) = \sum_{n=1}^{\infty} -\frac{4}{n\pi}(-1)^n \sin\frac{n\pi x}{2}$$

$$= \frac{4}{\pi}\left[\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \frac{1}{4}\sin\frac{4\pi x}{2} + \ldots\ldots\right].$$

Example 2. Expand $f(x) = \sin x$, $0 < x < \pi$ in a Fourier cosine series.

Sol. We know that the Fourier cosine series in $0 < x < \pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad \ldots(28)$$

Where $\quad a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x \; dx$$

$$= \frac{2}{\pi}[-\cos x]_0^{\pi}$$

$$= \frac{-2}{\pi}[\cos\pi - \cos 0]$$

$$= -\frac{2}{\pi}[(-1) - 1]$$

$$= \frac{4}{\pi}$$

and $\quad a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx \, dx$

$$= \frac{1}{\pi}\int_0^{\pi} 2\cos nx \sin x \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} [\sin(n+1)x - \sin(n-1)x]dx$$

$$= \frac{1}{\pi}\left[\left(\frac{-\cos(n+1)x}{n+1}\right)_0^{\pi} - \left(-\frac{\cos(n-1)x}{n-1}\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{-\cos(n+1)\pi}{n+1} + \frac{1}{n+1} + -\frac{\cos(n-1)x}{n-1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi(n+1)}\left[-(-1)^{n+1} + 1\right] + \frac{1}{\pi(n-1)}\left[(-1)^{n-1} - 1\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^{n+1} - 1}{n+1} - \frac{(-1)^{n-1} - 1}{n-1}\right] n \geq 2.$$

Note. $\quad$ *cos (A + B) = cos A Cos B – sinA sinB* *cos (A- B) = cos A cos B + sinA sinB*

2cos A sin B = sin (A + B) - sin (A - B)

2sin A sin B = cos (A – B) – cos (A + B)

Also $a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x \, dx$

$$= \frac{1}{\pi}\int_0^{\pi} \sin 2x \, dx$$

$$= \frac{1}{\pi}\left(-\frac{\cos 2x}{2}\right)_0^{\pi}$$

$$= -\frac{1}{2\pi}(\cos 2\pi - \cos 0)$$

$$= -\frac{1}{2\pi}(1-1)$$

$$= 0$$

Now putting these values of a_0, a_n in the equation (28), we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{4}{\pi} - \sum_{n=2}^{\infty} \frac{1}{\pi}\left[\frac{(-1)^{n+1}-1}{n+1} - \frac{(-1)^{n-1}-1}{n-1}\right]\cos nx$$

$$= \frac{4}{\pi} - \frac{1}{\pi}\left[\left(\frac{-2}{3} - \frac{-2}{1}\right)\cos 2x + \left(\frac{-2}{5} - \frac{-2}{3}\right)\cos 4x + \ldots\right]$$

$$= \frac{4}{\pi} - \frac{1}{\pi}\left[\left(\frac{-2}{3} - 2\right)\cos 2x + \left(\frac{-2}{5} + \frac{2}{3}\right)\cos 4x + \left(\frac{-2}{5} + \frac{2}{5}\right)\cos 6x + \ldots\ldots\right]$$

$$= \frac{4}{\pi} - \frac{1}{\pi}\left[\frac{4}{3}\cos 2x + \frac{4}{15}\cos 4x + \frac{4}{35}\cos 4x + \frac{4}{35}\cos 6x + \ldots\right]$$

or $\sin x = \frac{4}{\pi} - \frac{4}{\pi}\left[\frac{\cos 2x}{1.3} + \frac{\cos 4x}{3.5} + \frac{\cos 6x}{5.7} + \ldots\right]$

Example 3. Find the Fourier series for f(x)= x^2 - 2 in $-2 \le x \le 2$.

Sol. It is given that $f(x) = x^2 - 2$ …(29)

Since given function is even function, so we have $b_n = 0$.

Then the Fourier series for *f(x)* is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \qquad \ldots(30)$$

Where $a_0=\frac{1}{l}\int_{-l}^{l} f(x)dx$

$$=\frac{1}{2}\int_{-2}^{2}\left(x^2-2\right)dx$$

$$=\int_{0}^{2}\left(x^2-2\right)dx$$

$$=\left[\frac{x^3}{3}-2x\right]_0^2$$

$$=\frac{8}{3}-4$$

$$=-\frac{4}{3}$$

and $a_n=\frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx$

$$=\frac{1}{2}\int_{-2}^{2}\left(x^2-2\right)\cos\frac{n\pi x}{2}dx$$

$$=\int_{-2}^{2}\left(x^2-2\right)\cos\frac{n\pi x}{2}dx$$

$$=\left[\left(x^2-2\right)\left\{\frac{\sin\left(n\pi x/2\right)}{\left(n\pi/2\right)}\right\}-\left(2x\right)\left\{\frac{\sin\left(n\pi x/2\right)}{\left(n^2\pi^2/4\right)}\right\}+\left(2\right)\left\{-\frac{\sin\left(n\pi x/2\right)}{\left(n^3\pi^3/2\right)}\right\}\right]_0^2$$

$$=\left[2.\frac{\sin n\pi}{\left(n\pi/2\right)}+\frac{4}{\left(n^2\pi^2/4\right)}\cos n\pi-2\frac{\sin n\pi}{n^3\pi^3/8}\right]$$

$$=\frac{16}{n^2\pi^2}\cos n\pi$$

$$=\frac{16}{n^2\pi^2}\left(-1\right)^n$$

Now putting these values of a_0, a_n in the equation (30), we get

$$f(x)=-\frac{2}{3}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{2}$$

$$= -\frac{2}{3}+\sum_{n=1}^{\infty}\left[\frac{16}{n^2\pi^2}(-1)^n \cos\frac{n\pi x}{2}\right]$$

$$= -\frac{2}{3}+\frac{16}{\pi^2}\left[\cos\frac{\pi x}{2}-\frac{1}{4}\cos\pi x+\frac{1}{9}\cos\frac{3\pi x}{2}-......\right].$$

Example.4. Find the Fourier series for f(x)= e^{-x} in (-l, l).

Sol. We know that the Fourier series for *f(x)* in interval (-*l*, *l*) is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\frac{n\pi x}{l}+b_n\frac{\sin n\pi x}{l}\right] \qquad ...(31)$$

where $$a_0=\frac{1}{l}\int_{-l}^{l} f(x)dx$$

$$=\frac{1}{l}\int_{-l}^{l} e^{-x}\,dx$$

$$=\frac{1}{l}\left[\frac{e^{-x}}{-1}\right]_{-l}^{l}$$

$$=\frac{1}{l}\left[-e^{-l}+e^{l}\right]$$

$$=\frac{1}{l}\left[e^{l}-e^{-l}\right]$$

$$=\frac{2}{l}\sin hl.$$

and $$a_n=\frac{1}{l}\int_{-l}^{l} e^{-x}\frac{\cos n\pi x}{l}dx$$

$$=\frac{1}{l}\int_{-l}^{l} e^{-x}\frac{\cos n\pi x}{l}dx$$

$$\left\{\because \int e^{ax}\cos bx\,dx=\frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)\right\}$$

$$=\frac{1}{l}\left\{\frac{e^{-x}}{1+\frac{n^2\pi^2}{l^2}}\left(-\cos\left(\frac{n\pi x}{l}\right)+\frac{n\pi}{l}\sin\left(\frac{n\pi x}{l}\right)\right)\right\}_{-l}^{l}$$

$$=\frac{l}{l^2+n^2\pi^2}\left[e^{-x}\left\{-\cos\frac{n\pi x}{l}+\frac{n\pi}{l}\sin\frac{n\pi x}{l}\right\}\right]_{-l}^{l}$$

$$=\frac{l}{l^2+n^2\pi^2}\left[e^{-l}\left(-\cos n\pi+\frac{n\pi}{l}\sin n\pi\right)-e^{l}\left(-\cos n\pi-\frac{n\pi}{l}\sin n\pi\right)\right]_{-l}^{l}$$

$$= \frac{l}{l^2+n^2\pi^2}\left[\left(e^l - e^{-l}\right)\cos n\pi\right]$$

$$= \frac{2l}{l^2+n^2\pi^2}\ \cos n\pi . \sin hl$$

Also $b_n = \frac{1}{l}\int_{-l}^{l} f(x)\frac{\sin \pi x}{l}dx$

$$= \frac{1}{l}\int_{-l}^{l} e^{-l}\sin\frac{n\pi x}{l}dx$$

$$= \frac{1}{l}\left\{\frac{e^{-x}}{1+\frac{n^2\pi^2}{l^2}}\left(-\sin\frac{n\pi x}{l} - \frac{n\pi}{l}\cos\frac{n\pi x}{l}\right)\right\}_{-l}^{l}$$

$$\left\{\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right)\right\}$$

$$= \frac{l}{l^2+n^2\pi^2}\left[e^{-l}\left(-\sin n\pi + \frac{n\pi}{l}\cos n\pi\right) - e^{l}\left(\sin n\pi - \frac{n\pi}{l}\cos n\pi\right)\right]$$

$$= \frac{l}{l^2+n^2\pi^2}\left[\frac{n\pi}{l}\cos n\pi\left(e^l - e^{-l}\right)\right]$$

$$= \frac{2n\pi}{l^2+n^2\pi^2}\cos n\pi . \sinh l.$$

Now putting these values of a_0, a_n in the equation (31), we get

$$f(x) = \frac{\sin hl}{l} + \sum_{n=1}^{\infty}\left[\frac{2l\cos n\pi . \sin hl}{l^2+n^2\pi^2}\ .\cos nx + \frac{2n\pi\cos n\pi . \sinh l}{l^2+n^2\pi^2}.\sin nx\right]$$

Example.5. Find the Fourier series $f(x) = \begin{cases} \frac{1}{2}+x & -\frac{1}{2} < x \le 0 \\ \frac{1}{2}-x & 0 < x < \frac{1}{2} \end{cases}$ with period 1.

Sol. We know that the Fourier series for *f(x)* in interval (-*l*, *l*) is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\ \frac{n\pi x}{l} + b_n\ \frac{\sin n\pi x}{l}\right] \qquad \ldots(32)$$

where $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$

Here l = 1/2, so we have

$$a_0 = \frac{1}{1/2}\int_{-1/2}^{1/2} f(x)dx$$

$$= 2\int_{-1/2}^{0}\left(\frac{1}{2}+x\right)dx + 2\int_{0}^{1/2}\left(\frac{1}{2}-x\right)d\,x$$

$$= 2\left[\frac{x}{2}+\frac{x^2}{2}\right]_{-1/2}^{0} + 2\left[\frac{x}{2}-\frac{x^2}{2}\right]_{0}^{-1/2}$$

$$= 2\left[\frac{1}{4}-\frac{1}{8}\right] + 2\left[\frac{1}{4}-\frac{1}{8}\right]_{0}^{-1/2}$$

$$= 2\left[\frac{1}{8}\right] + 2\left[\ \frac{1}{8}\right]$$

$$=\frac{1}{2}$$

And $$a_n = \int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\ dx$$

$$= \frac{1}{1/2}\int_{-1/2}^{1/2} f(x)\cos\frac{n\pi x}{1/2}\ dx$$

$$= 2\int_{-1/2}^{0}\left(\frac{1}{2}+x\right)\cos 2n\pi x\ dx + 2\int_{0}^{\pi/2}\left(\frac{1}{2}-x\right)\cos 2n\pi x\ dx$$

$$= 2\left[\left(\frac{1}{2}+x\right)\ \frac{\sin 2n\pi x}{2n\pi}\ \ dx - 1\ .\left(-\frac{\cos 2n\pi x}{2n^2\pi^2}\right)\right]_{-1/2}^{0}$$

$$+\ 2\left[\left(\frac{1}{2}-x\right)\ \frac{\sin 2n\pi x}{2n\pi}\ -1(-1)\ \left(-\frac{\cos 2n\pi x}{4n^2\pi^2}\right)\right]_{0}^{1/2}$$

$$= 2\left[\frac{1}{4n^2\pi^2} - \frac{\cos 2n\pi}{4n^2\pi^2}\right] + 2\left[-\frac{\cos n\pi}{4n^2\pi^2} + \frac{1}{4n^2\pi^2}\right]$$

$$= \frac{1}{n^2\pi^2} - \frac{(-1)^n}{n^2\pi^2}$$

$$= \frac{1}{\pi^2} - \left[\frac{1}{n^2} - \frac{(-1)^n}{n^2}\right]$$

$$= \begin{cases} \dfrac{2}{n^2\pi^2} & if\ \ n\ is\ odd \\ 0 & if\ \ n\ is\ even \end{cases}$$

Similarly, $$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{1/2}\int_{-1/2}^{1/2} f(x)\sin\frac{n\pi x}{1/2}\,dx$$

$$= 2\int_{-1/2}^{0}\left(\frac{1}{2}+x\right)\sin 2n\pi x\,dx + 2\int_{0}^{1/2}\left(\frac{1}{2}-x\right)\sin 2n\pi x\,dx$$

$$= 2\left[\left(\frac{1}{2}+x\right)\left(-\frac{\cos 2n\pi x}{2n\pi}\right) - (1)\left(-\frac{\sin 2n\pi x}{4n^2\pi^2}\right)\right]_{-1/2}^{0}$$

$$+\ 2\left[\left(\frac{1}{2}-x\right)\left(-\frac{\cos 2n\pi x}{2n\pi}\right) - (-1)\left(-\frac{\sin 2n\pi x}{4n^2\pi^2}\right)\right]_{0}^{1/2}$$

$$= -\left[\frac{1}{2n\pi}\right] + \left[\frac{1}{2n\pi}\right]$$

$$= 0$$

Now putting these values of a_0, a_n and b_n in the equation (32), we get

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\left[\frac{2}{n^2\pi^2}\cos\frac{n\pi x}{l} + 0\,.\sin\frac{n\pi x}{l}\right]$$

$$= \frac{1}{4} + \frac{2}{\pi^2}\left[\sum_{n=1}^{\infty}\frac{1}{n^2}\,.\cos\frac{n\pi x}{1/2}\right]$$

$$= \frac{1}{4} + \frac{2}{\pi^2}\left[\frac{\cos 2\pi x}{1^2} + \frac{\cos 6\pi x}{3^2} + \frac{\cos 10\pi x}{5^2} + \ldots\ldots\right].$$

Example 6. Find the Fourier series for f(x) = x^2 in ($-\pi$, π). Also prove that $\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$ using Perseval's theorem.

Sol. Given that $f(x) = x^2$

Since the given function is even so $b_n = 0$ …(33)

And the Fourier series for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad \ldots(34)$$

where $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\, dx = \frac{2}{\pi}\int_{0}^{\pi} x^2\, dx = \frac{2}{\pi}\left[\frac{x^3}{2}\right]_0^x = \frac{2\pi^2}{3}$$

and $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos \mathrm{nx}\, \mathrm{dx}$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos \mathrm{nx}\, \mathrm{dx}$$

$$= \frac{2}{\pi}\left[\frac{x^2 \sin nx}{n} + \frac{2x.\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2\pi.\cos n\pi}{n^2}\right]$$

$$= \frac{4}{n^2}(-1)^n.$$

By equation (34), we have

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

Now by Perseval’s theorem, we have

$$\int_{-\pi}^{\pi}[f(x)]^2\, dx = \pi\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)\right]$$

$$\Rightarrow \quad \int_{-\pi}^{\pi}[f(x)]^2\, dx = \pi\left[\frac{4.\pi^4}{2.9} + \sum_{n=1}^{\infty}\left(\frac{16}{n^4} + 0\right)\right]$$

or $$\left[\frac{x^5}{5}\right]_{-\pi}^{\pi} = \pi\left[\frac{2\pi^4}{9} + \sum_{n=1}^{\infty}\frac{16}{n^4}\right]$$

or $$\left[\frac{\pi^5}{5} - \frac{(-\pi)^5}{5}\right] = \frac{2\pi^5}{9} + \pi\sum_{n=1}^{\infty}\frac{16}{n^4}$$

or $$\frac{2\pi^5}{5} = \frac{2\pi^5}{9} + \pi\sum_{n=1}^{\infty}\frac{16}{n^4}$$

or $$\frac{2\pi^5}{5} - \frac{2\pi^5}{9} = \pi\sum_{n=1}^{\infty}\frac{16}{n^4}$$

or $$\frac{16\pi^5}{90} = \pi\sum_{n=1}^{\infty}\frac{16}{n^4}$$

or $$\frac{16\pi^5}{90} = \sum_{n=1}^{\infty} \frac{16}{n^4}$$

or $$\frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

Hence $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$

10.5 SUMMARY

1. Now to find the Fourier series for $f(x)$ in $-l < x < l$. Changing the variable z to x, we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

where $a_0 = \frac{l}{l}\int_{-l}^{l} f(x)dx \left\{\because z = \frac{\pi}{l}x\right\}$

$a_n = \frac{1}{\pi}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx$, Similarly, $b_n = \frac{1}{\pi}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx$

3. The Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty}$ $(a_n \cos nx + b_n \sin nx)$ converges uniformly to $f(x)$ at every points of $(0, 2\pi)$ then $\frac{1}{\pi}\int_0^{2\pi}[f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$

10.6 TERMINAL QUESTIONS

Q.1 Explain the concept of half range expansions in fourier series..

Q.2 State and prove Parseval's identity.

Q.3 Find the half-range sine series for e^x in (0, 1).

Q.4 Find the half-range cosine series for x in (0, 2).

Q.5 If $f(x) = \begin{cases} \sin x & \text{for} \quad 0 \le x \le \pi/4 \\ \cos x & \text{for} \quad \pi/4 \le x \le \pi/2 \end{cases}$. Expand $f(x)$ in series of sine terms.

Q.6 Expand $f(x) = \begin{cases} \frac{1}{4} - x, & if \ 0 < x < \frac{1}{2} \\ x - \frac{3}{4}, & if \ \frac{1}{2} < x < 1 \end{cases}$ as the Fourier series of sine terms.

Q.7 Find the half-range sine series to represent $f(x) = x(\pi - x)$ for $0 \leq x \leq \pi$.

Q.8 If $f(x) = \begin{cases} \pi x & 0 \leq x \leq 1 \\ \pi(2-x) & 1 \leq x \leq 2 \end{cases}$. Show that in the interval (0, 2);

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + ...\right]$$

Q.9 Find the Fourier series for $f(x) = x - x^2$ in (-1, 1).

Q.10 Find the Fourier series for $f(x) = \begin{cases} x, & -1 \leq x \leq 1 \\ x+2 & 0 \leq x \leq 1 \end{cases}$ and hence deduce the sum of the series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$$

Answer

3. $2\pi\left[\frac{1+e}{1+\pi^2}\sin \pi x + \frac{2(1-e)}{1+4\pi^2}\sin 2\pi x + \frac{3(1+e)}{1+9\pi^2}\sin 3\pi x + ...\right]$

4. $1 - \frac{8}{\pi^2}\left[\frac{\cos \pi x/2}{1^2} + \frac{\cos 3\pi x/2}{3^2} + \frac{\cos 5\pi x/2}{5^2}\right]$

5. $\frac{8}{\pi}\cos\frac{\pi}{4}\left[\frac{\sin 2x}{1.3} + \frac{\sin 6x}{5.7} + \frac{\sin 10x}{9.11} + ...\right]$

6. $\left(\frac{1}{\pi} - \frac{4}{\pi^2}\right)\sin \pi x + \left(\frac{1}{3\pi} + \frac{4}{3^2\pi^2}\right)\sin 3x + \left(\frac{1}{5\pi} - \frac{4}{5^2\pi^2}\right)\sin 5\pi x + ...$

7. $\frac{8}{\pi}\left(\sin x + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + ...\right)$

9. $-\frac{1}{3} + \frac{4}{\pi^2}\left[\frac{\cos \pi x}{1^2} - \frac{\cos 2\pi x}{2^2} + \frac{\cos 3\pi x}{3^2} - ....\right] + \frac{2}{\pi}\left[\frac{\sin \pi x}{1} - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} - ....\right]$

10. $1 + \frac{2}{\pi^2}\left[3\sin \pi x - \frac{1}{2}\sin \pi x + \frac{1}{3}(3)\sin 3\pi x - \frac{1}{4}\sin 4\pi x + \frac{1}{5}(3)\sin 5\pi x - \frac{1}{6}\sin 6\pi x + ...\right]$

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Advanced Real Analysis
And Integral Equations

BLOCK

4

INTEGRAL EQUATIONS

UNIT-11

Classifications of Integral Equations

UNIT-12

Fredholm Integral Equations-I

UNIT-13

Fredholm Integral Equations-II

UNIT-14

Volterra Integral Equations

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BLOCK INTRODUCTION

Integral equations are mathematical equations where an unknown function appears under one or more integral signs. They are extensively utilized to model a broad spectrum of phenomena across numerous disciplines such as physics, engineering, economics, and biology. These equations can be categorized into two main types: Fredholm Integral Equations and Volterra Integral Equations. Integral equations have widespread applications, spanning electromagnetics, acoustics, heat transfer, fluid dynamics, quantum mechanics, signal processing, and image processing. They offer a robust mathematical framework for representing real-world phenomena and addressing practical engineering challenges.

Integral equations find applications across numerous fields due to their ability to model diverse phenomena and solve practical problems. Integral equations are a fundamental part of mathematical analysis and are extensively studied in pure mathematics. They provide insights into the behavior of functions and help in understanding the properties of various mathematical operators. Integral equations are used in biology and medicine to model biological processes, population dynamics, and the spread of diseases. They help in understanding the dynamics of ecosystems, predicting the effects of interventions, and analyzing medical imaging data. Hence the integral equations provide a powerful mathematical framework for modeling complex systems, analyzing data, and solving practical problems across a wide range of disciplines.

In the eleventh unit, we shall discuss about introduction and classifications of integral equations. In unit twelveth we shall discuss about Fredholm Integral equation, Fredholm first theorem, Fredholm second theorem, Fredholm third theorem. Fredholm Integral equation, resolved kernel for Fredholm integral equation and separable kernel are discussed in unit thirteen. In unit fourtheen we shall discussed the Volterra Integral equation, Solution of non-homogeneous Volterra integral equation of second kind by the method of successive substitution and successive approximation, iterated kernels.

UNIT-11 CLASSIFICATIONS OF INTEGRAL EQUATIONS

Structure

11.1 Introduction

11.2 Objectives

11.3 Integral Equation

11.4 Types of Integral Equations

11.5 Linear Integral Equations

11.6 Volterra Integral Equations

11.7 Fredholm Integral Equations

11.8 Non-Linear Integral Equations

11.9 Singular Integral Equations

11.10 Types of Kernels

11.11 Conversion of multiple integral into a single ordinary integral

11.12 Summary

11.13 Terminal Questions

11.1 INTRODUCTION

An integral equation is an equation in which the unknown function occurs under the integral sign. The name "integral equation" for any equations involving the unknown function $\emptyset(x)$ under the integral sign was introduced by du Bois-Reymond in 1888. In 1782, Laplace used the integral transform $\boldsymbol{f(x)} = \int_{\mathbf{0}}^{\infty} \boldsymbol{e}^{\xi x}\boldsymbol{f}(\xi)\boldsymbol{d}\xi$ to solve the linear difference equations and differential equations. In 1826, Abel solved the integral equation and named after him having the form of $\boldsymbol{f(x)} = \int_{\mathbf{0}}^{x}(\boldsymbol{x}-\xi)^{-\boldsymbol{\alpha}}\,\emptyset(\xi)\boldsymbol{d}\xi$**,** where $\boldsymbol{f(x)}$ is a continuous function satisfying $f(a) = 0$ *and* $0 < \alpha < 1.$ Huygens solved the Abel's integral equation for $\alpha = 1/2$.

In 1826, Poisson obtained an integral equation of the type $\emptyset(\boldsymbol{x}) = \boldsymbol{f(x)} + \boldsymbol{\lambda}\int_{\mathbf{0}}^{x}\mathbf{K(x,\xi)}\emptyset(\boldsymbol{\xi})\mathbf{d\xi}$ in which the unknown function $\emptyset(\xi)$ occurs outside as well as before the integral sign and the variable x appears as one of the limits of the integral. Dirichlet's problem, which is the determination of a function Ψ having prescribe values over a certain boundary surface S and satisfying Laplace Equations $\nabla^2\Psi = 0$ within the region enclosed by S, we shown by heuman in 1870 to be equivalent to the solution of an integral equation. He solved the integral equation by an expansion in powers of a certain parameter λ. In 1896, Volterra gave the first general solution of class of linear integral equation in variable x appearing as the upper limit of the integral. In 1990, Fredholms have discussed a more general class of linear Integral equation and defined as $\emptyset(\boldsymbol{x}) = \boldsymbol{f(x)} + \boldsymbol{\lambda}\int_{\mathbf{a}}^{\boldsymbol{b}}\mathbf{K(x,\xi)}\emptyset(\boldsymbol{\xi})\mathbf{d\xi}$.

11.2 OBJECTIVES

After reading this unit the learner should be able to understand about:

- the integral equations and its types
- type of kernels
- Conversion of multiple integral into a single ordinary integral

11.3 INTEGRAL EQUATIONS

An equation which contains unknown function under one or more integral signs is known as integral equation.

$$f(x) = \int_a^x K(x,t)\emptyset(\xi)\boldsymbol{d}\xi \qquad \ldots\ldots. (1)$$

$$\emptyset(x) = f(x) + \lambda \int_a^x K(x,\xi)\emptyset(\xi)d\xi \qquad \ldots\ldots (2)$$

$$\emptyset(x) = \int_a^b K(x,\ \xi)[\emptyset(\xi)]^2 d\xi \qquad \ldots\ldots (3)$$

Where $\emptyset(x)$ is unknown function and $f(x)$, $K(x,\xi)$ are known functions $\lambda,\ a\ and\ b$ are constants.

Examples

Example.1. Verify that the given function $u(x) = \frac{1}{2}$ is the solution of the integral equation $\int_0^x \frac{u(t)}{\sqrt{(x-t)}}dt = \sqrt{x}$.

Solution: Here, the given integral equation is

$$\int_0^x \frac{u(t)}{\sqrt{(x-t)}}dt = \sqrt{x}$$

Putting value $u(x) = \frac{1}{2}$ in the given equation, we get

$$\frac{1}{2}\int_0^x \frac{dt}{\sqrt{(x-t)}} = \sqrt{x}$$

i.e., $$-\frac{1}{2}\left(2.\sqrt{x-t}\,\right)_0^x = \sqrt{x}$$

or $$\sqrt{x} = \sqrt{x},$$ Which is an identity in x.

Hence the function $u(x) = \frac{1}{2}$ is the solution of the integral equation.

11.4 TYPES OF INTEGRAL EQUATIONS

There are three types of integral equations:

(i) Linear integral equations

(ii) Non linear integral equations

(iii) Singular integral equations.

11.5 LINEAR INTEGRAL EQUATIONS

An integral equation is called linear integral equation if there is only linear function as unknown function under the integral sign.

For example: $\alpha(x).\emptyset(x) = f(x) + \lambda\int_a^b \mathrm{K(x,\xi)}\emptyset(\xi)\mathrm{d\xi}$ is linear as the unknown function $\mathrm{K(x,\xi)}\emptyset(\xi)$ is linear.

This Linear integral equation is further has been divided into two parts.

(i) Volterra integral equation and (ii) Fredholm integral equation.

11.6 VOLTERRA INTEGRAL EQUATIONS

An integral equation is said to be volterra integral equation if the upper limit of integration is variable.

For example: The equation $\alpha(x).\emptyset(x) = f(x) + \lambda\int_a^x K(x,\xi)\emptyset(\xi)d\xi$ …(1)

Here upper limit is *x* which is variable.

Case-1: If α = 0 then from equation (1), we get

$$f(x) = -\lambda\int_a^x K(x,\xi)\emptyset(\xi)d\xi,$$

which is called Volterra integral of first kind.

Case-II: If α = 1 then from equation (1), we get

$$\emptyset(x) = f(x) + \lambda\int_a^x K(x,\xi)\emptyset(\xi)d\xi,$$

which is called Volterra integral equation of second kind.

Case-III: If α = 1, *f*(*x*) = 0 then from equation (1), we get

$$\emptyset(x) = \lambda\int_a^x K(x,\xi)\emptyset(\xi)d\xi,$$

which is called homogeneous Volterra integral equation.

11.7 FREDHOLM INTEGRAL EQUATIONS

An integral equation is called Fredholm integral equation if both the limits are constants [domain of integration is fixed.

For example: The equation $\alpha(x)\emptyset(x) = f(x) + \lambda\int_a^b K(x,\xi)\emptyset(\xi)d\xi$ ….(1)

is called Fredholm integral equation.

Case-1: If α = 0 then from equation (1), we get

$F(x) = \lambda \int_a^b K(x,\xi)\emptyset(\xi)d\xi,$

which is called Fredholm integral equation of first kind.

Case-II: If α = 1 then from equation (1), we get

$\emptyset(x) = f(x) + \lambda \int_a^b K(x,\xi)\emptyset(\xi)d\xi,$

which is called Fredholm integral equation of second kind.

Case-III: If α = 1, *f*(*x*) = 0 then from equation (1), we get

$\emptyset(x) = \lambda \int_a^b K(x,\xi)\emptyset(\xi)d\xi,$

which is called homogeneous Fredholm integral equation.

11.8 NON-LINEAR INTEGRAL EQUATIONS

If the unknown function appears under an integral sign to some power greater than one, is known as non-linear integral equation.

For example: $\emptyset(x) = f(x) + \lambda \int_a^b K(x,\xi)\emptyset^n(\xi)d\xi$ (n> 1).

11.9 SINGULAR INTEGRAL EQUATIONS

When one or both limits of integration are infinite or the kernel $K(x,\xi)$ becomes infinite at one or more points within the range of integration, is called singular integral equation.

For example: $\emptyset(x) = f(x) + \lambda \int_{-\infty}^{\infty} \exp\{-|x-\xi|\}\emptyset(\xi)d\xi$

and $\emptyset(x) = f(x) = \int_{-\infty}^{\infty} \frac{1}{(x-\xi)^\alpha}\ \emptyset(\xi)d\xi.$

Note: Convolution integration. If the kernel of the integral equation is of the form $K(x,\xi) = K(x-\xi)$

For example: $\emptyset(x) = e^x + \lambda \int_a^b [(x-\xi)^2 + 3(x-\xi)]\emptyset(\xi)d\xi.$

Examples

Example.2: Given that $\emptyset(x) = \left(1 - x + \frac{x^3}{6}\right) + \int_0^x \left[sin\xi - (x-\xi)\left(cos\xi + e^{\xi}\right)\right]\emptyset(\xi)d\xi$

Determine the values of $\emptyset'(x) and\ \emptyset''(x)$.

Solution: It is given that

$$\emptyset(\boldsymbol{x}) = \left(\mathbf{1} - \boldsymbol{x} + \frac{\boldsymbol{x}^3}{\mathbf{6}}\right) + \int_{\mathbf{0}}^{\boldsymbol{x}} \left[\boldsymbol{sin\xi} - (\boldsymbol{x}-\boldsymbol{\xi})\left(\boldsymbol{cos\xi} + \boldsymbol{e}^{\xi}\right)\right]\emptyset(\boldsymbol{\xi})\boldsymbol{d\xi} \qquad ...(1)$$

To determine the values of $\emptyset'(x)\ and\ \emptyset''(x)$

Differentiating equation (1) with respect to x, we have

$$\emptyset'(x) = \left(0 - 1 + \frac{3x^2}{6}\right)$$
$$+ \int_0^x \left[\frac{\partial}{\partial x}\{sin\xi - (x-\xi)\left(cos\xi + e^{\xi}\right)\}\emptyset(\xi)\right] d\xi$$
$$+ \left[\{sin\xi - (x-\xi)\left(cos\xi + e^{\xi}\right)\}\emptyset(\xi)\right]_{\xi=x} \frac{d}{dx}(x)$$
$$- \left[\{sin\xi - (x-\xi)\left(cos\xi + e^{\xi}\right)\}\emptyset(\xi)\right]_{\xi=0} \frac{d}{dx}(0)$$

$$\emptyset'(x) = \left(-1 + \frac{x^2}{2}\right) + \int_0^x \left[\{0 - \left(cos\xi + e^{\xi}\right)\}\emptyset(\xi)\right] d\xi + [sinx - (x-x)(cosx + e^x)]\emptyset(x).1 - 0$$

$$\left[\because \frac{d}{dx}(0) = 0\right] \emptyset'(x) = -1 + \frac{x^2}{2} - \int_0^x \left[\left(cos\xi + e^{\xi}\right)\emptyset(\xi)\right] d\xi + sinx.\emptyset(x)$$

Again differentiating with respect to x, we have

$$\emptyset''(x) = -0 + \frac{2x}{2}$$
$$- \int_0^x \left[\frac{\partial}{\partial x}\left\{\left(cos\xi + e^{\xi}\right)\emptyset(\xi)\right\}\right] d\xi + \left\{\left(cos\xi + e^{\xi}\right)\emptyset(\xi)\right\}_{\xi=x} \frac{d}{dx}(x)$$
$$- \left\{\left(cos\xi + e^{\xi}\right)\emptyset(\xi)\right\}_{\xi=0} \frac{d}{dx}(0) + sinx.\emptyset'(x) + \emptyset(x)cosx$$
$$\emptyset''(x) = x - 0 - (e^x + cosx)\,\emptyset(x) + \ sinx.\emptyset'(x) + \emptyset(x)cosx$$
$$\emptyset''(x) = x - e^x\,\emptyset(x) - cosx.\emptyset(x) + \ sinx.\emptyset'(x) + \emptyset(x)cosx$$
$$\emptyset''(x) - \ sinx.\emptyset'(x) + e^x\,\emptyset(x) = x$$

Example.3: Given that

$$\boldsymbol{\emptyset(x) = (1 - x - 4sinx) + \int_0^x [3 - 2(x-\xi)]\emptyset(\xi)d\xi}$$

Determine the values of $\emptyset'(x)$ *and* $\emptyset''(x)$.

Solution: It is given that

$$\emptyset(x) = (1 - x - 4sinx) + \int_0^x [3 - 2(x-\xi)]\emptyset(\xi)d\xi \qquad \ldots(1)$$

Differentiating equation (1) with respect to x, we have

$$\emptyset'(x) = (0 - 1 - 4cosx) + \int_0^x \left[\frac{\partial}{\partial x}\left\{\left(3 - 2(x-\xi)\right)\emptyset(\xi)\right\}\right] d\xi + [\{3 - 2(x-\xi)\}\emptyset(\xi)]_{\xi=x}\frac{d}{dx}(x)$$

$$- +[\{3 - 2(x-\xi)\}\emptyset(\xi)]_{\xi=0}\frac{d}{dx}(0)\emptyset'(x)$$

$$= (-1 - 4cosx) + \int_0^x -2\emptyset(\xi)d\xi + [\{3 - 2(x-x)\}\emptyset(x)]\ \ .1 - 0$$

$$\emptyset'(x) = (-1 - 4cosx) - 2\int_0^x \emptyset(\xi)d\xi + 3\emptyset(x)$$

Again differentiating with respect to x, we have

$$\emptyset''(x) = 0 + 4sinx - [2\int_0^x \left\{\frac{\partial}{\partial x}\emptyset(\xi)\right\}d\xi + [\emptyset(\xi)]_{\xi=x}\frac{d}{dx}(x) - [\emptyset(\xi)]_{\xi=0}\frac{d}{dx}(0)]\ + 3\emptyset'(x)$$

$$\emptyset''(x) = 4sinx - 2[0 + \emptyset(x) - 0] + 3\emptyset'(x)$$

$$\emptyset''(x) - 3\emptyset'(x) + 2\emptyset(x) = 4sinx.$$

Example.4: If $\emptyset(x) = 3 + \int_0^x (5x - 3\xi)\emptyset(\xi)d\xi$ then determine the values of $\emptyset'(x)\ and\ \emptyset''(x)$.

Solution: It is given that

$$\emptyset(x) = 3 + \int_0^x (5x - 3\xi)\emptyset(\xi)d\xi \qquad \ldots(1)$$

Differentiating equation (1) with respect to *x*, we have

$$\emptyset'(x) = 0 + \int_0^x \left[\frac{\partial}{\partial x}\{(5x - 3\xi)\emptyset(\xi)\}\right]d\xi + \{(5x - 3\xi)\emptyset(\xi)\}_{\xi=x}\frac{d}{dx}(x)$$

$$-\{(5x - 3\xi)\emptyset(\xi)\}_{\xi=0}\frac{d}{dx}(0)\emptyset'(x) = \int_0^x (5 - 0)\emptyset(\xi)d\xi + (5x - 3x)\emptyset(x)$$

$$\emptyset'(x) = \int_0^x 5.\emptyset(\xi)d\xi + 2x.\emptyset(x)$$

Again differentiating with respect to *x*, we have

$$\emptyset''(x) = \int_0^x \left[\frac{\partial}{\partial x}\{5.\emptyset(\xi)\}\right]d\xi + \{5\emptyset(\xi)\}_{\xi=x}\frac{d}{dx}(x) - \{5\emptyset(\xi)\}_{\xi=0}\frac{d}{dx}(0) + 2x.\emptyset'(x) + 2\emptyset(x)$$

$$\emptyset''(x) = 0 + 5\emptyset(x) - 0 + 2x.\emptyset'(x) + 2\emptyset(x)$$

$\emptyset''(x) - 2x.\emptyset'(x) - 7\emptyset(x) = 0.$

Example.5: If $\emptyset(x) = \int_0^x (x + \xi)\emptyset(\xi)d\xi$ then determine the values of $\emptyset'(x) and\ \emptyset''(x)$.

Solution: It is given that

$$\emptyset(x) = \int_0^x (x+\xi)\emptyset(\xi)d\xi \qquad \text{... (1)}$$

Differentiating equation (1) with respect to x, we have

$$\emptyset'(x) = \int_0^x \left[\frac{\partial}{\partial x}\{(x+\xi)\emptyset(\xi)\}\right] d\xi + [(x+\xi)\emptyset(\xi)]_{\xi=x}\frac{d}{dx}(x) - [(x+\xi)\emptyset(\xi)]_{\xi=0}\frac{d}{dx}(0)$$

$$\emptyset'(x) = \int_0^x \emptyset(\xi)d\xi + [(x+x).\emptyset(x)].1 - 0$$

$$\emptyset'(x) = \int_0^x \emptyset(\xi)d\xi + 2x\,\emptyset(x)$$

Again differentiating with respect to x, we have

$$\emptyset''(x) = \int_0^x \left[\frac{\partial}{\partial x}\emptyset(\xi)\right] d\xi + 2x.\emptyset'(x) + 2\emptyset(x) + \emptyset(x)$$

$$\emptyset''(x) = 0 + 2x.\emptyset'(x) + 3\emptyset(x)$$

or $$\emptyset''(x) - 2x.\emptyset'(x) - 3\emptyset(x) = 0.$$

Note: $\int_a^x f(\xi)d\xi^n = \int_a^x \frac{(x-\xi)^{n-1}}{(n-1)!} f(\xi)d\xi.$

11.10 TYPES OF KERNELS

Symmetric Kernel- A Kernel $k(x,t)$ is Symmetric (or complex symmetric or Hermitian) if $k(x,t) = \overline{k(x,t)}$, where bar denotes the complex conjugate.

A real kernel $k(x,t)$ is symmetric if

$$k(x,t) = k(t,x) .$$

For example: $sin(x+t), e^{xt}, x^3t^3 + x^2t^2 + xt + 1$ are all symmetric kernels.

Separable or Degenerate Kernel- A kernel which is particularly useful in solving the Fredholm equation has the form

$k(x,t) = \sum_{i=1}^{n} a_i(x)b_i(t)$, where n is finite and a_i, b_i are linearly independent sets of functions. Such a kernel is known as separable or degenerate kernel.

Remark: A degenerate kernel has a finite number or characteristic values.

Transposed Kernel- The kernel $k^T(x,t) = k(t,x)$ is called the transposed kernel of $k(x,t)$.

Iterated Kernel

(i) Consider Fredholm integral equation of the second kind

$$u(x) = f(x) + \lambda \int_a^b K(x,\xi)\emptyset(\xi)d\xi \qquad \ldots(1)$$

Then, the iterated kernels $k_n(x,t), n = 1,2,3, \ldots$ ...are defined as follows

$$k_1(x,t) = k(x,t)$$

and $\quad k_n(x,t) = \int_a^b k(x,s)k_{n-1}(s,t)ds, n = 2,3 \ldots$

(ii) Consider Volterra integral equation of the second kind

$$u(x) = f(x) + \lambda \int_a^b K(x,\xi)\emptyset(\xi)d\xi \qquad \ldots(2)$$

Then, the iterated kernels $k_n(x,t), n = 1,2,3, \ldots$ ...are defined as follows

$$k_1(x,t) = k(x,t)$$

and $k_n(x,t) = \int_a^b k(x,s)k_{n-1}(s,t)ds, n = 2,3 \ldots$

Resolvent Kernel or Reciprocal Kernel- Consider the integral equations

$$u(x) = f(x) + \lambda \int_a^b K(x,\xi)\emptyset(\xi)d\xi \qquad \ldots(1)$$

and $$u(x) = f(x) + \lambda \int_a^b K(x,\xi)\emptyset(\xi)d\xi \qquad \ldots(2)$$

Let the solution of equations (1) and (2) be given by

$$u(x) = f(x) + \lambda \int_a^b R(x,\xi,\lambda)\emptyset(\xi)d\xi \qquad \ldots(3)$$

and $$u(x) = f(x) + \lambda \int_a^b \Gamma(x,\xi,\lambda)\emptyset(\xi)d\xi \qquad \ldots(4)$$

Then, $R(x,\xi,\lambda)$ or $\Gamma(x,\xi,\lambda)$ is called the resolvent kernel or reciprocal kernel.

11.11 CONVERSION OF MULTIPLE INTEGRAL INTO A SINGLE ORDINARY INTEGRAL

Consider the integral

$$I_n(x) = \int_a^x (x-t)^{n-1} f(t)dt \qquad \ldots..(1)$$

where t is a positive integer and a is a constant.

Differentiating equation (1) using Leibnitz's rule, we get

$$\frac{dI_n}{dx} = (n-1)\int_a^x (x-t)^{n-2} f(t)dt + [(x-t)^{n-1}f(t)]_{t=x}$$

$$\frac{dI_n}{dx} = (n-1)I_{n-1}, n > 1 \qquad \ldots.(2)$$

From equation (1), we get

$$I_1(x) = \int_a^x f(t)dt$$

or $$\frac{dI_1}{dx} = f(x) \qquad \ldots..(3)$$

Now, differentiating equation (2) successively m times, we get

$$\frac{d^m I_n}{dx^m} = (n-1)(n-2)(n-3)\ldots\ldots\ldots(n-m)I_{n-m,}\ n > m$$

In particular, we have

$$\frac{d^{n-1}I_n}{dx^{n-1}} = (n-1)!\ I_1(x)$$

$$\frac{d}{dx}\left(\frac{d^{n-1}I_n}{dx^{n-1}}\right) = (n-1)!\frac{dI_1}{dx}$$

$$\frac{d^n I_n}{dx^n} = (n-1)!\, f(x) \qquad \text{....(4)}$$

Thus, we have

$I_1(x) = \int_a^x f(x_1)dx_1$ [from equation (3)]

And $\frac{dI_2}{dx} = I_1 = \int_a^x f(x_1)dx_1$

$\Rightarrow\ I_2(x) = \int_a^x \int_a^{x_2} f(x_1)dx_1 dx_2$ [from equation (2)]

In general, we have

$$I_n(x) = (n-1)! \int_a^x \int_a^{x_n} \ldots\ldots\ldots \int_a^{x_3} \int_a^{x_2} f(x_1)dx_1 dx_2 \ldots. dx_{n-1}dx_n \qquad \text{...(5)}$$

Using equations (1) and (5), we conclude that

$$\int_a^x \int_a^{x_n} \ldots\ldots\ldots \int_a^{x_3} \int_a^{x_2} f(x_1)dx_1 dx_2 \ldots. dx_{n-1}dx_n$$

$$= \frac{1}{(n-1)!} I_n(x)$$

$$= \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)dt$$

On integrating $(n-1)$ times, we have

$$\int_a^x f(t)dt^n = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} f(t)dt.$$

11.12 SUMMARY

An equation which contains unknown function under one or more integral signs is known as integral equation.

An integral equation is called linear integral equation if there is only linear function as unknown function under the integral sign.

An integral equation is said to be volterra integral equation if the upper limit of integration is variable.

An integral equation is called Fredholm integral equation if both the limits are constants i.e., domain of integration is fixed.

If the unknown function appears under an integral sign to some power greater than one, is called non – linear integral equation.

When one or both limits of integration are infinite or the kernel $K(x, \xi)$ becomes infinite at one or more points within the range of integration, is called singular integral equation.

11.13 TERMINAL QUESTIONS

Q.1 What do you mean by Integral equation?

Q.2 Explain the types of Integral equation.

Q.3 Verify that the given function $u(x) = \frac{1}{2}$ is the solution of the integral equation $\int_0^x \frac{u(t)}{\sqrt{(x-t)}} dt = \sqrt{x}$

Q.4. Show that the function $u(x) = 1$ is the solution of the Fredholm integral equation.

$$u(x) + \int_0^1 x(e^{xt} - 1)u(t)dt = e^x - x$$

Q.5. Verify or check that the given functions are solutions of the corresponding integral equations:

(a) $u(x) = xe^x$; $\quad u(x) = e^x sinx + 2\int_0^x \cos(x-t)\, u(t)dt$

(b) $u(x) = x - \frac{x^3}{6}$; $\quad u(x) = x - \int_0^x \sinh(x-t)\, u(t)dt$

Q.6. Show that the function $u(x) = 1 - x$ is a solution of the integral equation $\int_0^x e^{x-t} u(t)dt = x$.

Q.7. From an integral equation corresponding to the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$, with initial conditions $y(0) = 0, y'(0) = -1$.

UNIT-12 FREDHOLM INTEGRAL EQUATIONS-I

Structure

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12.1 INTRODUCTION

Integral equations offer a rich array of solution techniques, including successive approximations, separation of variables, variation of parameters, and numerical methods like finite element, collocation, and quadrature techniques. These equations find broad applications across physics (e.g., quantum mechanics), engineering (e.g., heat transfer), and mathematical modeling. Named after the Finnish mathematician Ivar Fredholm, Fredholm integral equations feature the unknown function appearing both inside and outside the integral sign. Their versatility allows them to describe relationships between variables over continuous domains, making them indispensable in various fields. Fredholm integral equations provide alternative approaches to solving boundary value problems in differential equations, particularly useful for problems with intricate boundary conditions and non-standard geometries. Fredholm integral equations are instrumental in studying conduction in materials with complex geometries or non-uniform properties.

Fredholm integral equations play a pivotal role in physics, describing phenomena like electromagnetic scattering, wave propagation, and quantum mechanics. In quantum mechanics, they emerge in the analysis of scattering problems, elucidating the interaction between particles and potentials. This comprehensive utility underscores the significance of Fredholm integral equations across various scientific and engineering disciplines.

12.2 OBJECTIVES

After reading this unit the learner should be able to understand about:

- the Fredholm integral equation and their solution
- the Fredholm First, second and third theorem

12.3 FREDHOLM INTEGRAL EQUATIONS

We have determined the solution of the Fredholm integral equations as a power series the parameterλ, uniformly convergent for $|\lambda|$ sufficiently small. Fredholm obtained the solution of the integral equation in the general form, if possible, for all values of the parameterλ.

In the theory of integral equation, the well-known theorems of linear algebra, which are related to the solution of the system of algebraic equation, play by a leading role. Now we shall discuss the solution of the non-homogeneous Fredholm integral equation of second kind by replacing the integral, appearing in the equation with a sum of which reduces the equation to a system of linear equations and assuming the number of terms of the sum tends to infinitely. The Fredholm integral equation is $\phi(x) = F(x) + \lambda \int_a^b K(x,\xi)\phi(\xi)d\xi$.

12.4 FREDHOLM FIRST THEOREM

The non- homogeneous Fredholm integral equation of second kind

$$\phi(x) = F(x) + \lambda \int_a^b K(x,\xi)\phi(\xi)d\xi,$$

Under the assumption that the function F(x) and K(x,ξ) are integrable has a unique solution, is of the form

$$\phi(x) = F(x) + \lambda \int_a^b R(x,\xi,\lambda)F(\xi)d\xi$$

Where the Resolvent kernel R is a meromorphic † function of the parameterλ, being the ratio of two entire function of the parameter λ

$$R(x,\xi,\lambda) = \frac{D(x,\xi,\lambda)}{D(\lambda)}, \ D(\lambda) \neq 0$$

Defined by Fredholm's series of the form

$$D(\lambda) = 1+\sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} \int_a^b \ldots\ldots \int_a^b \int_a^b K\begin{pmatrix} \xi_1 & \xi_2 & \cdot & \cdot & \xi_m \\ \xi_1 & \xi_2 & \cdot & \cdot & \xi_m \end{pmatrix} d\xi_1 d\xi_2 \ldots d\xi_m$$

And $$D(x,\xi,\lambda) = K(x,\xi)+\sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} \int_a^b \ldots \int_a^b \int_a^b K\begin{pmatrix} x & \xi_1 & \xi_2 \ldots & \xi_m \\ \xi & \xi_1 & \xi_2 \ldots & \xi_m \end{pmatrix} d\xi_1 d\xi_2 \ldots d\xi_m$$

These series converge for all values of λ. In particular, the solution of the homogeneous integral equation is zero.

Consider the Fredholm integral equation;

$$\phi(x) = F(x) + \lambda \int_a^b K(x,\xi)\phi(\xi)d\xi \qquad \text{.... (1)}$$

With a Riemann integral in a given interval (a,b).In accordance with Fredholm method, we consider partition of the interval (a,b) into n equal parts by the points:

$a= \xi_0, \xi_1, \xi_2, \xi_3, \dots \dots .. \xi_{n-1}, \xi_n = b$

Where $\xi_0 = a, \xi_1 = a + h, \xi_2 = a + 2h, \dots \dots, \xi_n = a + nh$

And $h=\xi_{v+1} - \xi_v = \frac{b-a}{n}$ ... (2)

Replace the definite integral in (1) by the sum corresponding to the points of division,

We have,

$$\phi(x) = F(x) + \lambda h \sum_{v=1}^{n} K(x,\xi_v)\phi(\xi_v) \qquad \dots (3)$$

$\phi(x) - \lambda h[K(x,\xi_1)\phi(\xi_1) + \cdots \dots \dots \dots \dots + K(x,\xi_n)\phi(\xi_n)] = F(x)$

Since the equation (3) holds for every value of x, it must be satisfied at its n points of division $x=\xi_1, \xi_2, \dots \xi_n$

Thus we obtain a set of n linear equation with n unknown values of the function $\phi(\xi_1), \phi(\xi_2), \dots \dots \phi(\xi_n)$:

$$\phi(\xi_1) - \lambda h[K(\xi_1,\xi_1)\phi(\xi_1) + K(\xi_1,\xi_2)\phi(\xi_2) + \cdots + K(\xi_1,\xi_n)\phi(\xi_n) = F(\xi_1),$$

$$\phi(\xi_2) - \lambda h[K(\xi_2,\xi_1)\phi(\xi_1) + K(\xi_2,\xi_2)\phi(\xi_2) + \cdots + K(\xi_2,\xi_n)\phi(\xi_n) = F(\xi_2),$$

$$\phi(\xi_3) - \lambda h[K(\xi_3,\xi_1)\phi(\xi_1) + K(\xi_3,\xi_2)\phi(\xi_2) + \cdots + K(\xi_3,\xi_n)\phi(\xi_n) = F(\xi_3),$$

…… ……. ……. ……. …….. ……… ………

$$\phi(\xi_n) - \lambda h[K(\xi_n,\xi_1)\phi(\xi_1) + K(\xi_n,\xi_2)\phi(\xi_2) + \cdots + K(\xi_n,\xi_n)\phi(\xi_n) = F(\xi_n) \dots . (4)$$

With the notations $F(\xi_i) = F_{i,}\phi(\xi_i) = \phi_i, K(\xi_i,\xi_j) = K_{ij}$

The system of equation (4) reduce to

$$(1 - \lambda h K_{11})\phi_1 - \lambda h K_{12}\phi_2 - \lambda h K_{13}\phi_3 - \cdots . -\lambda h K_{1n}\phi_n = F_1$$

$$-\lambda h K_{21}\phi_1 + (1 - \lambda h K_{22})\phi_2 - \lambda h K_{23}\phi_3 - \cdots - \lambda h K_{2n}\phi_n = F_2$$

$$-\lambda h K_{31}\phi_1 - \lambda h K_{32}\phi_2 + (1 - \lambda h K_{33})\phi_3 - \cdots - \lambda h K_{3n}\phi_n = F_3$$

… … … … … … … … … … …

… … … … … … … … … … …

$$-\lambda h K_{n1}\phi_1 - \lambda h K_{n2}\phi_2 + (-\lambda h K_{n3})\phi_3 - \cdots (1 - \lambda h K_{nn})\phi_n = F_n \qquad \dots (5)$$

The solution $\varphi_1, \varphi_2, \dots . \varphi_n$ of the system of equations may be expressed in the form of the ratios of certain determinants by the common characteristic determinant:

$$D_n(\lambda)=\begin{vmatrix} (1-\lambda h K_{11}) & -\lambda h K_{12} & -\lambda h K_{13} & \dots .. & -\lambda h K_{1n} \\ -\lambda h K_{21} & (1-\lambda h K_{22}) & -\lambda h K_{23} & \dots .. & -\lambda h K_{2n} \\ \dots .. & \dots .. & \dots .. & \dots .. & \dots .. \\ \dots .. & \dots .. & \dots .. & \dots .. & \dots .. \\ -\lambda h K_{n1} & -\lambda h K_{n2} & -\lambda h K_{n3} & \dots .. & (1-\lambda h K_{nn}) \end{vmatrix} \qquad \dots (6)$$

Provided that $D_n(\lambda) \neq 0$.

Now We shall expand the determinant (6) in powers of the factor $-\lambda h$.

The first term not containing this factor is obviously equal to unity. The term containing

($-\lambda h$) in the first power is the sum determined as

$= -\lambda h K_{vv}$, v=1,2,3......n.

The term containing the factor $(-\lambda h)^2$ is the sum of all determinants having two columns with that factor,i.e.,the sum of the determinants of the form

$$=(-\lambda h)^2\begin{vmatrix} Krr & Krs \\ Ksr & Kss \end{vmatrix}$$

Where (r,s) is an arbitrary pair of integers taken from the sequence 1,2,3,.....n with r<s.

Similarly, the term containing $(-\lambda h)^3$ is the sum of the determinants of the form

$$=(-\lambda h)^3\begin{vmatrix} Krr & Krs & Krt \\ Ksr & Kss & Kst \\ Ktr & Kts & Ktt \end{vmatrix}$$

Where r,s,t are the arbitrary integers taken from the sequence 1,2,3......n with r<s<t.

Thus we conclude that the expansion of the determinant (6) may be expressed in the form

$$D_n(\lambda)=1-\lambda h\sum Kvv+\frac{(-\lambda h)^2}{2!}\sum\begin{vmatrix} Krr & Krs \\ Ksr & Kss \end{vmatrix}+\frac{(-\lambda h)^3}{3!}\sum\begin{vmatrix} Krr & Krs & Krt \\ Ksr & Kss & Kst \\ ktr & Kts & Ktt \end{vmatrix}+....$$

$$+\frac{(-\lambda h)^n}{n!}\sum\begin{vmatrix} K\alpha1\alpha2 & K\alpha1\alpha2 & ...K\alpha1\alpha n \\ K\alpha2\alpha1 & K\alpha2\alpha2 & ...K\alpha2\alpha n \\ ..... & .... & ..... \\ K\alpha n\alpha1 & K\alpha n\alpha2 & ...K\alpha n\alpha n \end{vmatrix} \qquad \ldots (7)$$

Let h→0 and n→∞,then each of the terms of sum (7) reduces to some single, double or triple integral etc.Thus,we have

$$D(\lambda)=1-\lambda\int_a^b K(\xi_1,\xi_1)d\xi1 +\frac{\lambda^2}{2!}\int_a^b\int_a^b\begin{vmatrix} K(\xi_1,\xi_1) & K(\xi_1,\xi_2) \\ K(\xi_2,\xi_1) & K(\xi_2,\xi_2) \end{vmatrix}d\xi_1 d\xi_2-$$

$$\frac{\lambda^3}{3!}\int_a^b\int_a^b\int_a^b\begin{vmatrix} K(\xi_1,\xi_1) & K(\xi_1,\xi_2) & K(\xi_1,\xi_3) \\ K(\xi_2,\xi_1) & K(\xi_2,\xi_2) & K(\xi_2,\xi_3) \\ K(\xi_3,\xi_1) & K(\xi_3,\xi_2) & K(\xi_3,\xi_3) \end{vmatrix}d\xi_1 d\xi_2 d\xi_3+....... \qquad \ldots (8)$$

Where D(λ) is called the Fredholm's determinant.

Similarly, the power series analogous to the series (8) may be written as

$$D(\lambda)=1+\sum\frac{(-\lambda)^m}{m!}\int_a^b\int_a^b...\int_a^b K\begin{pmatrix}\xi_1,\xi_2 \ldots.\xi m \\ \xi_1,\xi_2,\ldots.\xi m\end{pmatrix}d\xi_1 d\xi_2.....d\xi_m. \qquad ... \qquad (9)$$

where $$K\begin{pmatrix}\xi1,\xi2,\ldots.\xi m \\ \xi1,\xi2,\ldots.\xi m\end{pmatrix}=\begin{vmatrix} K(\xi_1,\xi_1) & K(\xi_1,\xi_2) & K(\xi_1,\xi_m) \\ K(\xi_2,\xi_1) & K(\xi_2,\xi_2) & K(\xi_2,\xi_m) \\ ..... & ..... & ..... \\ K(\xi_m,\xi_1) & K(\xi_m,\xi_2) & K(\xi_m,\xi_m) \end{vmatrix}$$

This is called Fredholm's first series.

Fredholm assumed that the solution φ(x) of the integral equation are to be sought for arbitrary λ in the form of a ratio of two power series in the parameter λ, where the series D(λ) is to be the divisor. We know that the solution of the Fredholm integral equation for sufficiently small |λ| is of the form

$$\emptyset(x)=F(x)+\lambda\int_a^b R(x,\xi;\lambda)F(\xi)d\xi \quad \dots (10)$$

Where the function $R(x,\xi;\lambda)$ is called the resolvent kernel,

$$R(x,\xi;\lambda) = D(x,\xi;\lambda)/D(\lambda) \quad \text{---}(11)$$

Where D(λ) is the Fredholm first series and $D(x,\xi;\lambda)$ is the sum of some functional series yet to be determined.This Theorem refers to the case where λ is not a zero of the function $D(\lambda)$.The study of the case where λ is a zero of the entire function $D(\lambda)$ gives rise to the Fredholm second and the third theorems.

We know that equation (10) is a solution of the integral equation (1) if the Resolvent Kernel $R(x,\xi;\lambda)$ satisfies the equation

$$R(x,\xi;\lambda)=K(x,\xi)+\lambda\int_a^b K(x,\xi 1)R(\xi 1,\xi;\lambda)d\xi_1 \quad \dots (12)$$

In view of the form (11) of the Resolvent kernel, the numerator D(x, ξ; λ) should satisfy the integral equation

$$D(x,\xi;\lambda)/D(\lambda) = K(x,\xi) + \lambda\int_a^b K(x,\xi_1)\, D(\xi_1,\xi;\lambda)/D(\lambda)d\xi_1$$

$$D(x,\xi;\lambda) = K(x,\xi)D(\lambda)+\lambda\int_a^b K(x,\xi_1)D(\xi_1,\xi;\lambda)d\xi_1 \quad \dots (13)$$

The solution of the equation in the form of power series in the parameter λ is given by

$$D(x,\xi;\lambda) = B_0(x,\xi) + \sum_{m=1}^{\infty}(-1)^m(\lambda^m/m!)B_m(x,\xi) \quad \dots (14)$$

We know that

$$D(\lambda) = 1 + \sum_{m=1}^{\infty}(-1)^m(\lambda^m/m!)C_m \quad \dots (15)$$

Where $C_m=\int_a^b \dots \int_a^b \int_a^b K\begin{pmatrix}\xi_1 & \xi_2 & \dots . \xi m\\ \xi_1 & \xi_2 & \dots . \xi m\end{pmatrix}d\xi_1 d\xi_2 \dots\dots. d\xi_m$

from the equations (14) and (15), the equation (13) reduces

$$B_0(x,\xi)+\sum_{m=1}^{\infty}(-1)^m(\lambda^m/m!)B_m(x,\xi) = K(x,\xi)$$

$$\{1+\sum_{m=1}^{\infty}(-1)^m(\lambda^m/m!)C_m\} + \lambda\int_a^b K(x,\xi_1)\{B_0(\xi_1,\xi) +\sum_{m=1}^{\infty}(-1)^m(\lambda^m/m!)B_m(\xi_1,\xi)\}\, d\xi_1$$

Substituting $\lambda=0 \Rightarrow B_0(x,\xi) = K(x,\xi)$

To determine the coefficients $B_m(x,\xi)$ equating the coefficients of λ^m , we have

$$((-1)^m/m!)B_m(x,\xi) = \{(-1)^m/m!\}C_mK(x,\xi)+\{(-1)^{m-1}/(m-1)!\}\int_a^b K(x,\xi 1)B_{m-1}(\xi_1,\xi)\}\, d\xi_1$$

$$\Rightarrow B_m(x,\xi) = C_mK(x,\xi)- m\int_a^b K(x,\xi 1)B_{m-1}(\xi_1,\xi)\}\, d\xi_1 \quad \dots(16)$$

Which is a recursive relation between the consecutive functions B_m and B_{m-1}

Thus for $m = 1$, we have

$$B_1(x,\xi) = C_1K(x,\xi)- \int_a^b K(x,\xi_1)B_0(\xi_1,\xi)\}\, d\xi_1$$

$$B_1(x,\xi) = K(x,\xi)\int_a^b K(\xi_1,\xi_1)\, d\xi_1 - \int_a^b K(x,\xi_1)\, K(\xi_1,\xi)\, d\xi_1$$

$$B_1(x,\xi) = \int_a^b \begin{vmatrix} K(x,\xi) & K(x,\xi_1) \\ K(\xi_1,\xi) & K(\xi_1,\xi_1) \end{vmatrix} d\xi_1 = \int K\begin{pmatrix} x & \xi \\ \xi & \xi \end{pmatrix} d\xi_1 \qquad \dots (17)$$

In general, we shall prove that

$$B_m(x,\xi) = \int_a^b \dots \int_a^b \int_a^b K\begin{pmatrix} x & \xi_1 & \xi_2 \dots & \xi_m \\ \xi & \xi_1 & \xi_2 \dots & \xi_m \end{pmatrix} d\xi_1 d\xi_2 \dots\dots d\xi_m \qquad \dots (18)$$

where ξ, ξ_1, $\xi_2 \dots\dots \xi_m$ are the variables of integration

expanding the determinant under the integral sign in (13), we get

$$K\begin{pmatrix} x & \xi_1 & \xi_2 \dots & \xi_m \\ \xi & \xi_1 & \xi_2 \dots & \xi_m \end{pmatrix} = \begin{vmatrix} K(x,\xi) & K(x,\xi_1) & K(x,\xi_2) & K(x,\xi m) \\ K(\xi_1,\xi) & K(\xi_1,\xi_1) & K(\xi_1,\xi_2) & K(\xi_1,\xi m) \\ \dots & \dots & \dots & \dots \\ K(\xi m,\xi) & K(\xi m,\xi_1) & K(\xi m,\xi_2) & K(\xi m,\xi m) \end{vmatrix}$$

$$\text{Or} \quad K\begin{pmatrix} x & \xi_1 & \xi_2 \dots & \xi_m \\ \xi & \xi_1 & \xi_2 \dots & \xi_m \end{pmatrix} = K(x,\xi)\, K\begin{pmatrix} \xi_1, \xi_2, \dots . \xi m \\ \xi_1, \xi_2, \dots . \xi m \end{pmatrix}$$

$$- K(x,\xi_1)\, K\begin{pmatrix} \xi_1, \xi_2, \dots . \xi m \\ \xi_1, \xi_2, \dots . \xi m \end{pmatrix}$$

$$- K(x,\xi_2)\, K\begin{pmatrix} \xi_1, \xi_2, \dots . \xi m \\ \xi_1, \xi_2, \dots . \xi m \end{pmatrix}$$

$$- K(x,\xi_m)\, K\begin{pmatrix} \xi_1, \xi_2, \dots . \xi m \\ \xi_1, \xi_2, \dots . \xi m \end{pmatrix}$$

by integrating both sides of the equality n times with regard to the variable ξ_1, $\xi_2 \dots\dots \xi_m$, we obtain

$$\text{or} \quad \int_a^b \int_a^b \dots \dots \int_a^b \int_a^b \int_a^b K\begin{pmatrix} x & \xi_1 & \xi_2 \dots & \xi_m \\ \xi & \xi_1 & \xi_2 \dots & \xi_m \end{pmatrix} d\xi_1 d\xi_2 \dots\dots d\xi_m$$

$$= C_m K(x,\xi) - m\int_a^b K(x,s) \left\{ \int_a^b K\begin{pmatrix} s & \xi_1 & \xi_2 \dots & \xi_m \\ \xi & \xi_1 & \xi_2 \dots & \xi_m \end{pmatrix} d\xi 1 d\xi 2 \dots d\xi m-1 \right\} ds$$
$\dots$ (19)

We notice from the equality (17) that the relation (18) holds for m=1. Thus we conclude by induction that it holds for all values of m. Hence the series (14) takes the following form

$D(x, \xi; \lambda) = K(x, \xi)$

$$+\sum_{m=1}^{\infty} (-\lambda^m)/m! \int_a^b \int_a^b \dots \dots \int_a^b \int_a^b \int_a^b K\begin{pmatrix} x & \xi_1 & \xi_2 \dots & \xi_m \\ \xi & \xi_1 & \xi_2 \dots & \xi_m \end{pmatrix} d\xi_1 d\xi_2 \dots\dots d\xi_m \qquad \dots (20)$$

or $\qquad D(x, \xi; \lambda) = K(x, \xi) + \sum_{m=1}^{\infty} ((-1)^m/m!)\, \lambda^m B_m(x, \xi)$,

$$\text{where } C_m = \int_a^b \dots \int_a^b \int_a^b K\begin{pmatrix} x & \xi_1 & \xi_2 \dots & \xi_m \\ \xi & \xi_1 & \xi_2 \dots & \xi_m \end{pmatrix} d\xi_1 d\xi_2 \dots\dots d\xi_m \qquad \dots (21)$$

This is called Fredholm second series and the function

$D(x, \xi; \lambda)$ is said to be the Fredholm minor.

Corollary: The Fredholm homogeneous integral equation is $\emptyset(x) = \lambda \int_a^b K(x,\xi)\emptyset(\xi)d\xi$

has only one and only one solution given by $\emptyset(x) = 0$, if $\mathrm{D}(\lambda) \neq 0$.

Substituting $F(x) = 0$ it follows that if $\mathrm{D}(\lambda) \neq 0$ then the homogeneous integral equation contains only the trivial solution $\emptyset(x) = 0$ in an interval (a, b).

12.5 NON-HOMOGENOUS FREDHOLM EQUATION

We know that

$$\emptyset(x) = F(x) + \lambda \int_a^b R(x,\xi;\lambda)F(\xi)d\xi$$

of the non-homogeneous Fredholm equation is unique for any λ provided by $\mathrm{D}(\lambda) \neq 0$

we know that

$$R(x,\xi;\lambda) = K(x,\xi) + \lambda \int_a^b R(x,s;\lambda)K(s,\xi)ds \qquad \dots (1)$$

when modulus of λ is sufficiently small.

We shall show that the above relation is satisfied by the Fredholm Resolvent kernel of the form

$R(x,\xi;\lambda) = \frac{D(x,\xi;\lambda)}{D(\lambda)}$, $\mathrm{D}(\lambda) \neq 0$.

It follows that both side of the above equation are meromorphic functions of the parameter λ. If they are equal in a region where in which modulus of λ is sufficiently small then they are also equal for all λ, provided that $\mathrm{D}(\lambda) \neq 0$. The Fredholm second series is given by

$$\mathrm{D}(x,\xi;\lambda) = K(x,\xi) + \sum \frac{(-\lambda)^m}{m!} \int_a^b \int_a^b \dots . \int_a^b K\begin{pmatrix} x & \xi_1 & - & \xi_m \\ \xi & \xi_1 & - & \xi_m \end{pmatrix}$$

$$\mathrm{d}\xi_1\xi_2\xi_3 - - - \xi_m \qquad \dots (2)$$

by expanding the determinant under the integral sign, with regard to the elements of the first column, we obtain

$$K\begin{pmatrix} x & \xi_1 & - & \xi_m \\ \xi & \xi_1 & - & \xi_m \end{pmatrix} = K(x,\xi)K\begin{pmatrix} \xi_1 & \xi_2 & - & \xi_m \\ \xi_1 & \xi_2 & - & \xi_m \end{pmatrix} +$$

$$\sum_{q=1}^{m}(-1)^{q+2} K(\xi_q,\xi)\ K\begin{pmatrix} x, & \xi_1 & - - -\xi_{q+1} - - & \xi_m \\ \xi, & \xi_1 & - - -\xi_{q+1} - - & \xi_m \end{pmatrix}$$

By transposing ξ_q to the first place we see that the integral with regard to ξ_q of each of the components of the above sum have the same value

$$\int_a^b (-1)^{q+2} K(\xi_q,\xi)K\begin{pmatrix} x, & \xi_1 & - - -\xi_{q+1} - - & \xi_m \\ \xi, & \xi_1 & - - -\xi_{q+1} - & \xi_m \end{pmatrix} d\,\xi_q$$

$$=(-1)^{2q+1} \int_a^b K(s,\xi)K\begin{pmatrix} x & \xi_1 & - - -\xi_{q+1} - - & \xi_m \\ \xi & \xi_1 & - - -\xi_{q+1} - - & \xi_m \end{pmatrix} d\,s$$

Consequently have, $\frac{(-\lambda)^m}{m!} \iint_a^b - - - \int_a^b K\begin{pmatrix} x & \xi_1 & - & \xi_m \\ \xi & \xi_1 & - & \xi_m \end{pmatrix} \mathrm{d}\xi_1\xi_2\xi_3 - - - \xi_m$

$= K(x,\xi)\frac{(-\lambda)^m}{m!}\iint_a^b ----\int_a^b K\begin{pmatrix}\xi_1 & \xi_2 & - & \xi_m\\ \xi_1 & \xi_2 & - & \xi_m\end{pmatrix} d\xi_1\xi_2\xi_3 --- \xi_m + \lambda\int_a^b[\frac{(-\lambda)^{m-1}}{m-1!}\iint_a^b --$
$-\int_a^b K\begin{pmatrix}x & \xi_1 & - & \xi_m\\ s & \xi_1 & - & \xi_m\end{pmatrix} d\xi_1\xi_2\xi_3 --- \xi_{m-1}]\, K(s,\xi)ds$ ... (3)

Upon summing the terms of second Fredholm series, we obtain

$D(x,\xi;\lambda) = K(x,\xi)D(\lambda) + \lambda\int_a^b D(x,s;\lambda)K(s,\xi)ds$ … (4)

Or, $\frac{D(x,\xi;\lambda)}{D(\lambda)} = K(x,\xi) + \lambda\int_a^b \frac{D(x,\xi;\lambda)}{D(\lambda)}K(s,\xi)ds$, $D(\lambda) \neq 0$

Or, $R(x,\xi;\lambda) = K(x,\xi) + \lambda\int_a^b R(x,s;\lambda)K(s,\xi)ds$

Which is the same as equation (1).

In order to prove that the solution obtained by Fredholm equation is unique, suppose that $\emptyset(x)$ is the given solution of the Fredholm equation

$\emptyset(x) = F(s) + \lambda\int_a^b K(s,\xi)\emptyset(\xi)d\xi$ … (5)

Multiplying both the sides of equation 5 by the Resolvent kernel and integrating with respect to s, we get

$\int_a^b R(x,s;\lambda)\emptyset(s)ds = \int_a^b R(x,s;\lambda)\,F(s)ds + \lambda\int_a^b\left\{\int_a^b R(x,s;\lambda)K(s,\xi)\,ds\right\}\emptyset(\xi)d\xi$

From the equation 5 we get

$$\int_a^b K(x,\xi)\emptyset(\xi)d\xi = \int_a^b R(x,s;\lambda)\,F(s)ds$$

Since $\emptyset$ is the given solution of the equation 5, it follows that the Fredholm equation is .

$$\emptyset(x) = F(x) + \int_a^b R(x,s;\lambda)\,F(s)ds$$

Hence , Fredholm solution is unique for all λ proved $D(\lambda) \neq 0$.

12.6 EVERY ZERO OF FREDHOLM FUNCTION D(λ) IS A POLE OF THE RESOLVENT KERNEL

We have know that

R(x,ξ;λ) = D(x,ξ;λ) /D(λ)

The order of this pole is at most equal to the order of the zero of the denominator D(x,ξ;λ) and is a single pole of the resolvent kernel. The zeroes of the Fredholm D(λ). By interchanging the indices of the variables of integration in the Fredholm first series, we get

$D'(\lambda) = -\int_a^b K(s,s)ds - \sum_{m=2}^{\infty}\frac{(-\lambda)^{m+1}}{(m-1)!}\times$

$$\int_a^b\{\int_a^b ...\int_a^b K\begin{pmatrix}s & \xi_1 & - & \xi_{m-1}\\ s & \xi_1 & - & \xi_{m-1}\end{pmatrix} d\xi_1\xi_2\xi_3 --- \xi_{m-1}$$

This we have the fundamental relation D'(λ)=$-\int_a^b D(s,s;\lambda)ds$

If λ_0 is a zero of order n of the function D(λ), then it is a zero of the order (n-1) of its derivative D'(λ). The point λ_0 may be a zero of the order at most (n-1) of the function D(x,ξ;λ). This is a pole of the ratio,

R(x,ξ;λ) = D(x,ξ;λ) /D(λ) of order at most n.

In particular, when λ_0 is single zero, we have

$D(\lambda_0)=0$, $D'(\lambda_0)\neq 0$.

So λ_0 cannot be zero of the function D(λ) are called the Eigen values of the kernel K(x,ξ). Since D(0) =1, therefore, zero is never an Eigen value.

The set of all Eigen value of this kernel is known the spectrum of the integral equation.

Note: If a Real Kernel K(x,ξ) has a Complex Eigen Value λ_0= u+iv, then it also contains the conjugate Eigen Value to λ_0 = u-iv:

Let the Complex Eigen value be λ_0 = u+iv, then its conjugate eigenvalue will be $\bar{\lambda}_0$ = u-iv. The entire function D(λ) takes real value on the real axis of the kernel K(x,ξ) is real. The value of power function D(λ) at points symmetrical with regard to the real axis are complex conjugates. It follows that,

If, D(u+iv)=0, then D(u-iv)=0

We know that, $\frac{D'(\lambda)}{D(\lambda)} = -\int_a^b D(s,s;\lambda)/D(\lambda)ds$ ... (1)

Or, $\frac{d}{d\lambda}\log[D(\lambda)] = -\int_a^b R(s,s;\lambda)ds, D(\lambda)\neq 0$

Since |λ| is sufficiently small, the relation (1) may be represented as,

$$\frac{d}{d\lambda}\log[D(\lambda)] = -\int_a^b \sum_{n=0} \lambda^n Kn+1\,(s,s)ds,$$

Where the series on the R.H.S. is convergent, hence,

$$\log D(\lambda) = -\sum_{n=0}^{\infty}\frac{\lambda^{n+1}}{n+1}\int_a^b Kn+1(s,s)ds \qquad ... (2)$$

Since D(0)=1, the integrals of the iterated kernels

$$\int_a^b Kn+1(s,s)ds$$

are called the traces of the kernel K(x,ξ). The radius of congruence of series (2) is equal to the smallest modulus of the Eigen value. If a kernel possesses no Eigen values, then the series (1) is convergent for each value of λ.

12.7 FREDHOLM SECOND THEOREM

If λ_0 is a zero of multiplicity m of the function D(λ), then the homogeneous integral equation,

$$\phi(x) = \lambda_0\int_a^b K(x,\xi)\,\phi(\xi)d\xi$$

Possesses at least one, and at most m, linearly independent solutions,

$$\phi_i(x) = D_r\begin{pmatrix} x_1 & x_2 & - & \xi_v \\ \xi_1 & \xi_2 & - & \xi_v \end{pmatrix}; \lambda 0)$$

(i=1,2,3,……….v; 1≤ v ≤ m not identically zero, and any other solution of this equation is a linear combination of these solutions.

Fredholm first theorem does not hold when h is a root of the equation

D(h) =0 .Consider the Fredholm homogeneous integral equation of the form

$$\emptyset(x) = \lambda \int_a^b k(x,\xi)\,\emptyset(\xi)d\xi \qquad \dots (1)$$

We shall determine the existence of non –zero solutions of the homogeneous equation (1) where D(λ) = 0 has a certain number of solution different from zero.

Let $\lambda = \lambda_0$ is a simple zero of the function D(λ) , where

$$D(\lambda_0) = 0\ ,\ D'(\lambda_0) \neq 0 \Rightarrow D(x,\xi,\lambda) \neq 0$$

Is not identically equal to zero.

The resolvent kernel satisfied by Fredholm entire function for all h is given by

$$D(x,\xi,\lambda) = k(x,\xi)D(\lambda) + \lambda \int_a^b k(x,s)\,D(s,\xi:\lambda_0)ds \qquad \dots (2)$$

For $\lambda \to \lambda_0$, we have

$$D(x,\xi:\lambda_0) = \lambda_0 \int_a^b k(x,s)\,D(s,\xi:\lambda_0)ds$$

Assuming a particular value $\xi = \xi_0$ such that the function D(x, ξ_0: λ) be non zero it follows that the equation (1) possesses a non –zero solution.

$$\emptyset(x) = D(x,\xi_0:\lambda_0) \qquad \dots (3)$$

Similarly, the function AD(x, ξ_0: λ_0) is also a solution of the homoeneous equation (1) where is an arbitrary constant. In general, let $\lambda = \lambda_0$ is a zero of arbitrary multiplicity m ,i.e,

$$D(\lambda_0) = 0,\dots,\ D^v(\lambda_0) = 0,\dots,\ D^m(\lambda_0) \neq 0\ ,\ v=1,2,3,\dots(m-1) \qquad \dots (4)$$

Fredholm introduced the concept of mnors for the existence of non –zero solution of the homogeneous integral equation . A Fredholm minor of order n relative to the kernel k(x, ξ) denoted by

$$D_n\begin{pmatrix} x_1,x_2,x_3,\dots & x_n, & \\ \xi_1,\xi_2,\xi_3,\dots & \xi_n & :\lambda \end{pmatrix}$$

Is the sum of the power series in the parameter h , i.e,

$$D_n\begin{pmatrix} x_1,x_2,x_3,\dots & x_n, & \\ \xi_1,\xi_2,\xi_3,\dots & \xi_n & :\lambda \end{pmatrix} = k\begin{pmatrix} x_1,x_2,x_3,\dots & x_n, & \\ \xi_1,\xi_2,\xi_3,\dots & \xi_n & :\lambda \end{pmatrix} + \sum_{p=1}^{\infty}\frac{(-\lambda)^p}{p!}$$
$$\int_a^b \dots \int_a^b k\begin{pmatrix} x_1,x_2,x_3,\dots x_n, & s_1,s_2,s_3,\dots s_p \\ \xi_1,\xi_2,\xi_3,\dots & s_1,s_2,s_3,\dots s_p \end{pmatrix} ds_1\dots, ds_p \qquad \dots (5)$$

Where $x_1,x_2,x_3,\dots x_n$ and $\xi_1,\xi_2,\xi_3,\dots\ \xi_n$ are two sequences of arbitrary variables.

The series is converges for all values of the parameter λ .

Differentiating the Fredholm first series , n times, we have

$$\frac{d^n}{d\lambda^n}D(\lambda) = (-1)^n \int_a^b \dots \int_a^b k\begin{pmatrix} s_1,s_2,s_3,\dots & ,\dots s_n \\ s_1,s_2,s_3,\dots & ,\dots s_n \end{pmatrix} ds_1\dots ds_n$$

$$+(-1)^n \sum_{p=1}^{\infty}\frac{(-\lambda)^p}{p!}\int_a^b \dots \int_a^b k\begin{pmatrix} s_1 \dots s_n, & s_{n+1},\dots s_{n+p} \\ s_1 \dots s_n, & s_{n+1},\dots s_{n+p} \end{pmatrix} ds_1\dots ds_{n+p} \qquad \dots (6)$$

By comparing the series (5) and (6) , we have

$$\frac{d^n}{d\lambda^n}\mathrm{D}(\lambda) = (-1)^n \int_a^b \dots \int_a^b D_n \begin{pmatrix} x_1, x_2, x_3, \dots & x_n, \\ \xi_1, \xi_2, \xi_3, \dots & \xi_n \end{pmatrix} : \lambda \Big) \, \mathrm{d}x_1 \, \mathrm{d}x_2 \dots \mathrm{d}x_n, \qquad \dots (7)$$

Which represents a relation between nth derivative of the Fredholm function and Fredholm minor of order n, where n is an arbitrary positive integer.

From the relation (7), we notice that if λ_0 is a zero of order m of the function $\mathrm{D}(\lambda)$ then the minor of order m becomes

$$D_m \left(\begin{matrix} x_1, x_2, x_3, \dots & x_n, \\ \xi_1, \xi_2, \xi_3, \dots & \xi_n \end{matrix} : \lambda_0 \right)$$

For that value of $\lambda_0 \neq 0$, since then $D^m(\lambda_0) \neq 0$. It follows that minors of lower order than m also do not identically vanish.

A relation between the minors that corresponds to the Resolvent kernel is determined by expanding the Fredholm determinant under the integral sign in equation (5), with respect to the elements of the first row

$$K \left(\begin{matrix} x_1, x_2, x_3, \dots x_n, & s_1, s_2, s_3, \dots s_p \\ \xi_1, \xi_2, \xi_3, \dots \xi_n & s_1, s_2, s_3, \dots s_p \end{matrix} \right)$$

$$= \begin{vmatrix} k(x_1, \xi_1) & k(x_1, \xi_2) \dots k(x_1, \xi_n) & k(x_1, s_1) \dots & k(x_1, s_p) & \\ k(x_2, \xi_1) & k(x_2, \xi_2) \dots k(x_2, \xi_n) & k(x_2, s_1) \dots & k(x_2, s_p) & \dots \\ & \dots & \dots & \dots & \\ k(x_n, \xi_1) & k(x_n, \xi_2) \dots k(x_n, \xi_n) & k(x_n, s_1) \dots & k(x_n, s_p) & \\ k(s_1, \xi_1) & k(s_1, \xi_2) \dots k(s_1, \xi_n) & k(s_1, s_1) \dots & k(s_1, s_p) & \dots \\ \dots k(s_p, \xi_1) & k(s_p, \xi_2) \dots k(s_p, \xi_n) & k(s_p, s_1) & k(s_p, s_p) & \end{vmatrix} \qquad \dots (8)$$

By integrating with regard to $s_1, s_2, \dots s_p \; for \; p \geq 1$, p times, we obtain

$$\int_a^b \dots \int_a^b K \begin{pmatrix} x_1, x_2, \dots, x_n, s_1, s_2, \dots, s_p \\ \xi_1, \xi_2, \dots, \xi_n, s_1, s_2, \dots, s_p \end{pmatrix} ds_1, ds_2, \dots, ds_p$$

$$= \sum_{k=1}^{n} (-1)^{k+1} K(x_1, \xi_k) \int_a^b \dots \int_a^b K \begin{pmatrix} x_2, \dots, x_n, s_1, s_2, \dots, s_p \\ \xi_1, \dots, \xi_{k-1}, \xi_{k+1}, \xi_n, s_1, s_2, \dots, s_p \end{pmatrix} ds_1 \dots ds_p$$

$$+ \sum_{k=1}^{p} (-1)^{k+n+1} \int_a^b \dots \int_a^b K(x_1, s_k)$$

$$\times K \begin{pmatrix} x_2, \dots, x_n, s_1, s_2, \dots, s_k, \dots, s_p \\ \xi_1, \dots, \xi_{n-1}, \dots, \xi_n, s_1, s_2, \dots, s_{k-1}, s_{k+1}, s_p \end{pmatrix} ds_1 \dots ds_p \qquad \dots (9)$$

The determinant K on the RHS of relation (9) do not contain the variable x_1 in the upper sequence and the variable $\xi_k \; or \; s_k$ in the lower sequence. Further, all the terms of the letter of the above sums have the same value. Now, by transposing the variables s_k in the upper sequence to the first place [by means of $(k + n - 2)$ transpositions] and omitting the index k, we may represents each of the terms of the second sum in the form

$$\int_a^b K(x_1, s) \{ \int_a^b \dots \int_a^b K \begin{pmatrix} s, x_2, \dots, x_n, s_1, \dots, s_{p-1} \\ \xi_1, \dots, \xi_n, s_1, \dots, s_{p-1} \end{pmatrix} ds_1 .. ds_{p-1} \} ds$$

Thus the equation (9) may be written as

$$\int_a^b \dots \int_a^b K \begin{pmatrix} x_1, \dots x_n, s_1, \dots s_p \\ \xi_1, \dots \xi_n, s_1, \dots s_p \end{pmatrix} ds_1 \dots ds_p$$

$$= \sum_{k=1}^{n} (-1)^{k+1} K(x_1, \xi_k) \int_a^b \dots \int_a^b K \begin{pmatrix} x_2, \dots x_n, s_1, \dots, s_p \\ \xi_1, \dots, \xi_{k-1}, \xi_{k+1}, \dots, \xi_n, s_1, \dots, s_p \end{pmatrix}$$

$$ds_1 \dots \dots . \, ds_p$$

$$-p \int_a^b K(x_1, s) \left\{ \int_a^b \dots \int_a^b K \begin{pmatrix} s, x_2, \dots, x_n, s_1, \dots, s_{p-1} \\ \xi_1, \xi_2, \dots, \xi_n, s_1, \dots, s_{p-1} \end{pmatrix} ds_1 \dots . \, ds_{p-1} \right\} ds \qquad \dots (10)$$

From the relation (5) and (10) we have

$$D_n \begin{pmatrix} x_1, x_2, \dots, x_n \\ \xi_1, \xi_2, \dots, \xi_n \end{pmatrix} ; \lambda \Big)$$

$$= \sum_{k=1}^{n} (-1)^{k+1} K(x_1, \xi_k) D_{n-1} \begin{pmatrix} x_2, \dots \dots . , x_n \\ \xi_1, \dots . \xi_{k-1}, \xi_{k+1}, \dots . , \xi_n \end{pmatrix} ; \lambda \Big)$$

$$+\lambda \int_a^b K(x_1, s) \, D_n \begin{pmatrix} s, x_2, \dots, x_n \\ \xi_1, \xi_2, \dots, \xi_n \end{pmatrix} ; \lambda \Big) ds \qquad \dots (11)$$

Expanding the determinant (8) with regard to an arbitrary i-th row and i-th column , where $1 \leq i \leq n$.

$$D_n \begin{pmatrix} x_1, x_2, \dots . , x_n \\ \xi_1, \xi_2, \dots, \xi_n \end{pmatrix} ; \lambda \Big)$$

$$= \sum_{k=1}^{n} (-1)^{k+1} K(x_i, \xi_k) D_{n-1} \begin{pmatrix} x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots \dots, x_n \\ \xi_1, \xi_2, \dots, \xi_{k-1}, \xi_{k+1}, \dots, \xi_n \end{pmatrix} ; \lambda \Big) \qquad \dots (12)$$

And $D_n \begin{pmatrix} x_1, x_2, \dots, x_n \\ \xi_1, \xi_2, \dots, \xi_n \end{pmatrix} ; \lambda \Big)$

$$= \sum_{k=1}^{n} (-1)^{k+1} K(x_k, \xi_i) D_{n-1} \begin{pmatrix} x_1, \dots . , x_{k-1}, x_{k+1}, \dots . , x_n \\ \xi_1, \dots . , \xi_{i-1}, \xi_{i+1}, \dots . , \xi_n \end{pmatrix} ; \lambda \Big)$$

$$+\lambda \int_a^b K(s, \xi_1) \, D_n \begin{pmatrix} x_1, \dots \dots \dots, x_n \\ \xi_1, \dots, \xi_{i-1}, s, \xi_{i+1}, \dots, \xi_n \end{pmatrix} ; \lambda \Big) ds \qquad \dots (13)$$

The relation (12)and (13) holds for all values of λ.The relation (12) provides the solution of homogeneous integral equation in the cases when $\lambda = \lambda_0$ is an eigenvalue of the kernel.

Consider $\lambda = \lambda_0$ is a zero of order m of the function $D(\lambda)$.Then as defined in the relation (7), the minor D_m does not identically vanish and the minors $D_1, D_2, \dots . , D_{m-1}$ need not be identically equal to zero. Let D_α be the first minor in the sequence $D_1, D_2, \dots . , D_{m-1}$which does not vanish identically.The number v is equal at least to the unity and is at most the order m of zero λ_0.It follows that $D_{v-1} = 0$, the relation (12) shows that the minor D_v satisfies the homogeneous integral equation

$$D_v \begin{pmatrix} x_1, \dots . , x_i, \dots . , x_v \\ \xi_1, \dots \dots \dots . , \xi_v \end{pmatrix} ; \lambda_0 \Big)$$

$$= \lambda_0 \int_a^b K(x_i, s) D_v \begin{pmatrix} x_1, \dots, x_{i-1}, s, x_{i+1}, \dots \dots, x_v \\ \xi_1, \dots \dots \dots, \xi_v \end{pmatrix} ; \lambda_0 \Big) ds \qquad \dots (14)$$

$i = 1,2,3 \dots \dots . \, v$

Thus the function

$$\phi_i(x) = D_v \begin{pmatrix} x_1, \dots \dots, x_{i-1}, x, x_{i+1}, \dots \dots, x_v \\ \xi_1, \dots \dots \dots \dots \dots, \xi_v \end{pmatrix} ; \lambda_0 \Big) \qquad \dots (15)$$

is a solution of the homogeneous equation (1) not identically equal to zero for

some chosen fixed values of the remaining variables $x_1, x_2, \dots, x_v$ and $\xi_1, \xi_2, \dots, \xi_v$

Substituting x for x_i at v different points in the minor (15), we obtain v non-trivial solutions $\phi_1(x), \phi_2(x),\ldots, \phi_r(x)$ of the homogeneous equation (1, not Identically equal to zero, and hence may be written as

$$\phi_i(x)=\frac{D_v\begin{pmatrix} x1,\ldots,xi-1,x,xi+1,\ldots,xn \\ \xi 1,\ldots\ldots\ldots\ldots\xi n & ;\lambda 0\end{pmatrix}}{D_v\begin{pmatrix} x1,\ldots,xi-1,x,xi+1,\ldots,xn \\ \xi 1,\ldots\ldots\ldots\ldots\xi n & ;\lambda 0\end{pmatrix}},\ i=1,2,\ldots,v \qquad \ldots (16)$$

where the numbers $x_1,x_2,\ldots,x_v$ and $\xi_1,\xi_2,\ldots,\xi_v$ are selected in such a manner so that the denominator does not vanish.

Now we shall show that the solution ϕ_i as determined by (16) are linearly independent, i.e. if there exists arbitrary constant $c_1,c_2,\ldots,c_v$ such that

$$c_1\phi_1(x)+c_2\phi_2(2)+\ldots+c_v\phi_v(x)=0 \qquad \ldots (17)$$

$$c_1=c_2=\ldots=c_v=0$$

then in fact, if relation (17) were to hold with not all $c_i=0$, then we obtain

$$c_1\phi_1(x)=0,\ c_2\phi_2(2)=0,\ \ldots,\ cv\phi v(x)=0$$

as $$\phi_i(x_k)=\begin{cases}0, i\neq k\\ 1,\ i\neq k\end{cases}$$

it follows that $c_1=c_2=\ldots=c_v=0$, which is contrary to the hypothesis,

Therefore, the solution of the homogeneous integral equation (1), not identically equal to zero, is called the characteristic solution of that equation corresponding to a given characteristic value λ_0 of the kernel $K(X,\xi)$. This system is known as the fundamental system of the characteristic solutions.

Any linear combination of solutions (16) of the form

$$\phi(x)=c_1\phi_1(x)+c_2\phi_2(2)+\ldots+c_v\phi_v(x),$$

where $c_1,c_2,\ldots,c_v$ are constants, is also a solution of the homogeneous integral equation.

Converse. Every solution $\phi(x)$ of integral equation (1) is some linear combination of characteristic solutions

$$\phi_1(x), \phi_2(x),\ldots, \phi_v(x)$$

Assuming an auxiliary function $H(x,\xi)$ of the form

$$H(x,\xi)=\frac{D_{v+1}\begin{pmatrix} x & x1,\ldots,\ldots, & xv \\ \xi & \xi 1,\ldots\ldots\ldots\ldots\xi v & ;\lambda 0\end{pmatrix}}{D_v\begin{pmatrix} x1,\ldots & ,\ldots\ .,xv \\ \xi 1,\ldots\ldots\ldots\ldots\xi v & ;\lambda 0\end{pmatrix}} \qquad \ldots (18)$$

Multiplying equation $H(x,\xi)$, both the sides, we have

$$\int_a^b H(x,\xi)\phi(\xi)d\xi=\lambda_0\int_a^b[\int_a^b K(s,\xi)H(x,s)ds]\phi(\xi)d\xi$$

Multiplying both the sides by λ_0 and adding term by term, we have

$$\phi(x)=\lambda_0\int_a^b K(x,\xi)\phi(\xi)d\xi==\lambda_0\int_a^b \Gamma(x,\xi)\phi(\xi)d\xi \qquad \ldots (19)$$

where $\Gamma(x,\xi)=K(x,\xi)-H(x,\xi)+\lambda_0\int_a^b K(s,\xi)H(x,s)ds \qquad \ldots (20)$

Now

$$D v+1\begin{pmatrix} x, x1, \ldots \ldots & \ldots \ldots, \quad x v \\ \xi\ \xi 1, \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . . \xi v & ; \lambda 0\end{pmatrix}$$

$$=K(x \xi) D v\begin{pmatrix} x 1, \ldots . ., x i-1, x, x i+1, \ldots . ., x v \\ \xi 1, \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . . \xi v & ; \lambda 0\end{pmatrix}$$

$$+\sum_{k=1}^{v}(-1)^{k} \mathrm{K}(\mathrm{x}_{\mathrm{k}}, \xi)\, D v\begin{pmatrix} x & x 1, \ldots . ., x k-1, x, x k+1, \ldots . ., & x v \\ s & \xi 1, \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots\ . . \xi v & ; \lambda 0\end{pmatrix}$$

$$+\lambda_{0} \int_{a}^{b} K(s, \xi) D v+1\begin{pmatrix} x\ x 1, \ldots, x i-1, x, x i+1, \ldots, x n \\ s\ \xi 1, \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . . \xi n & ; \lambda 0\end{pmatrix} \mathrm{ds} \qquad \ldots (21)$$

In every minor D_v we transpose the variable x from the first place to between the variables x_{k-1} and x_{k+1} and divide both sides of (21) by the constant

$$D v\begin{pmatrix} x 1, x 2 \ldots . . \ldots . ., x v \\ \xi 1, \xi 2 \ldots \ldots \ldots . . \xi v & ; \lambda 0\end{pmatrix} \neq 0$$

We obtain

$$\mathrm{H}(\mathrm{x}, \xi)=\mathrm{K}(\mathrm{x}, \xi)+\sum_{k=1}^{v} \mathrm{K}(\mathrm{x}_{\mathrm{k}}, \xi) \phi_{\mathrm{k}}(\mathrm{x})+\lambda_{0} \int_{a}^{b} K(s, \xi) \mathrm{H}(\mathrm{x}, \mathrm{s}) \mathrm{ds} \qquad \ldots(22)$$

From the relation (20) and (22), we have

$$\Gamma(x, \xi)=-\sum_{k=1}^{v} \mathrm{K}(\mathrm{x}_{\mathrm{k}}, \xi) \phi_{\mathrm{k}}(\mathrm{x}) \phi(\xi) \mathrm{d} \xi \qquad \ldots (23)$$

Thus the equation (19) reduces to

$$\phi(\mathrm{x})=-\lambda_{0} \sum_{k=1}^{v} \int_{a}^{b} \mathrm{K}(\mathrm{x}_{\mathrm{k}}, \xi) \phi_{\mathrm{k}}(\mathrm{x}) \phi(\xi) \mathrm{d} \xi \qquad \ldots (24)$$

If we omit the function $\phi_k(x)$ from under the integral sign then each term on the R.H.S. the function $\phi(x)$ has the form

$$\phi(x)=c_1\phi_1(x)+c_2\phi_2(2)+\ldots\ldots+c_v\phi_v(x)$$

where $c_1, c_2, \ldots, c_v$ are constants. It follows that the function $\phi(x)$ is a linear combination of characteristic solutions $\phi_i(x)$.

12.8 CHARACTERISTIC SOLUTIONS

Corresponding to distinct characteristic values of Fredholm's integral equation and its associate equation , are orthogonal.

Since $\phi(x)$ is a characteristic solution of the homogeneous equation

$$\phi(x)=\lambda_{0} \int_{a}^{b} K(x, \xi) \phi(\xi) d \xi \qquad \ldots (1)$$

Corresponding to the characteristic value λ_0.

Let $\Psi(x)$ be a characteristic solution of the associate equation

$$\Psi(x)=\lambda_{1} \int_{a}^{b} K(\xi, x) \Psi(\xi) d \xi \qquad \ldots (2)$$

Corresponding to the characteristic value $\lambda_1$where $\lambda_0 \neq \lambda_1$.

Multiplying (1) by $\lambda_1\, \Psi(x)$ and (2) by $\lambda_0\, \phi(x)$, integrating and then subtracting , we have

$$(\lambda_{1}-\lambda_{0}) \int_{a}^{b} \phi(x) \Psi(x) d x$$

$=\lambda_0\lambda_1 \int_a^b \int_a^b K(x,\xi)\Psi(x)\phi(\xi)d\xi dx - \lambda_0\lambda_1 \int_a^b \int_a^b K(\xi,x)\Psi(\xi)\phi(x)d\xi dx = 0$

Upon interchanging x and ξ, the second integral is identical to the first one. Thus

$$\int_a^b \phi(x)\Psi(x)dx = 0, \lambda_1 \neq \lambda_0$$

This implies that the characteristic solutions, corresponding to distinct characteristic values of Fredholm integral and its associate equation, are orthogonal.

12.9 FREDHOLM'S THIRD THEOREM

For the non-homogeneous integral equation of second kind

$$\phi(x) = f(x) + \lambda_0 \int_a^b K(x,\xi)\phi(\xi)d\xi$$

To possess a solution in the case D(λ_0)=0, it is necessary and sufficient that the given function f(x) be orthogonal to all the eigen solutions $\phi_i(x)$,i=1, 2,…,v of the associate homogeneous equation corresponding to the eigenvalue λ_0 and forming the fundamental system.

Consider the non-homogeneous integral equation

$$\phi(x) = f(x) + \lambda_0 \int_a^b K(x,\xi)\phi(\xi)d\xi \qquad \ldots (1)$$

Where λ_0 is an eigenvalue, i.e., D(λ_0)=0.

Let $\Psi(x)$ be an eigenfunction of the associated equation

$$\Psi(x) = \lambda_0 \int_a^b K(\xi,x)\Psi(\xi)d\lambda \qquad \ldots (2)$$

Corresponding to the eigenvalue λ_0.

Multiplying (1) by $\Psi(x)$, both the sides and integrating, we have

$$\int_a^b \Psi(x)\phi(x)dx = \int_a^b f(x)\Psi(x)dx + \lambda_0 \int_a^b \int_a^b K(x,\xi)\Psi(x)\,\phi(\xi)dx\,d\xi$$

or

$$\int_a^b f(x)\Psi(x)dx = \int_a^b \Psi(x)\phi(x)dx - \lambda_0 \int_a^b \int_a^b K(x,\xi)\Psi(x)\,\phi(\xi)dx\,d\xi$$

By Permuting the variables x and ξ, we have

$$\int_a^b f(x)\Psi(x)dx = \int_a^b \left[\Psi(x) - \lambda_0 \left\{\int_a^b K(\xi,x)\Psi(\xi)\,d\xi\right\}\right]\phi(x)dx = 0 \qquad \ldots(3)$$

It follows that the integral equation (1) does not always have a solution when D(λ_0)=0, but a necessary condition for the existence of a solution is the orthogonality of the known function f(x) to all the eigen function $\Psi(x)$, of the associated equation

$$\int_a^b f(x)\Psi(x)dx = 0 \qquad \ldots (4)$$

Since every eigen function of the integral equation is a linear combination of the basic solutions, so a necessary condition for the existence of a solution of the non-homogeneous equation (1) is the orthogonality of the known function f(x) to v fundamental solution of the associated equation

$$\int_a^b f(x)\Psi_i(x)dx = 0\,;\, i = 1,2,\ldots, v \qquad \ldots (5)$$

Corresponding to an eigen value λ_0 i.e., D(λ_0)=0

Further, the relation (5) is the sufficient condition for the existence of a solution of the non-homogeneous equation (1). Multiplying (1) by the auxiliary function $H(x,\xi)$ and integrating, we have

$$\int_a^b H(x,\xi)\phi(\xi)d\xi \int_a^b H(x,\xi)f(\xi)d\xi + \lambda_0 \int_a^b \left\{\int_a^b H(x,s)K\,(s,\xi)ds\right\}\phi(\xi)d\xi \qquad \ldots(6)$$

Where $H(x,\xi)$ is the ratio of the two minors

$$H(x,\xi)= \frac{D_{v+1}\begin{pmatrix} x, x_1, \ldots\ldots\ldots, x_v \\ \xi, \xi_1 \ldots\ldots\ldots \xi_v & ;\lambda_0 \end{pmatrix}}{D_v\begin{pmatrix} x_1, \ldots\ldots\ldots, x_v \\ \xi_1, \ldots\ldots\ldots \xi_v & ;\lambda_0 \end{pmatrix}} \qquad \ldots (7)$$

Multiplying both sides of equation (6) by λ_0, we have

$$\phi(x) = f(x) + \lambda_0 \int_a^b H(x,\xi)f(\xi)d\xi + \lambda_0 \int_a^b T(x,\xi)\phi(\xi)d\xi \qquad \ldots (8)$$

where $T(x,\xi) = K(x,\xi) - H(x,\xi) + \lambda_0 \int_a^b H(x,s)K(x,\xi)ds \qquad \ldots (9)$

Thus, the solution of the non-homogeneous integral equation (1) exists in the form

$$\phi(x) = f(x) + \lambda_0 \int_a^b \frac{D_{v+1}\begin{pmatrix} x, x_1, \ldots\ldots\ldots, x_v \\ \xi, \xi_1 \ldots\ldots\ldots \xi_v & ;\lambda_0 \end{pmatrix}}{D_v\begin{pmatrix} x_1, \ldots\ldots\ldots, x_v \\ \xi_1, \ldots\ldots\ldots \xi_v & ;\lambda_0 \end{pmatrix}} f(\xi)d\xi + \sum_{k=1}^{v} C_k \phi_k(x) \qquad \ldots(10)$$

Where $T(x,\xi) = \sum_{k=1}^{v} K(x_k,\xi)\phi(x)$ and C_k *are constants*.

The function (10) is a solution of equation (1) if the orthogonality condition is satisfied. The third term of (10) being a linear combination of the fundamental solution $\phi_k(x)$ is a solution of the homogeneous equation.

Examples

Example.1: Find D(λ) (Fredholm determinants) and $R(x,\xi;\lambda)$ (Resolvent kernel) of the following $K(x,\xi) = xe^{\xi}$, $a = 0, b = 1$.

Solution: we know

$$D(\lambda) = 1 + \sum \frac{(-\lambda)^m}{m!} C_m$$

$$R(x,\xi;\lambda) = \frac{D(x,\xi;\lambda)}{D(\lambda)}\ ,\ D(\lambda) \neq 0$$

$$D(x,\xi;\lambda) = B_o(x,\xi) + \sum \frac{-1^m \lambda^m}{m!} B_m(x,\xi)$$

$$B_o(x,\xi) = K(x,\xi) = xe^{\xi}$$

$B_1(x,\xi) = \int_0^1 K\begin{pmatrix} x & \xi_1 \\ \xi & \xi_1 \end{pmatrix}\mathrm{d}\xi_1$

$$= \int_0^1 \begin{vmatrix} K(x,\xi) & K(x,\xi_1) \\ K(\xi_1,\xi) & K(\xi_1,\xi_1) \end{vmatrix} d\xi_1$$

$=\int_0^1 \begin{vmatrix} xe^{\xi} & xe^{\xi_1} \\ \xi_1 e^{\xi} & \xi_1 e^{\xi_1} \end{vmatrix} d\xi_1$

$=\int_0^1 (x\xi_1 e^{\xi+\xi_1} - x\xi_1 e^{\xi+\xi_1})\, d\xi_1$

$= 0$

$B_2(x,\xi) = \iint_0^1 K\begin{pmatrix} x & \xi_1 & \xi_2 \\ \xi & \xi_1 & \xi_2 \end{pmatrix} d\xi_1 d\xi_2$

$$= \iint_0^1 \begin{vmatrix} K(x,\xi) & K(x,\xi_1) & K(x,\xi_1) \\ K(\xi_1,\xi) & K(\xi_1,\xi_1) & K(\xi_1,\xi_2) \\ K(\xi_2,\xi) & K(\xi_2,\xi_1) & K(\xi_2,\xi_2) \end{vmatrix} d\xi_1 d\xi_2$$

$B_2(x,\xi) = \iint_0^1 \begin{vmatrix} xe^{\xi} & xe^{\xi_1} & xe^{\xi_2} \\ \xi_1 e^{\xi} & \xi_1 e^{\xi_1} & \xi_1 e^{\xi_2} \\ \xi_2 e^{\xi} & \xi_2 e^{\xi_1} & \xi_2 e^{\xi_2} \end{vmatrix} d\xi_1 d\xi_2$

$=\iint_0^1 e^{\xi+\xi_1+\xi_2} \begin{vmatrix} x & x & x \\ \xi_1 & \xi_1 & \xi_1 \\ \xi_2 & \xi_2 & \xi_2 \end{vmatrix} d\xi_1 d\xi_2$

$=\iint_0^1 e^{\xi+\xi_1+\xi_2} D_3 \; d\xi_1 d\xi_2$

$D_3 = \begin{vmatrix} x & x & x \\ \xi_1 & \xi_1 & \xi_1 \\ \xi_2 & \xi_2 & \xi_2 \end{vmatrix}$

$= \begin{vmatrix} x & 0 & 0 \\ \xi_1 & 0 & 0 \\ \xi_2 & 0 & 0 \end{vmatrix}, \qquad C_2 \to C_2 - C_1 \,, C_3 \to C_3 - C_1$

$= 0$

It implies $B_2(x,\xi) = 0$

Now, $C_1 = \int_0^1 K(\xi_1,\xi_1)\mathrm{d}\xi_1$

$= \int_0^1 \xi_1 e^{\xi_1}\mathrm{d}\xi_1$

$=\left|\xi_1 e^{\xi_1}\right| - \int_0^1 1e^{\xi_1}\mathrm{d}\xi_1$

$= 1$

$C_2 = \iint_0^1 K\begin{pmatrix} \xi_1 & \xi_2 \\ \xi_1 & \xi_2 \end{pmatrix} d\xi_1 d\xi_2$

$=\int\int_0^1 \begin{vmatrix} K(\xi_1,\xi_1) & K(\xi_1,\xi_2) \\ K(\xi_2,\xi_1) & K(\xi_2,\xi_2) \end{vmatrix} d\xi_1 d\xi_2$

$$=\int\int_0^1 \begin{vmatrix} \xi_1 e^{\xi_1} & \xi_1 e^{\xi_2} \\ \xi_2 e^{\xi_1} & \xi_2 e^{\xi_2} \end{vmatrix} d\xi_1 d\xi_2$$

$$=\int\int_0^1 (\xi_1\xi_2 e^{\xi_2+\xi_1} - \xi_1\xi_2 e^{\xi_2+\xi_1})\, d\xi_1 \mathrm{d}\xi_2$$

$$= 0$$

We already know that $\mathrm{D}(\lambda) = 1 + \sum \frac{(-\lambda)^m}{m!} C_m$

It implies $\mathrm{D}(\lambda) = 1 - \frac{\lambda}{1!} C_1 + \frac{\lambda^2}{2!} C_2 - \frac{\lambda^3}{3!} C_3 + ----$

$$= (1-\lambda)\ ,\ C_2 = C_3 = 0 ----$$

$$\mathrm{D}(x,\xi;\lambda) = B_o(x,\xi) + \sum \frac{-1^m \lambda^m}{m!} B_m(x,\xi)$$

$$= \mathrm{K}(x,\xi) - \frac{\lambda}{1!} B_1(x,\xi) + \frac{\lambda^2}{2!} B_2(x,\xi) ------$$

$$= xe^{\xi}$$

$$\Rightarrow \qquad R(x,\xi;\lambda) = \frac{D(x,\xi;\lambda)}{D(\lambda)} = \frac{xe^{\xi}}{(1-\lambda)}$$

This is the required solution.

Example 2. Find $\mathrm{D}(\lambda)$ (Fredholm determinants) and $R(x,\xi;\lambda)$ (Resolvent kernel) of the following $\mathrm{K}(x,\xi) = 2x - \xi\ , a = 0, b = 1.$

Solution: $\mathrm{D}(\lambda) = 1 + \sum \frac{(-\lambda)^m}{m!} C_m$

$R(x,\xi;\lambda) = \frac{D(x,\xi;\lambda)}{D(\lambda)}$, $\mathrm{D}(\lambda) \neq 0$

$$\mathrm{D}(x,\xi;\lambda) = B_o(x,\xi) + \sum \frac{-1^m \lambda^m}{m!} B_m(x,\xi)$$

$$= \mathrm{K}(x,\xi) - \frac{\lambda}{1!} B_1(x,\xi) + \frac{\lambda^2}{2!} B_2(x,\xi) ------$$

$$B_1(x,\xi) = \int_a^b K\begin{pmatrix} x & \xi_1 \\ \xi & \xi_1 \end{pmatrix} \mathrm{d}\xi_1$$

$$= \int_a^b \begin{vmatrix} K(x,\xi) & K(x,\xi_1) \\ K(\xi_1,\xi) & K(\xi_1,\xi_1) \end{vmatrix} d\xi_1$$

$$=\int_0^1 \begin{vmatrix} 2x-\xi & 2x-\xi_1 \\ 2\xi_1-\xi & 2\xi_1-\xi_1 \end{vmatrix} d\xi_1$$

$$=\int_0^1 \begin{vmatrix} 2x-\xi & \xi-\xi_1 \\ 2\xi_1-\xi & \xi-\xi_1 \end{vmatrix} d\xi_1 \ ,\ C_2 \to C_2 - C_1$$

$$=\int_0^1 (\xi-\xi_1) \begin{vmatrix} 2x-\xi & 1 \\ 2\xi_1-\xi & 1 \end{vmatrix} d\xi_1$$

$$=\int_0^1 (\xi-\xi_1)(2x-\xi-2\xi_1+\xi) d\xi_1$$

$$=\int_0^1 2(\xi-\xi_1)(x-\xi_1) d\xi_1$$

$$=\int_0^1 2(x\xi - \xi\xi_1 + {\xi_1}^2 - x\xi_1) d\xi_1$$

$= 2[x\xi - \frac{\xi}{2} - \frac{x}{2} + \frac{1}{3}]$

$= 2\,x\xi - \xi - x + \frac{2}{3}$

$$B_2(x,\xi) = \iint_a^b K\begin{pmatrix} x & \xi_1 & \xi_2 \\ \xi & \xi_1 & \xi_2 \end{pmatrix} d\xi_1 d\xi_2$$

$$= \iint_a^b \begin{vmatrix} K(x,\xi) & K(x,\xi_1) & K(x,\xi_1) \\ K(\xi_1,\xi) & K(\xi_1,\xi_1) & K(\xi_1,\xi_2) \\ K(\xi_2,\xi) & K(\xi_2,\xi_1) & K(\xi_2,\xi_2) \end{vmatrix} d\xi_1 d\xi_2$$

$$B_2(x,\xi) = \iint_0^1 \begin{vmatrix} 2x-\xi & 2x-\xi_1 & 2x-\xi_2 \\ 2\xi_1-\xi & 2\xi_1-\xi_1 & 2\xi_1-\xi_2 \\ 2\xi_2-\xi & 2\xi_2-\xi_1 & 2\xi_2-\xi_2 \end{vmatrix} d\xi_1 d\xi_2$$

$$= \iint_0^1 \begin{vmatrix} 2x-\xi & \xi-\xi_1 & \xi-\xi_2 \\ 2\xi_1-\xi & \xi-\xi_1 & \xi-\xi_2 \\ 2\xi_2-\xi & \xi-\xi_1 & \xi-\xi_2 \end{vmatrix} d\xi_1 d\xi_2$$

Obtained by applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$

$$= \iint_0^1 (\xi-\xi_1)(\xi-\xi_2) \begin{vmatrix} 2x-\xi & 1 & 1 \\ 2\xi_1-\xi & 1 & 1 \\ 2\xi_2-\xi & 1 & 1 \end{vmatrix} d\xi_1 d\xi_2$$

= 0 as two columns are equal

Similarly, other further terms of B also become zero.

Now, $C_1 = \int_a^b K(\xi_1,\xi_1)\mathrm{d}\xi_1$

$= \int_0^1 (2\xi_1 - \xi_1)\mathrm{d}\xi_1$

$= \int_0^1 \xi_1\,\mathrm{d}\xi_1$

$= \frac{1}{2}$

$$C_2 = \iint_a^b K\begin{pmatrix} \xi_1 & \xi_2 \\ \xi_1 & \xi_2 \end{pmatrix} d\xi_1 d\xi_2$$

$$= \int\int_a^b \begin{vmatrix} K(\xi_1,\xi_1) & K(\xi_1,\xi_2) \\ K(\xi_2,\xi_1) & K(\xi_2,\xi_2) \end{vmatrix} d\xi_1 d\xi_2$$

$$= \int\int_0^1 \begin{vmatrix} 2\xi_1-\xi_1 & 2\xi_1-\xi_2 \\ 2\xi_2-\xi_1 & 2\xi_2-\xi_2 \end{vmatrix} d\xi_1 d\xi_2$$

$$= \int\int_0^1 \begin{vmatrix} \xi_1 & 2\xi_1-\xi_2 \\ 2\xi_2-\xi_1 & \xi_2 \end{vmatrix} d\xi_1 d\xi_2$$

$$= \int\int_0^1 [\xi_1\xi_2 - (2\xi_1-\xi_2)(2\xi_2-\xi_1)] d\xi_1 d\xi_2$$

$$= \int\int_0^1 \left[\xi_1\xi_2 - 4\xi_1\xi_2 - \xi_1\xi_2 + 2{\xi_1}^2 + 2{\xi_2}^2\right] d\xi_1 d\xi_2$$

$$= \int_0^1 \left[\frac{2}{3} - 2\xi_2 + 2{\xi_2}^2\right] d\xi_2$$

$=\frac{1}{3}$

Now by similar method we find C_3 which comes out to be zero and further terms will also become zero.

Now further we got,

$$D(x,\xi;\lambda) = B_o(x,\xi) + \sum \frac{-1^m \lambda^m}{m!} B_m(x,\xi)$$

$$= K(x,\xi) - \frac{\lambda}{1!} B_1(x,\xi) + \frac{\lambda^2}{2!} B_2(x,\xi) ------$$

$$= 2x - \xi - \lambda(2\,x\xi - \xi - x + \tfrac{2}{3})$$

$$D(\lambda) = 1 - \frac{\lambda}{1!} C_1 + \frac{\lambda^2}{2!} C_2 - \frac{\lambda^3}{3!} C_3 + ----$$

$$= 1 - \frac{\lambda}{2} + \frac{\lambda^2}{6}$$

$$R(x,\xi;\lambda) = \frac{D(x,\xi;\lambda)}{D(\lambda)} = \frac{2x - \xi - \lambda(2\,x\xi - \xi - x + \frac{2}{3})}{1 - \frac{\lambda}{2} + \frac{\lambda^2}{6}}$$

This is the required solution.

Example 3. Find $R(x,\xi,\lambda)$ of the following $K(x,\xi) = x^2\xi - x\xi^2, 0 \leq x \leq 1, 0 \leq \xi \leq 1.$

Solution: Given$K(x,\xi) = x^2\xi - x\xi^2, a = 0, b = 0 \ldots (1)$

To find $R(x,\xi,\lambda) = \frac{D(x,\xi,\lambda)}{D(\lambda)}$

where $D(x,\xi,\lambda) = B_0(x,\xi) - \frac{\lambda}{\llcorner 1} B_1(x,\xi) + \frac{\lambda^2}{\llcorner 2} B_2(x,\xi) - \frac{\lambda^3}{\llcorner 3} B_3(x,\xi) + \ldots\ldots\ldots\ldots$

$$B_0(x,\xi) = K(x,\xi) = x^2\xi - x\xi^2$$

$$B_1(x,\xi) = \int_0^1 K\begin{vmatrix} x & \xi_1 \\ \xi & \xi_1 \end{vmatrix} d\xi_1 = \int_0^1 \begin{vmatrix} K(x,\xi) & K(x,\xi_1) \\ K(\xi_1,\xi) & K(\xi_1,\xi_1) \end{vmatrix} d\xi_1$$

$$B_1(x,\xi) = \int_0^1 \begin{vmatrix} x^2\xi - x\xi^2 & x^2\xi_1 - x{\xi_1}^2 \\ {\xi_1}^2\xi_1 - \xi_1\xi^2 & 0 \end{vmatrix} d\xi_1$$

$$B_1(x,\xi) = -\int_0^1 (\xi_1^2\xi - \xi^2)\,(x^2\xi_1 - x\xi_1^2) d\xi_1$$

$$B_1(x,\xi) = -\int_0^1 \left[(\xi_1^2\xi - \xi_1\xi^2)\,(x^2\xi_1) - (\xi_1^2\xi - \xi_1\xi^2)(x{\xi_1}^2)\right] d\xi_1$$

$$B_1(x,\xi) = -\int_0^1 (x^2\xi\xi_1^3 - \xi^2\xi_1^2x^2 - x\xi_1^4\xi + \xi_1^3\xi^2x)\, d\xi$$

$$B_1(x,\xi) = -\left[\frac{x^2\xi}{4} - \frac{x^2\xi^2}{3} - \frac{x\xi}{5} + \frac{x\xi^2}{4}\right]$$

$$B_1(x,\xi) = -x\xi\left[\frac{x+\xi}{4} - \frac{x\xi}{3} - \frac{1}{5}\right]$$

$$B_2(x,\xi) = \int_0^1 \int_0^1 K\begin{vmatrix} x & \xi_1 & \xi_2 \\ \xi & \xi_1 & \xi_2 \end{vmatrix} d\xi_1\, d\xi_2$$

$$= \int_0^1 \int_0^1 \begin{vmatrix} K(x,\xi) & K(x,\xi_1) & K(x,\xi_2) \\ K(\xi_1,\xi) & K(\xi_1,\xi_1) & K(\xi_1,\xi_2) \\ K(\xi_2,\xi_1) & K(\xi_2,\xi_1) & K(\xi_2,\xi_2) \end{vmatrix} d\xi_1\, d\xi_2$$

$$= \int_0^1 \int_0^1 \begin{vmatrix} x^2\xi - x\xi^2 & x^2\xi_1 - x\xi_1^2 & x^2\xi_2 - x\xi_2^2 \\ \xi_1^2\xi - \xi_1\xi^2 & \xi_1^2\xi_1 - \xi_1\xi_1^2 & \xi_1^2\xi_2 - \xi_1\xi_2^2 \\ \xi_2^2\xi - \xi_2\xi^2 & \xi_2^2\xi_1 - \xi_2\xi_1^2 & \xi_2^2\xi_2 - \xi_2\xi_2^2 \end{vmatrix} d\xi_1\, d\xi_2$$

$$= \int_0^1 \int_0^1 x\xi_1\xi_2 \begin{vmatrix} \xi(x-\xi) & \xi_1(x-\xi_1) & \xi_2(x-\xi_2) \\ \xi(\xi_1-\xi) & \xi_1(\xi_1-\xi_1) & \xi_2(\xi_1-\xi_2) \\ \xi_1(\xi_2-\xi) & \xi_1(\xi_2-\xi_1) & \xi_2(\xi_2-\xi_2) \end{vmatrix} d\xi_1\, d\xi_2$$

$$= \int_0^1 \int_0^1 x\xi_1\xi_1^2\xi_2^2 \begin{vmatrix} x-\xi & x-\xi_1 & x-\xi_2 \\ \xi_1-\xi & \xi_1-\xi_1 & \xi_1-\xi_2 \\ \xi_2-\xi & \xi_2-\xi_1 & \xi_2-\xi_2) \end{vmatrix} d\xi_1\, d\xi_2$$

$$= \int_0^1 \int_0^1 x\xi_1\xi_1^2\xi_2^2 \begin{vmatrix} x-\xi & \xi-\xi_1 & \xi_1-\xi_2 \\ \xi_1-\xi & \xi-\xi_1 & \xi_1-\xi_2 \\ \xi_2-\xi & \xi-\xi_1 & \xi_1-\xi_2 \end{vmatrix} d\xi_1\, d\xi_2 \; C_2 \to C_2 - C_1 \quad C_3 \to C_3 - C_1$$

$$= \int_0^1 \int_0^1 x\xi_1\xi_1^2\xi_2^2(\xi-\xi_1)(\xi_1-\xi_2) \begin{vmatrix} x-\xi & 1 & 1 \\ \xi_1-\xi & 1 & 1 \\ \xi_2-\xi & 1 & 1 \end{vmatrix} d\xi_1\, d\xi_2 = 0$$

as two columns are same.

Similarly $B_3(x,\xi) = 0 = B_4(x,\xi) = \ldots\ldots..$

$$C_1 = \int_0^1 K\,(\xi_1,\xi_1)\, d\xi_1 = \int_0^1 ({\xi_1}^2\xi_1 - \xi_1{\xi_1}^2)\, d\xi_1 = 0$$

$$C_2 = \int_0^1 \int_0^1 K\begin{vmatrix} \xi_1 & \xi_2 \\ \xi_1 & \xi_2 \end{vmatrix} d\xi_1\xi_2 = \int_0^1 \int_0^1 \begin{vmatrix} K(\xi_1\xi_1) & K(\xi_1\xi_2) \\ K(\xi_2\xi_1) & K(\xi_2\xi_2) \end{vmatrix} d\xi_1 d\xi_2$$

$$C_2 = \int_0^1 \int_0^1 \begin{vmatrix} {\xi_1}^2\xi_1 - \xi_1{\xi_1}^2 & {\xi_1}^2\xi_2 - \xi_1{\xi_2}^2 \\ {\xi_2}^2\xi_1 - \xi_2{\xi_1}^2 & {\xi_2}^2\xi_2 - \xi_2{\xi_2}^2 \end{vmatrix} d\xi_1 d\xi_2$$

$$C_2 = \int_0^1 \int_0^1 \begin{vmatrix} 0 & \xi_1\xi_2(\xi_1-\xi_2) \\ \xi_1\xi_2(\xi_2-\xi_1) & 0 \end{vmatrix} d\xi_1 d\xi_2$$

$$C_2 = \int_0^1 \int_0^1 + {\xi_1}^2{\xi_2}^2(\xi_1 - \xi_2)^2 d\xi_1 d\xi_2$$

$$C_2 = \int_0^1 \int_0^1 {\xi_1}^2{\xi_2}^2({\xi_1}^2 + {\xi_2}^2 - 2\xi_1\xi_2)\, d\xi_1 d\xi_2$$

$$C_2 = \int_0^1 \int_0^1 {\xi_1}^4{\xi_2}^2 + {\xi_1}^2 + {\xi_2}^4 - 2{\xi_1}^3{\xi_2}^3)\, d\xi_1 d\xi_2$$

$$C_2 = \int_0^1 \left(\frac{1}{5}.\xi_2{}^2 + \frac{1}{3}\xi_2{}^4 - \frac{2}{4}\xi_2{}^3) \,d\xi_2\right)$$

$$C_2 = \frac{1}{15} + \frac{1}{15} - \frac{1}{2}.\frac{1}{4} = \frac{1}{120}$$

$$C_3 = \int_0^1\int_0^1\int_0^1 \mathrm{K}\begin{vmatrix} \xi_1 & \xi_2 & \xi_3 \\ \xi_1 & \xi_2 & \xi_3 \end{vmatrix} d\xi_1\, d\xi_2\, d\xi_3$$

$$C_3 = \int_0^1\int_0^1\int_0^1 \begin{vmatrix} \mathrm{K}(\xi_1\xi_1) & \mathrm{K}(\xi_1\xi_2) & \mathrm{K}(\xi_1\xi_3) \\ \mathrm{K}(\xi_2\xi_1) & \mathrm{K}(\xi_2\xi_2) & \mathrm{K}(\xi_2\xi_3) \\ \mathrm{K}(\xi_3\xi_1) & \mathrm{K}(\xi_3\xi_2) & \mathrm{K}(\xi_3\xi_3) \end{vmatrix} d\xi_1\, d\xi_2\, d\xi_3$$

$$C_3 = \int_0^1\int_0^1\int_0^1 \begin{vmatrix} (\xi_1{}^2\xi_1 - \xi_1\xi_1{}^2) & (\xi_1{}^2\xi_2 - \xi_1\xi_2{}^2) & (\xi_1{}^2\xi_3 - \xi_1\xi_3{}^2) \\ (\xi_2{}^2\xi_1 - \xi_2\xi_1{}^2) & (\xi_2{}^2\xi_2 - \xi_2\xi_2{}^2) & (\xi_2{}^2\xi_3 - \xi_2\xi_3{}^2) \\ (\xi_3{}^2\xi_1 - \xi_3\xi_1{}^2) & (\xi_3{}^2\xi_2 - \xi_3\xi_2{}^2) & (\xi_3{}^2\xi_3 - \xi_3\xi_3{}^2) \end{vmatrix} d\xi_1\, d\xi_2\, d\xi_3$$

$$C_3 = \int_0^1\int_0^1\int_0^1 \begin{vmatrix} (\xi_1\xi_1(\xi_1 - \xi_1) & (\xi_1\xi_2(\xi_1 - \xi_2) & (\xi_1\xi_3(\xi_1 - \xi_3) \\ (\xi_2\xi_1(\xi_2 - \xi_1) & (\xi_2\xi_2(\xi_2 - \xi_2) & (\xi_2\xi_3(\xi_2 - \xi_3) \\ (\xi_3\xi_1(\xi_3 - \xi_1) & (\xi_3\xi_2(\xi_3 - \xi_2) & (\xi_3\xi_3(\xi_3 - \xi_3) \end{vmatrix} d\xi_1\, d\xi_2\, d\xi_3$$

$$C_3 = \int_0^1\int_0^1\int_0^1 \xi_1\xi_2\xi_3 \begin{vmatrix} (\xi_1(\xi_1 - \xi_1) & (\xi_1(\xi_1 - \xi_2) & (\xi_1(\xi_1 - \xi_3) \\ (\xi_2(\xi_2 - \xi_1) & (\xi_2(\xi_2 - \xi_2) & (\xi_2(\xi_2 - \xi_3) \\ (\xi_3(\xi_3 - \xi_1) & (\xi_3(\xi_3 - \xi_2) & (\xi_3(\xi_3 - \xi_3) \end{vmatrix} d\xi_1\, d\xi_2\, d\xi_3$$

$$C_3 = \int_0^1\int_0^1\int_0^1 \xi_1{}^2\xi_2{}^2\xi_3{}^2 \begin{vmatrix} \xi_1 - \xi_1 & \xi_1 - \xi_2 & \xi_1 - \xi_3 \\ \xi_2 - \xi_1 & \xi_2 - \xi_2 & \xi_2 - \xi_3 \\ \xi_3 - \xi_1 & \xi_3 - \xi_2 & \xi_3 - \xi_3 \end{vmatrix} d\xi_1\, d\xi_2\, d\xi_3$$

$$C_3 = \int_0^1\int_0^1\int_0^1 \xi_1{}^2\xi_2{}^2\xi_3{}^2 \begin{vmatrix} \xi_1 - \xi_1 & \xi_1 - \xi_2 & \xi_1 - \xi_3 \\ \xi_2 - \xi_1 & \xi_2 - \xi_2 & \xi_2 - \xi_3 \\ \xi_3 - \xi_1 & \xi_3 - \xi_2 & \xi_3 - \xi_3 \end{vmatrix} d\xi_1\, d\xi_2\, d\xi_3$$

$C_{2\rightarrow}C_2 - C_1$

$C_3 \rightarrow C_3 - C_1$

$$C_3 = \int_0^1\int_0^1\int_0^1 \xi_1{}^2\xi_2{}^2\xi_3{}^2(\xi_1 - \xi_2)(\xi_1\xi_3) \begin{vmatrix} 0 & 1 & 1 \\ \xi_2 - \xi_1 & 1 & 1 \\ \xi_3 - \xi_1 & 1 & 1 \end{vmatrix} d\xi_1\, d\xi_2\, d\xi_3$$

$C_3 = 0$ Since two columns are equals.

Similarly $C_4 = 0 = C_5 =$………..

$$\mathrm{R}(x,\xi,\lambda) = \frac{\mathrm{D}(x,\xi,\lambda)}{\mathrm{D}(\lambda)}$$

$$\left[\frac{K(x,\xi) - \frac{\lambda}{\llcorner 1}\mathrm{B}_1(x,\xi) + \frac{\lambda^2}{\llcorner 2}\mathrm{B}_2(x,\xi) - \frac{\lambda^3}{\llcorner 3}\mathrm{B}_3(x,\xi) + \ldots}{1 - \frac{\lambda}{\llcorner 1}\mathrm{C}_1 + \frac{\lambda^2}{\llcorner 2}\mathrm{C}_2 - \frac{\lambda^3}{\llcorner 3}\mathrm{C}_3 + \ldots\ldots}\right]$$

$$R(x,\xi,\lambda) = \frac{x^2\xi - x\xi^2 + \lambda x\xi\left[\frac{x+\xi}{4} - \frac{x\xi}{3} - \frac{1}{5}\right]}{1 - \frac{\lambda}{\lfloor 1}.0 + \frac{\lambda^2}{\lfloor 2}.\frac{1}{120} + 0}$$

$$R(x,\xi,\lambda) = \frac{x\xi(x-\xi) + \lambda x\xi\left[\frac{x+\xi}{4} - \frac{x\xi}{3} - \frac{1}{5}\right]}{(1 + \frac{\lambda^2}{240})}$$

Example 4. Find $R(x,\xi,\lambda)$ of the following $K(x,\xi) = \sin x \cos \xi, 0 \le x \le 2\pi, 0 \le \xi \le 2\pi$

Given $K(x,\xi) = \sin x \cos \xi, a = 0, b = 2\pi$.

Solution: We know that

$$R(x,\xi;\lambda) = \frac{K(x,\xi) - \frac{\lambda}{\lfloor 1}B_1(x,\xi) + \frac{\lambda^2}{\lfloor 2}B_2(x,\xi) - \frac{\lambda^3}{\lfloor 3}B_3(x,\xi) + \ldots}{1 - \frac{\lambda}{\lfloor 1}C_1 + \frac{\lambda^2}{\lfloor 2}C_2 - \frac{\lambda^3}{\lfloor 3}C_3 + \ldots..} \qquad \ldots(1)$$

Now we have

$$B_1(x,\xi) = \int_0^{2\pi} K\begin{vmatrix} x & \xi_1 \\ \xi & \xi_1 \end{vmatrix} d\xi_1 = \int_0^{2\pi} \begin{vmatrix} K(x,\xi) & K(x,\xi_1) \\ K(\xi_1,\xi) & K(\xi_1,\xi_1) \end{vmatrix} d\xi_1$$

$$B_1(x,\xi) = \int_0^{2\pi} \begin{vmatrix} \sin x \cos \xi & \sin x \cos \xi_1 \\ \sin \xi_1 \cos \xi & \sin \xi_1 \cos \xi_1 \end{vmatrix} d\xi_1$$

$$= \sin x \int_0^{2\pi} \begin{vmatrix} \cos \xi & \cos \xi_1 \\ \sin \xi_1 \cos \xi & \sin \xi_1 \cos \xi_1 \end{vmatrix} d\xi_1 = 0$$

Similarly $\quad B_2(x,\xi) = 0 = B_3(x,\xi) = \ldots..$

$$C_1 = \int_0^{2\pi} K(\xi_1,\xi_1)d\xi_1 = \int_0^{2\pi} \sin \xi_1 \cos \xi_1 \, d\xi_1 = 0$$

Similarly $C_2 = 0 = C_3 = \ldots\ldots..$

So $R(x,\xi;\lambda) = \frac{\sin x \cos \xi}{1}$.

Example 5. Find fredholm determinant and resolvent kernel of $k(x, \xi) = \sin x - \sin \xi$, $a = 0$, $b = 2\pi$.

Solution: We know fredholm determinant

$D(\lambda) = 1 - \frac{\lambda}{1!} C_1 + \frac{\lambda.\lambda}{2!} C_2 - \ldots\ldots\ldots\ldots\ldots$ …(1)

Resolve kernel

$R(x,\xi; \lambda) = \frac{D(x,\xi;\lambda)}{D(\lambda)}$ …(2)

where $D(x,\xi; \lambda) = B_0(x, \xi) - \frac{\lambda}{1!} B_1(x, \xi) + \ldots\ldots\ldots$ …(3)

Here $B_0(x, \xi) = k(x, \xi) = \sin x - \sin \xi$ (given) …(4)

Now we have

$$B_1(x, \xi) = \int_0^{2\pi} k\begin{pmatrix} x & \xi_1 \\ \xi & \xi_1 \end{pmatrix} d\xi_1 = \int_0^{2\pi} . \begin{vmatrix} k(x,\xi) & k(x,\xi_1) \\ k(\xi_1, \xi) & k(\xi_1, \xi_1) \end{vmatrix} d\xi_1 \qquad \ldots(5)$$

Consider

$$D_1 = \begin{vmatrix} k(x,\xi) & k(x,\xi_1) \\ k(\xi_1, \xi) & k(\xi_1, \xi_1) \end{vmatrix}$$

$$= \begin{vmatrix} \sin x - \sin\xi & \sin x - \sin\xi_1 \\ \sin\xi_1 - \sin\xi & \sin\xi_1 - \sin\xi_1 \end{vmatrix}$$

$$= -(\sin x - \sin \xi)(\sin \xi_1 - \sin\xi)$$

$$= -\sin x . \sin \xi_1 + \sin^2\xi_1 - \sin x . \sin \xi - + \sin \xi . \sin\xi_1$$

$$\therefore\ B_1(x, \xi) = -\sin x \int_0^{2\pi} . \sin \xi_1\, d\xi_1 - \int_0^{2\pi} . \sin^2\xi_1 d\xi_1 + \sin x . \sin \xi \int_0^{2\pi} . d\xi_1 +$$

$$\sin\xi \int_0^{2\pi} . \sin \xi_1 d\xi_1$$

$$= 0 + \frac{1}{2}\int_0^{2\pi} . (1 - \cos 2\xi_1)\, d\xi_1 + \sin x \sin\xi . 2\pi + 0$$

$$= \pi(1 + 2 \sin x \sin \xi)$$

$$B_2(x, \xi) = \iint_0^{2\pi} k\begin{pmatrix} x & \xi_1 & \xi_2 \\ \xi & \xi_1 & \xi_2 \end{pmatrix} d\xi_1\, d\xi_2$$

Where $D_2(x, \xi_1, \xi_2) = k\begin{pmatrix} x & \xi_1 & \xi_2 \\ \xi & \xi_1 & \xi_2 \end{pmatrix}$

$$= \begin{pmatrix} k(x,\xi) & k(x,\xi_1) & k(x,\xi_2) \\ k(\xi_1,\xi) & k(\xi_1,\xi_1) & k(\xi_1,\xi_2) \\ k(\xi_2,\xi) & k(\xi_2,\xi_1) & k(\xi_2,\xi_2) \end{pmatrix}$$

$$= \begin{pmatrix} \sin x - \sin\xi & \sin x - \sin\xi & \sin x - \sin\xi_2 \\ \sin\xi_1 - \sin\xi & \sin\xi_1 - \sin\xi_1 & \sin\xi_1 - \sin\xi_2 \\ \sin\xi_2 - \sin\xi & \sin\xi_2 - \sin\xi_1 & \sin\xi_2 - \sin\xi_2 \end{pmatrix}$$

$C_2 \to C_2 - C_1 \qquad C_3 \to C_3 - C_1$

$$= \begin{pmatrix} \sin x - \sin\xi & \sin\xi - \sin\xi_1 & \sin\xi - \sin\xi_2 \\ \sin\xi_1 - \sin\xi & \sin\xi - \sin\xi_1 & \sin\xi - \sin\xi_2 \\ \sin\xi_2 - \sin\xi & \sin\xi - \sin\xi_1 & \sin\xi - \sin\xi_2 \end{pmatrix}$$

$$= \begin{pmatrix} \sin x - \sin\xi & 1 & 1 \\ \sin\xi_1 - \sin\xi & 1 & 1 \\ \sin\xi_2 - \sin\xi & 1 & 1 \end{pmatrix} = 0$$

$D_2(x, \xi_1, \xi_2) = 0$

$\therefore\ B_2(x, \xi) = 0$

Similarly, other terms are zero.

Now $C_1 = (\xi_1, \xi_1)\, d\xi_1 = \int_0^{2\pi}.(\sin \xi_1 - \sin\xi_1) d\xi_1 = 0$

$$C_2 = \iint_0^{2\pi} k\begin{pmatrix} \xi_1 & \xi_2 \\ \xi_1 & \xi_2 \end{pmatrix} d\xi_1 d\xi_2 = \iint_0^{2\pi}. \begin{vmatrix} k(\xi_1, \xi_1) & k(\xi_1, \xi_2) \\ k(\xi_2, \xi_1) & k(\xi_2, \xi_2) \end{vmatrix} d\xi_1 d\xi_2$$

$$= \iint_0^{2\pi}. \begin{vmatrix} \sin\xi_1 - \sin\xi_1 & \sin\xi_1 - \sin\xi_2 \\ \sin\xi_2 - \sin\xi_1 & \sin\xi_2 - \sin\xi_2 \end{vmatrix} d\xi_1 d\xi_2$$

$$= \iint_0^{2\pi}.(\sin\xi_1 - \sin\xi_2)^2 d\xi_1 d\xi_2$$

$$= \iint_0^{2\pi}.(\sin^2\xi_1 + \sin^2\xi_2 - \sin\xi_1\sin\xi_2) d\xi_1 d\xi_2$$

$$= \iint_0^{2\pi}.[1 - 2\sin\xi_1\sin\xi_2 - \tfrac{1}{2}(\cos2\xi_1 + \cos2\xi_2)] d\xi_1 \xi_2$$

$$= \iint_0^{2\pi} 1 d\xi_1 d\xi_2 - 2\iint_0^{2\pi}.\sin\xi_1 \sin\xi_2\, d\xi_1\, d\xi_2 - \tfrac{1}{2}\iint_0^{2\pi}.(\cos2\xi_1 + \cos2\xi_2) d\xi_1\, \xi_2$$

$$= \int_0^{2\pi} 2\pi d\xi_2 - 0 - 0$$

$$= 4\pi^2$$

$$C_3 = \iiint_0^{2\pi}.K\begin{pmatrix} x & \xi_1 & \xi_2 \\ \xi & \xi_1 & \xi_2 \end{pmatrix} d\xi_1\, d\xi_2\, d\xi_3$$

Where $D_3(\xi_1, \xi_2, \xi_3) = K\begin{pmatrix} x & \xi_1 & \xi_2 \\ \xi & \xi_1 & \xi_2 \end{pmatrix}$

$$= \begin{pmatrix} k(\xi_1, \xi_1) & k(\xi_1, \xi_2) & k(\xi_1, \xi_3) \\ k(\xi_2, \xi_1) & k(\xi_2, \xi_2) & k(\xi_2, \xi_3) \\ k(\xi_3, \xi_1) & k(\xi_3, \xi_2) & k(\xi_3, \xi_3) \end{pmatrix}$$

$$= \begin{pmatrix} \sin\xi_1 - \sin\xi_1 & \sin\xi_1 - \sin\xi_2 & \sin\xi_1 - \sin\xi_3 \\ \sin\xi_2 - \sin\xi_1 & \sin\xi_2 - \sin\xi_2 & \sin\xi_2 - \sin\xi_3 \\ \sin\xi_3 - \sin\xi_1 & \sin\xi_3 - \sin\xi_2 & \sin\xi_3 - \sin\xi_3 \end{pmatrix}$$

$C_2 \to C_2 - C_1$ $\qquad$ $C_3 \to C_3 - C_1$

$$= \begin{pmatrix} \sin\xi_1 - \sin\xi_1 & \sin\xi_1 - \sin\xi & \sin\xi_3 - \sin\xi \\ \sin\xi_2 - \sin\xi_1 & \sin\xi_2 - \sin\xi & \sin\xi_3 - \sin\xi \\ \sin\xi_3 - \sin\xi_1 & \sin\xi_2 - \sin\xi & \sin\xi_3 - \sin\xi \end{pmatrix}$$

$$= (\sin\xi_1 - \sin\xi)(\sin\xi_3 - \sin\xi) \begin{pmatrix} \sin\xi_1 - \sin\xi_1 & 1 & 1 \\ \sin\xi_2 - \sin\xi_1 & 1 & 1 \\ \sin\xi_3 - \sin\xi_1 & 1 & 1 \end{pmatrix} = 0$$

$D_3 = 0$

$\therefore C_3 = \iiint_0^{2\pi} 0\, d\xi_1\, \xi_2\, \xi_3 = 0$

Similarly, $C_4 = C_5 = C_6 = 0 = \ldots\ldots\ldots\ldots..$

$D(x, \xi; \lambda) = B_0(x, \xi) - \frac{\lambda}{1!} B_1(x, \xi) + \ldots\ldots$

$= \sin x - \sin \xi - \pi(1 + 2 \sin x \sin \xi) + 0 + 0 + \ldots\ldots$

$D(\lambda) = 1 - \frac{\lambda}{1!} C_1 + \frac{\lambda.\lambda}{2!} C_2 - \ldots\ldots\ldots\ldots\ldots\ldots$

$= 1 + \frac{1}{2}(4\pi^2\lambda^2)$

$= 1 + 2\pi^2\lambda^2$

$R(x, \xi; \lambda) = \frac{D(x,\xi;\lambda)}{D(\lambda)}$

$R(x, \xi; \lambda) = \{\sin x - \sin\xi - \pi(1 + 2 \sin x \sin \xi)\}/\{1 + 2\pi^2\lambda^2\}$

Which is required resolvent kernel.

12.10 SUMMARY

The non- homogeneous Fredholm integral equation of second kind

$$\phi(x) = F(x) + \lambda \int_a^b K(x, \xi)\phi(\xi)d\xi,$$

Under the assumption that the function F(x) and K(x, ξ) are integrable has a unique solution, is of the form

$$\phi(x) = F(x) + \lambda \int_a^b R(x, \xi, \lambda)F(\xi)d\xi$$

Where the Resolvent kernel R is a meromorphic † function of the parameter λ, being the ratio of two entire function of the parameter λ

$$R(x, \xi, \lambda) = \frac{D(x,\xi,\lambda)}{D(\lambda)}, D(\lambda) \neq 0.$$

We know that $\emptyset(x) = F(x) + \lambda \int_a^b R(x, \xi; \lambda)F(\xi)d\xi$

of the non-homogeneous Fredholm equation is unique for any λ provided by $D(\lambda) \neq 0$.

If λ_0 is a zero of multiplicity m of the function D(λ), then the homogeneous integral equation,

$$\phi(x) = \lambda_0 \int_a^b K(x, \xi)\, \phi(\xi)d\xi$$

Possesses at least one, and at most m, linearly independent solutions,

$$\phi_i(x) = D_r \begin{pmatrix} x_1 & x_2 & - & \xi_v \\ \xi_1 & \xi_2 & - & \xi_v \end{pmatrix}; \lambda_0 \Big)$$

(i=1,2,3,………v;1≤v≤m not identically zero, and any other solution of this equation is a linear combination of these solutions.

12.11 TERMINAL QUESTIONS

Q.1 Explain the Fredhlom integral equation.

Q.2 State and Prove Fredhlom First theorem.

Q.3 State and Prove Fredhlom Second theorem.

Q.4 State and Prove Fredhlom third theorem.

Q.5 Find $R(x, \xi, \lambda)$ of the following: $K(x, \xi) = 1 + 3x\xi$, $0 \le x \le 1$, $0 \le \xi \le 1$.

Q.6 Find the resolvent kernel of the following kernel $k(x, \xi) = x - 2\xi$, $0 \le x \le 1$, $0 \le \xi \le 1$.

Q.7 Find the reslovent kernel of $K(x, \xi) = x + \xi + 1$, $-1 \le x \le 1$, $-1 \le \xi \le 1$.

Q.8 Find the reslovent kernel of $K(x, \xi) = 4x\xi - x^2$, $0 \le x \le 1$, $0 \le \xi \le 1$.

Q.9 Find the reslovent kernel of $K(x, \xi) = 1$, $a = 0$, $b = 1$.

UNIT-13 FREDHOLM INTEGRAL EQUATIONS-II

Structure

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13.2 Objectives
13.3 Fredholm Integral Equation
13.4 Solution of Fredholm Integral Equation
13.5 Resolvent kernel for Fredholm integral equation
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13.7 Method to solve Fredholm integral equation
13.8 Eigen values and Eigen function
13.9 Symmetric kernel
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13.1 INTRODUCTION

Integral equations represent mathematical relationships where an unknown function is involved within an integral expression. These equations are prevalent across physics, engineering, and mathematics, particularly when addressing scenarios with continuous quantities. Named after the Swedish mathematician Ivar Fredholm, Fredholm integral equations feature the unknown function both inside and outside the integral sign. Their applications span diverse fields such as physics, engineering, and mathematics, offering valuable insights into physical phenomena and facilitating the analysis of engineering systems. Moreover, solutions to Fredholm integral equations aid in addressing specific boundary value problems. Extensively utilized in mathematical modeling, Fredholm integral equations play a crucial role in describing phenomena characterized by continuous interactions or distributions.

Across various disciplines, Fredholm integral equations contribute significantly to the analysis, modeling, and comprehension of complex systems and phenomena. In electrical engineering, Fredholm integral equations are instrumental in analyzing transmission lines, antennas, and electromagnetic wave propagation. They provide valuable insights into the behavior of electromagnetic fields within intricate structures and media, further enhancing the understanding and optimization of electrical systems. In this unit we shall discuss the vrious types for Fredholm integral equations, various methods to solve Fredholm integral equation of first and second kind are discussed. Resolvent kernels are used to solve Fredholm integral equations.

13.2 OBJECTIVES

After reading this unit the learner should be able to understand about:

- the Fredholm integral equation
- the solution of Fredholm integral equation
- the resolved kernel and separable kernel for Fredholm integral equation

13.3 FREDHOLM INTEGRAL EQUATION

A Fredholm Integral Equation is of the type

$$h(x)u(x) = f(x) + \int_a^b K(x,\xi)u(\xi)d\xi \;\; for\ all\ x \in [a,b]$$

(i) If h(x) = 0, the above equation reduces to:

$$-f(x) = \int_a^b K(x,\xi)u(\xi)d\xi$$

This equation is called Fredholm integral equation of first kind.

(ii) If h(x) = 1, the above equation becomes:

$$u(x) = f(x) + \int_a^b K(x,\xi)u(\xi)d\xi$$

This equation is called Fredholm integral equation of second kind.

(iii) If h(x) = 1, f(x) = 0 the above equation becomes:

$$u(x) = \int_a^b K(x,\xi)u(\xi)d\xi$$

This equation is called Homogeneous Fredholm integral equation of second kind.

Examples

Example 1. Reduce the boundary value problem to Fredholm equation

$$y" = xy = 1, y(0) = 0, y(1) = 0.$$

Solution: Given boundary value problem is

$$y" = 1 - xy \qquad \dots(1)$$

Integrating over 0 to x,

$$y'(x) = x - \int_0^x \xi\, y(\xi)d\xi + c_1$$

Again integrating over 0 to x,

$$y(x) = \left[\frac{x^2}{2}\right]_0^x - \int_0^x (x-\xi)\xi y(\xi)d\xi + c_1 x + c_2 \qquad \ldots (2)$$

Where c_1 and c_2 are constants to be determined by boundary value conditions.

Using y(0) = 0 in equation (2), we get

$$0 = 0 - 0 + 0 + c_2 \quad \Rightarrow \; c_2 = 0$$

So, equation (2) becomes

$$y(x) = \frac{x^2}{2} - \int_0^x (x-\xi)\xi y(\xi)d\xi + c_1 x \qquad \ldots (3)$$

Now, using y(1) = 0 in the equation (3), we get

$$0 = \frac{1}{2} - \int_0^1 (1-\xi)\xi y(\xi)d\xi + c_1$$

$$c_1 = \int_0^1 (1-\xi)\xi y(\xi)d\xi - \frac{1}{2} (4)$$

Putting value of c_1 in equation (3), we get

$$y(x) = \frac{x^2}{2} - \int_0^x (x-\xi)\xi y(\xi)d\xi + x\int_0^1 \xi(1-\xi)y(\xi)d\xi - \frac{x}{2}$$

$$or \qquad y(x) = \frac{x^2}{2} - \frac{x}{2} - \int_0^x \xi(x-\xi)y(\xi)d\xi + x\int_0^1 \xi(1-\xi)y(\xi)d\xi$$

To express this in standard form, we split the second integral into two integrals, as follows

$$y(x) = \frac{x^2}{2} - \frac{x}{2} - \int_0^x \xi(x-\xi)y(\xi)d\xi + x\int_0^x \xi(1-\xi)y(\xi)d\xi + x\int_x^1 \xi(1-\xi)y(\xi)d\xi$$

$$or \qquad y(x) = \frac{x^2}{2} - \frac{x}{2} + \int_0^x (x - x\xi - x + \xi)\xi y(\xi)d\xi + x\int_0^x \xi(1-\xi)y(\xi)d\xi$$

$$or \qquad y(x) = \frac{x^2}{2} - \frac{x}{2} + \int_0^x \xi(1-x)\xi y(\xi)d\xi + \int_0 \xi(1-\xi)y(\xi)d\xi$$

$$or \quad y(x) = f(x) + \int_0^1 \xi K(x,\xi)\, y(\xi)d\xi \; where \; f(x) = \frac{x^2}{2} - \frac{x}{2}$$

$$and \qquad K(x,\xi) = \begin{cases} (1-x)\xi & if \;\; 0 \le \xi \le x \\ (1-\xi)x & if \;\; x \le \xi \le 1 \end{cases}$$

Hence the solution.

Example 2. Reduce the boundary value problem,

$$y" + A(x)y' + B(x)y = g(x), a \leq x \leq b, y(a) = c_1, y(b) = c_2$$

to a Fredholm integral equation.

Solution: Given differential equation is

$$y" + A(x)y' + B(x)y = g(x)$$

$$y" = -A(x)y' - B(x)y + g(x)$$

Integrating w.r.t. x from a to x, we get

$$\frac{dy}{dx} = -\int_a^x A(\xi)y'(\xi)d\xi - \int_a^x B(\xi)y(\xi)d\xi + \int_a^x g(\xi)d\xi + \alpha_1$$

$$\Rightarrow \quad \frac{dy}{dx} = -[A(\xi)y(\xi)]_a^x + \int_a^x A'(\xi)y(\xi)d\xi - \int_a^x B(\xi)y(\xi)d\xi + \int_a^x g(\xi)d\xi + \alpha_1$$

$$\Rightarrow \quad \frac{dy}{dx} = \int_a^x [A'(\xi)y(\xi)]y(\xi)\, d(\xi) + \int_a^x g(\xi)d\xi - A(x)y(x) + A(a)c_1 + \alpha_1$$

Again integrating over a to x,

$$y(x) = \int_a^x (x - \xi)\, [A'(\xi) - B(\xi)]y(\xi)d\xi + \int_a^x (x - \xi)g(\xi)\, d\xi$$
$$- \int_a^x A(\xi)y(\xi) + (x - a)[\alpha_1 + A(a)c_1] + \alpha_2 \qquad (1)$$

Applying first boundary condition, $y(a) = c_1$, we get $\alpha_2 = c_1$

Again applying second boundary condition, $y(b) = c_2$, we have

$$c_2 = \int_a^b (b - \xi)\, [A'(\xi) - B(\xi)]y(\xi)d\xi + \int_a^b (b - \xi)g(\xi)\, d\xi$$
$$- \int_a^b A(\xi)y(\xi)d\xi + (b - a)[\alpha_1 + A(a)c_1] + c_1$$

$$\Rightarrow \quad \alpha_1 + c_1A(a) = \frac{1}{b - a}\left[c_2 - c_1\int_a^b [(b - \xi)\{A'(\xi) - B(\xi)\} - A(\xi)]\, y(\xi)d\xi - \int_a^b (b - \xi)g(\xi)d\xi\right]$$

$$\Rightarrow \ a_1 + c_1 A(a)$$

$$= \frac{1}{b-a}\left\{c_2 - c_1 - \int_a^x [(b-\xi)\{A'(\xi) - B(\xi)\} - A(\xi)]y(\xi)\,d\xi\right.$$

$$\left. - \int_x^b [(b-\xi)\{A'(\xi) - B(\xi)\} - A(\xi)]y(\xi)d\xi - \int_a^b (b-\xi)g(\xi)d\xi\right\}$$

Putting this value of $a_1 + c_1 A(a)$ in equation (1), we obtain

$$y(x) = c_1 + \int_a^x (x-\xi)g(\xi)d\xi + \frac{x-a}{b-a}\left[c_2 - c_1 - \int_a^b (b-\xi)g(\xi)\,d\xi\right]$$

$$+ \int_a^x [(x-\xi)\{A'(\xi) - B(\xi)\} - A(\xi)]\,y(\xi)d\xi - \int_a^x \frac{x-a}{b-a}[(b-\xi)\{A'(\xi) - B(\xi)\} - A(\xi)]y(\xi)d\xi$$

$$-\frac{x-a}{b-a}\int_x^b [(b-\xi)\{A'(\xi) - B(\xi)\} - A(\xi)]y(\xi)d\xi$$

$$or\ y(x) = f(x) + \int_a^x \left[\left\{(x-\xi) - \frac{(x-a)(b-\xi)}{b-a}\right\}\{A'(\xi) - B(\xi)\} + A(\xi)\left\{-1 + \frac{x-a}{b-a}\right\}\right]y(\xi)d\xi$$

$$-\frac{x-a}{b-a}\int_x^b [(b-\xi)\{A'(\xi) - B(\xi)\} - A(\xi)]y(\xi)d\xi$$

Now

$$(x-\xi) - \frac{(x-a)(b-\xi)}{b-a} = \frac{(x-b)(\xi-a)}{b-a} \ \ and \ -1 + \frac{x-a}{b-a} = \frac{x-b}{b-a}$$

Thus, the above equation becomes

$$y(x) = f(x) + \frac{x-b}{b-a}\int_a^x [A(\xi) - (a-\xi)\{A'(\xi) - B(\xi)\}]y(\xi)d\xi$$

$$-\frac{x-a}{b-a}\int_a^b [A(\xi) - (b-\xi)\{A'(\xi) - B(\xi)\}]y(\xi)d\xi$$

$$or \qquad\qquad y(x) = f(x) + \int_a^b K(x,\xi)y(\xi)d\xi$$

Where

$$f(x) = c_1 + \int_a^x (x-\xi)g(\xi)\,d\xi + \frac{x-a}{b-a}\left[c_2 - c_1 - \int_a^b (b-\xi)g(\xi)d\xi\right]$$

$$and \qquad K(x,\xi) = \begin{bmatrix} \dfrac{x-b}{b-a}[A(\xi) - (a-\xi)\{A'(\xi) - B(\xi)\}]\ x > \xi \\ \dfrac{x-a}{b-a}[A(\xi) - (b-\xi)\{A'(\xi) - B(\xi)\}]\ x < \xi \end{bmatrix}$$

This is complete solution.

Example 3. Covert the Fredholm integral equation

$$u(x) = \lambda \int_0^1 K(x,t)u(t)dt \ \ where\ K(x,t) = \begin{cases} x(1-t) & 0 \le x \le t \\ t(1-x) & t \le x \le 1 \end{cases}$$

into the boundary value problem $u" + \lambda u = 0,\ u(0) = 0, u(1) = 0.$

Solution. Write

$$u(x) = \lambda\left[\int_0^x K(x,t)u(t)dt + \int_x^1 K(x,t)u(t)dt\right]$$

$$= \lambda\left[\int_0^x t(1-x)u(t)dt + \int_x^1 x(1-t)u(t)dt\right]$$

$$= \lambda \int_0^x t(1-x)u(t)dt + \lambda \int_x^1 x(1-t)u(t)dt \qquad (1)$$

Differentiating (1) w.r.t. x and using Leibnitz formula

$$\frac{du}{dx} = \lambda \int_0^x -t\,u(t)dt + \lambda(x)(1-x)u(x) + \lambda \int_x^1 (1-t)u(t)dt - \lambda x(1-x)u(x)$$

$$so \qquad \frac{d^2u}{dx^2} = \lambda \int_0^x 0.(-t)u(t)dt + \lambda(-x)u(x) + \lambda \int_x^1 0.(1-t)u(t)dt - \lambda(1-x)u(x)$$

$$= -\lambda\, u(x)$$

$$\Rightarrow \qquad \frac{d^2u}{dx^2} + \lambda\, u(x) = 0$$

Also, from (1), we have, u(0) = 0 = u(1)

Hence the solution.

13.4 SOLUTION OF FREDHOLM INTEGRAL EQUATION

Let us Consider a Fredholm integral equation of second kind.

$$u(x) = f(x) + \lambda \int_0^1 K(x,\xi)u(\xi)d\xi \qquad (1)$$

We define an integral operator,

$$k[\phi(x)] = \int_a^b K(x,\xi)\phi(\xi)d\xi$$

$k^2[\phi(x)] = k[k\{\phi(x)\}]$ and so on.

Then, (1) can be written as

$$u(x) = f(x) + \lambda k[u(x)]$$

Theorem 1. If the Fredholm integral equation

$$u(x) = f(x) + \lambda \int_a^b K(x,\xi)u(\xi)d\xi \qquad (1)$$

Is such that

- **(i)** $K(x,\xi)$ is a non-zero real continuous function in the rectangle $R = I \times I$, where I = [a, b] and $|K(x,\xi)| < M$ in R.
- **(ii)** $f(x)$ is an non-zero real valued continuous function on I.
- **(iii)** λ is a constant satisfying the inequality, $|\lambda| < \frac{1}{M(b-a)}$.

Then (1) has one and only one continuous solution in the interval I and this solution is given by the absolutely and uniformly convergent series $u(x) = f(x) + \lambda k[f(x)] + \lambda^2 k^2[f(x)] + \cdots to\ \infty$.

Proof: We prove the result by the method of successive approximation. In this method we choose any continuous function say $u_0(x)$ defined on I as the zeroth approximation.

Then the first approximation, say $u_1(x)$, is given

$$u_1(x) = \mathrm{f(x)} + \lambda \int_a^b K(x,\xi)u_0(\xi)d\xi \qquad (2)$$

By substituting this approximation into R.H.S. of (1), we obtain next approximation, $u_2(x)$. Continuing like this, we observe that the successive approximations are determined by the recurrence formula

$$u_n(x) = \mathrm{f(x)} + \lambda \int_a^b K(x,\xi)u_{n-1}(\xi)d\xi \qquad (3)$$

$$= f(x) + \lambda k[u_{n-1}(x)]$$

$$= f(x) + \lambda k[f(x) + \lambda k\{u_{n-2}(x)\}]$$

$$= f(x) + \lambda k[f(x)] + \lambda^2 k^2 [f(x) + \lambda k\{u_{n-3}(x)\}]$$

Hence $u_n(x) = f(x) + \lambda k[f(x)] + \lambda^2 k^2 [f(x)] + \cdots \ldots . + \lambda^{n-1} k^{n-1} [f(x)] + R_n(x)$,

Where $R_n(x) = \lambda^n k^n [u_0(x)]$

As $u_0(x)$ is continuous, it is bounded that is, $|u_0(x)| \leq U$ in I

Now, $|R_n(x)| = |\lambda|^n \left[\int_a^b K(x,t) \int_a^b K(t,t_1) \ldots \int_a^b K(t_{n-2}, t_{n-1}) u_0(t_{n-1}) dt_{n-1} \ldots . dt\right]$

$$\leq |\lambda|^n M^n\, U\, (b-a)^n$$

$$= U[|\lambda| M(b-a)]^n \to 0 \; as\; n \to \infty \left(since, \qquad |\lambda| < \frac{1}{M(b-a)}\right)$$

$$\Rightarrow \qquad \lim_{n\to\infty} R_n(x) = 0$$

Thus, $\lim_{n\to\infty} u_n(x) = u(x) = f(x) + \lambda k f(x) + \lambda^2 k^2 f(x) + \cdots .. to\; \infty$

This can be easily verified by the virtue of M test that the above series is absolutely and uniformly convergent in I.

Uniqueness: Let v(x) be another solution of given integral equation then by choosing $u_0(x) = v(x)$, we get

$$u_n(x) = v(x) for\; all\; n$$

$$\Rightarrow \qquad \lim_{n\to\infty} u_n(x) = v(x) \Rightarrow u(x) = v(x)$$

This completes the proof.

Examples

Example 4. Find the first two approximation of the solution of Fredholm integral equation.

$$u(x) = 1 + \int_0^1 K(x,\xi) u(\xi) d\xi \quad where\; K(x,\xi) = \begin{cases} x & 0 \leq x \leq \xi \\ \xi & \xi \leq x \leq 1 \end{cases}$$

Solution: Let $u_0(x) = 1$ be the zeroth approximation. Then first approximation is given by

$$u_1(x) = 1 + \int_0^1 K(x,\xi) u_0(\xi) d\xi$$

$$= 1 + \int_0^x K(x,\xi) d\xi + \int_x^1 K(x,\xi) d\xi = 1 + \int_0^x \xi\, d\xi + \int_x^1 x d\xi$$

$$= 1 + \frac{x^2}{2} + x(1-x) = 1 + x - \frac{x^2}{2}$$

Now we have

$$u_2(x) = 1 + \int_0^1 K(x,\xi) u_1(\xi) d\xi$$

$$= 1 + \int_0^1 K(x,\xi)\left(1+\xi-\frac{\xi^2}{2}\right)d\xi$$

$$= 1 + \int_0^x \xi\left(1+\xi-\frac{\xi^2}{2}\right)d\xi + x\int_x^1 \left(1+\xi-\frac{\xi^2}{2}\right)d\xi$$

$$= 1 + \frac{4}{3}x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24}.$$

13.5 RESOLVENT KERNEL FOR FREDHOLM INTEGRAL EQUATION

Let us Consider the Fredholm integral equation

$$u(x) = f(x) + \lambda \int_a^b K(x,\xi)u(\xi)d\xi \qquad (1)$$

The integrated kernels are defined by $K_1(x,\xi) = K(x,\xi)$, and

$$K_{n+1}(x,\xi) = \int_a^b R(x,\xi:\lambda)f(\xi)d\xi$$

Where $R(x,\xi:\lambda) = K_1 + \lambda K_2 + \lambda^2 K_3 + \cdots..to\ \infty$

$$= \sum_{n=1}^{\infty} \lambda^{n-1}K_n(x,\xi)$$

Neumann series: The infinite series $K_1 + \lambda K_2 + \lambda^2 K_3 + \cdots$.. is called Neumann series and Resolvent Kernel: The function $R(x,\xi:\lambda)$ is called Resolvent Kernel.

Examples

Example.5: Obtain the resolvent kernel associated with the kernel $K(x,\xi) = 1 - 3x\xi$ in the interval (0, 1) and solve the integral equation

$$u(x) = 1 + \lambda \int_0^1 (1-3x\xi)u(\xi)d\xi$$

Solution: Here $K(x,\xi) = 1 - 3x\xi$ We know that the iterated kernel are given by the relation, $K_1(x,\xi) = K(x,\xi)$

and $K_{n+1}(x,\xi) = \int_a^b K(x,t)K_n(t,\xi)dt$

therefore, $K_1(x,\xi) = 1 - 3x\xi$

and $K_2(x,\xi) = \int_a^b K(x,t)K_1(t,\xi)dt$

$$= \int_0^1 (1-3xt)(1-3x\xi)dt$$

$$= \int_0^1 (1-3t\xi-3xt+9xt^2\xi)dt$$

$$= \left[1-\frac{3t^2\xi}{2}-\frac{3xt^2}{2}+3xt^2\xi\right]_0^1$$

$$= 1-\frac{3}{2}\xi-\frac{3}{2}x+3x\xi$$

$$K_3(x,\xi) = \int_0^1 K(x,t)\,K_2(t,\xi)dt$$

$$= \int_0^1 (1-3xt)\left(1-\frac{3}{2}t-\frac{3}{2}\xi+3t\xi\right)dt$$

$$= \frac{1}{4}(1-3x\xi)\ (on\ solving)$$

$$K_4(x,\xi) = \int_0^1 K(x,t)\,K_3(t,\xi)dt$$

$$= \frac{1}{4}\int_0^1 (1-3xt)\,(1-3t\xi)dt$$

$$= \frac{1}{4}\left[1-\frac{3\xi}{2}-\frac{3x}{2}+3x\xi\right]$$

The Resolvent Kernel $R(x,\xi:\lambda)$ is given by

$R(x,\xi:\lambda) = K_1+\lambda K_2+\lambda^2K_3+\lambda^3K_4+\cdots\ldots\ldots$..$(Here\ K_m\ means\quad K_m(x,\xi)$ etc)

$$= (1-3x\xi)+\lambda\left(1-\frac{3\xi}{2}-\frac{3x}{2}+3x\xi\right)+\frac{\lambda^2}{4}(1-3x\xi)+\frac{\lambda^3}{4}\left(1-\frac{3\xi}{2}-\frac{3x}{2}+3x\xi\right)+..$$

$$= (1-3x\xi)\left(1+\frac{\lambda^2}{4}\right)+\lambda\left(1-\frac{3\xi}{2}-\frac{3x}{2}+3x\xi\right)\left(1+\frac{\lambda^2}{4}\right)+\cdots..$$

$$= \left(1+\frac{\lambda^2}{4}+\cdots\right)\left[(1-3x\xi)+\lambda\left(1-\frac{3\xi}{2}-\frac{3x}{2}+3x\xi\right)\right]$$

$$= \left(\frac{1}{1-\frac{\lambda^2}{4}}\right)\left[(1-3x\xi)+\lambda\left(1-\frac{3\xi}{2}-\frac{3x}{2}+3x\xi\right)\right]$$

$$= \left(\frac{4}{4-\lambda^2}\right)\left[(1-3x\xi)+\lambda\left(1-\frac{3\xi}{2}-\frac{3x}{2}+3x\xi\right)\right]$$

Which provides the required result.

We know that the solution of an integral equation

$$u(x) = f(x) + \lambda \int_a^b K(x,\xi)u(\xi)\,d\xi \quad is\ given\ by$$

$$u(x) = f(x) + \lambda \int_a^b R(x,\xi:\lambda)f(\xi)\,d\xi$$

Here $K(x,\xi) = (1 - 3x\xi)$. Then

$$R(x,\xi:\lambda) = \left(\frac{4}{4-\lambda^2}\right)\left[(1-3x\xi) + \lambda\left(1 - \frac{3\xi}{2} - \frac{3x}{2} + 3x\xi\right)\right]$$

Thus, the solution of given integral equation is

$$u(x) = 1 + \frac{4\lambda}{4-\lambda^2}\int_0^1 \left[(1-3x\xi) + \lambda\left(1 - \frac{3\xi}{2} - \frac{3x}{2} + 3x\xi\right)\right].1d\xi$$

$$= 1 + \frac{4\lambda}{4-\lambda^2}\left[\xi - 3x\frac{\xi^2}{2} + \lambda\left(\xi - \frac{3\xi^2}{4} - \frac{3x\xi}{2} + \frac{3\xi^2}{2}\right)\right]_0^1$$

$$= 1 + \frac{4\lambda}{4-\lambda^2}\left[1 - \frac{3x}{2} + \lambda\left(1 - \frac{3}{4} - \frac{3x}{2} + \frac{3x}{2}\right)\right]$$

$$= 1 + \frac{4\lambda}{4-\lambda^2}\left(1 - \frac{3x}{2} + \frac{\lambda}{4}\right)$$

$$= \frac{4 + 4\lambda - 6x\lambda}{4-\lambda^2}, \lambda \neq \pm 2$$

This is the required solution of given integral equation.

13.6 SEPARABLE KERNEL

A kernel $K(x,\xi)$ of an integral equation is called separable if it can be expressed in the form

$$K(x,\xi) = \sum_{i=1}^{n} a_i(x)b_i(\xi) = a_1(x)b_1(\xi) + a_2(x)b_2(\xi) + \cdots. + a_n(x)b_n(\xi)$$

For example.

$(a)\ e^{x-\xi} = e^x.e^{-\xi} = a_1(x)b_1(\xi), n = 1$

$(b)\ x - \xi = x.1 + 1(-\xi) = a_1(x)b_1(\xi) + a_2(x)b_2(\xi), n = 2$

$(c)\ Similarly, \sin(x+\xi), 1 - 3x\,\xi$ are separable kernels.

$(d)\ x^{\xi}, \sin(x\,\xi)\,\xi$ are non – separable kernels.

13.7 ANOTHER METHOD TO SOLVE FREDHOLM INTEGRAL EQUATION

We have to solve the Fredholm integral equation of second kind with separable kernel.

Let the given integral equation be

$$u(x) = f(x) + \lambda \int_a^b K(x,\xi)u(\xi)\,d\xi \qquad (1)$$

$$where\ \ K(x,\xi) = \sum_{i=1}^{n} a_i(x)b_i(\xi) \qquad (2)$$

Thus, (1) can be written as

$$u(x) = f(x) + \lambda \int_a^b \left[\sum_{i=1}^{n} a_i(x)b_i(\xi)\right] u(\xi)\,d\xi$$

$$u(x) = f(x) + \lambda \sum_{i=1}^{n} a_i(x) \left[\int_a^b b_i(\xi)u(\xi)\,d\xi\right]$$

$$= f(x) + \lambda[c_1a_1(x) + c_2a_2(x) + \cdots .. + c_na_n(x)] \qquad (3)$$

$$where\ \ C_k = \int_a^b b_k(\xi)u(\xi)\,d\xi = \int_a^b b_k(x)u(x)\,dx \qquad (4)$$

Here (3) gives the solution of given Fredholm integral (1) provided the constants c_1, c_2, ….c_n are determined.

For this, we multiply (3) both sides $b_i(x)$ and then integrating w.r.t. x from a to b, we find

$$\int_a^b b_i(x)u(x)\,dx = \int_a^b f(x)b_i(x)\,dx + \lambda \sum_{k=1}^{n} C_k \int_a^b b_i(x)\,a_k(x)dx\ \ for\ \ i = 1,2,3,\ldots..n$$

$$\Rightarrow \qquad c_i = f_i + \lambda \sum_{k=1}^{n} \alpha_{ik}C_k \qquad \ldots.(5)$$

$$where\ \ f_i = \int_a^b f(x)b_i(x)\,dx\ \ and\ \ \alpha_{ik} = \int_a^b b_i(x)\,a_k(x)dx \qquad \ldots.(6)$$

From equation (5), we have

$$c_1 = f_1 + \lambda[c_{11}a_1 + c_{12}a_2 + \cdots .. + c_{1n}a_n]$$

$$c_2 = f_2 + \lambda[c_{21}a_1 + c_{22}a_2 + \cdots .. + c_{2n}a_n]$$

….   ………..   ………….   …………..

$$c_n = f_n + \lambda[c_{n1}a_1 + c_{n2}a_2 + \cdots .. + c_{nn}a_n]$$

In matrix form, $C = F + \lambda AC$ or $(I - \lambda A)C = F$ …. (7)

Where

$$C = \begin{bmatrix} c_1 \\ c_2 \\ .. \\ c_n \end{bmatrix}, \quad F = \begin{bmatrix} f_1 \\ f_2 \\ .. \\ f_n \end{bmatrix}, \quad A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots. & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots. & \alpha_{2n} \\ \dots. & \dots. & \dots. & \dots. \\ \alpha_{n1} & \alpha_{n2} & \dots. & \alpha_{nn} \end{bmatrix}$$

Let $|I - \lambda A| = \Delta(\lambda)$ …(8)

Now, we discuss the various cases

Case I: When $f(x) \neq 0$ and $F \neq 0$, that is, both integral equation as well as matrix equation are non-homogeneous. Then, from equation (7) has a unique solution if and only if $\Delta(\lambda) \neq 0$.

If $\Delta(\lambda) = 0$ for some value of λ, then from equation (7) has no solution or infinite solutions.

Case II: When f(x) = 0 that is, the Fredholm integral equation is homogeneous. In this case $f_i = 0$ for all i and consequently F = 0. Thus, equation (7) reduces to:

$$(I - \lambda A)C = 0 \quad \dots (9)$$

Subcase (a). If $\Delta(\lambda) \neq 0$, then from equation (9) has the trivial solution, C = 0 that is, $C_i = 0$ for all i .

Hence the equation (3) becomes, u(x) = 0 which is the solution of given integral equation.

Subcase (b). If $\Delta(\lambda_0) = 0$ for some scalar λ_0, then from equation (9) has infinitely many solutions. Consequently, the Fredholm integral equation $u(x) = \lambda_0 \int_a^b K(x,\xi)u(\xi)d\xi$ has infinitely many solutions.

Case III: When $f(x) \neq 0$ but F = 0. In this case also,

$$(I - \lambda A)C = 0 \quad \dots(10)$$

Subcase (a). If $\Delta(\lambda) \neq 0$, then (10) has only trivial solution, C = 0 that is, $C_i = 0$ for all i .

Hence the required solution of given equation becomes $u(x) = f(x) + 0 = f(x)$

Subcase (b). If $\Delta(\lambda_0) = 0$ for some scalar λ_0, then from equation (9) has infinitely many solutions. Consequently, the Fredholm integral equation $u(x) = \lambda_0 \int_a^b K(x,\xi)u(\xi)d\xi$ has infinitely many solutions.

13.8 EIGEN VALUES AND EIGEN FUNCTION

The values of λ for which $\Delta(\lambda) = 0$ are known as eigen values (or characteristic number) of Fredholm integral equation. The non-trivial solution corresponding to Eigen values are known as Eigen functions (or characteristic functions).

Remark: Separable kernels are also known as degenerate kernels.

Examples

Example.6. Solve the integral equation and discuss all its possible cases with the method of separable kernel

$$u(x) = f(x) + \lambda \int_0^1 (1 - 3x\xi)u(\xi)d\xi$$

Solution: The given equation is

$u(x) = f(x) + \lambda \int_0^1 (1 - 3x\xi)u(\xi)d\xi$ ... ... (1)

$u(x) = f(x) + \lambda[C_1 - 3xC_2]$ ......(2)

Where $c_1 = \int_0^1 u(\xi)d\xi$ ......(3)

and $c_2 = \int_0^1 \xi u(\xi)d\xi$ ......(4)

c_1 and c_2 are constant to be determined.

Integrating (2), w.r.t. x over the limit 0 to 1.

$$\int_0^1 u(x)dx = \int_0^1 f(x)dx + \lambda \int_0^1 (c_1 - 3xc_2)dx$$

$$c_1 = \int_0^1 f(x)dx + \lambda\left(c_1 - \frac{3}{2}c_2\right) \qquad [Using\ (3)]$$

Or $(1 - \lambda)c_1 + \frac{3}{2}\lambda c_2 = f_1$ ......(5)

Where $f_1 = \int_0^1 f(x)dx$

Now multiplying (2) with x and integrating w.r.t x between limits 0 and 1, we get

$$\int_0^1 xu(x)dx = \int_0^1 x\, f(x)dx + \lambda \int_0^1 (c_1 x - 3x^2 c_2)\, dx$$

Or $c_2 = f_1 + \lambda\left[c_1 \frac{x^2}{2} - x^3 c_2\right]_0^1$ [using (4)]

$$= f_1 + \lambda\left(\frac{c_1}{2} - c_2\right)$$

Or $-\frac{\lambda}{2}c_1 + (1 + \lambda)c_2 = f_2$ (6)

Where $f_2 = \int_0^1 x\, f(x)dx$

From (5) and (6), we get,

$$\Delta(\lambda) = \begin{vmatrix} 1 - \lambda & \frac{3\lambda}{2} \\ -\frac{\lambda}{2} & 1 + \lambda \end{vmatrix} = 1 - \lambda^2 + \frac{3\lambda^2}{4} = 1 - \frac{\lambda^2}{4}$$

Or

$$\Delta(\lambda) = \frac{4 - \lambda^2}{4}$$

Now, (5) and (6) can be written as

$$(I - \lambda A)C = f$$

Where $C = \begin{bmatrix} c_1 \\ c_1 \end{bmatrix}, F = \begin{bmatrix} f_1 \\ f_1 \end{bmatrix}$

Also, $|I - \lambda A| = \Delta(\lambda)$

Case I. When $f(x) \neq 0$ and $F \neq 0$ then equations (5) and (6) has a unique solution if $\Delta(\lambda) \neq 0$ that is, $\lambda \neq 2, -2$ When $\lambda = 2\ or - 2$, then these equations have either no solution or infinite many solutions.

(i) $\lambda = 2$

Then (5) and (6) reduce to

$$\left.\begin{matrix} -c_1 + 3c_2 = f_1 \\ -c_1 + 3c_2 = f_2 \end{matrix}\right] \tag{7}$$

These equation have no solution if $f_1 \neq f_2$ and have infinitely many solution when $f_1 = f_2$, that is

$$\int_0^1 f_1(x)dx = \int_0^1 xf(x)dx$$

or $$\int_0^1 (1 - x)f(x)dx = 0$$

Thus, the solution of given integral equation is

$$u(x) = f(x) + 2[c_1 a_1(x) + c_2 a_2(x)]$$
$$= f(x) + 2[c_1.1 + c_2(-3x)]$$
$$= f(x) + 2[3c_2 - f_1 - 3xc_2] \qquad from\ (7)$$
$$= f(x) + 6c_2(1 - x) - 2f_1$$

Or $u(x) = f(x) + 6c_2(1 - x) - 2\int_0^1 f(x)dx$ where c_2 is arbitrary.

(ii) $\lambda = -2$

As done above the solution is given by

$$u(x) = f(x) - 2(1 - 3x)c_2 - 2\int_0^1 xf(x)dx$$

Case II. When f(x) = 0, F = 0

In this case, the equation (5) and (6) becomes:

$$(1 - \lambda)c_1 + \frac{3\lambda}{2}c_2 = 0$$

$$-\frac{\lambda}{2}c_1 + (1 + \lambda)c_2 = 0$$

If $\lambda \neq 2, -2$, then system has only trivial solution $c_1 = 0 = c_2$. Thus u(x) = 0 is the solution of given integral equation.

(i) $\lambda = 2$

Then, (8) becomes

$$-c_1 + 3c_2 = 0 \rightarrow c_1 = 3c_2$$

Thus the solution of given integral equation is

$$u(x) = 0 + 2(3c_2 - 3xc_2) = 6c_2(1 - x)$$

(ii) $\lambda = -2$

Then, (8) becomes

$$c_1 - c_2 = 0 \rightarrow c_1 = c_2$$

Thus the solution is

$$u(x) = 0 - 2(c_2 - 3xc_2) = 2c_2(3x - 1)$$

Case III. When f(x) ≠ 0 and F = 0

If $\lambda \neq 2, -2$, the system (8) has only trivial solution $c_1 = c_2 = 0$and therefore u(x) = f(x) is the solution.

(i) $\lambda = 2$

Then $c_1 = 3c_2$ and the solution is

$$u(x) = f(x) + 2(3c_2 - 3xc_2) = f(x) + 6c_2(1 - x)$$

(ii) $\lambda = -2$

Then $c_1 = c_2$ and the solution is

$$u(x) = f(x) - 2(c_2 - 3xc_2) = f(x) - 2c_2(1 - 3x)$$

This completes the solution.

Example 7. Find the eigen values and eigen functions of the integral equation

$$u(x) = \lambda \int_0^{2\pi} \sin(x + t)u(t)dt$$

Solution: Eigen values are $\lambda = \pm\frac{1}{\pi}$. For $\lambda = \frac{1}{\pi}$,

Eigen function is $u(x) = A(\sin x + \cos x)$, where $A = \frac{c_1}{\pi}$ and for $\lambda = -\frac{1}{\pi}$,

Eigen function is $u(x) = B(\sin x - \cos x)$, where $B = \frac{c_2}{\pi}$.

13.9 SYMMETRIC KERNEL

The kernel $K(x, \xi)$ of an integral equation is said to be symmetric if

$K(x, \xi) = K(\xi, x)\ for\ all\ x\ and\ \xi.$

13.10 ORTHOGONALITY

Two function $\phi_1(x)$ and $\phi_2(x)$ continuous on an interval (a, b) are said to be orthogonal if

$$\int_a^b \phi_1(x)\,\phi_2(x)dx = 0.$$

Theorem 2. For the Fredholm integral equation $y(x) = \lambda \int_a^b K(x,\xi)y(\xi)d\xi$ with symmetric kernel, prove that:

(i) The eigen function corresponding to two different eigen values are orthogonal over (a, b).

(ii) The eigen values are real.

Proof: (i) Let λ_1 and λ_2 be two different eigen values of given integral equation

$$y(x) = \lambda \int_a^b K(x,\xi)y(\xi)d\xi \tag{1}$$

w.r.t. eigen function $y_1(x)$ and $y_2(x)$. We have to show that

$$\int_a^b y_1(x)\,y_2(x)dx = 0 \tag{2}$$

By definition we have,

$$y_1(x) = \lambda_1 \int_a^b K(x,\xi)\,y_1(\xi)d\xi \tag{3}$$

$$y_2(x) = \lambda_2 \int_a^b K(x,\xi)\,y_2(\xi)d\xi \tag{4}$$

Multiplying (3) by y2(x) and then integrating w.r.t. x over the interval a to b find

$$\int_a^b y_1(x)\,y_2(x)dx = \lambda_1 \int_a^b y_2(x)\left[\int_a^b K(x,\xi)\,y_1(\xi)d\xi\right]dx$$

Interchanging the order of integration

$$\int_a^b y_1(x)\,y_2(x)dx = \lambda_1 \int_a^b y_1(\xi)\left[\int_a^b K(x,\xi)\,y_2(x)dx\right]d\xi$$

$$= \lambda_1 \int_a^b y_1(\xi) \left[\int_a^b K(\xi, x)\, y_2(x) dx \right] d\xi \; [since\; K(x,\xi) = K(\xi, x)]$$

$$= \lambda_1 \int_a^b y_1(\xi) \frac{y_2(\xi)}{\lambda_2} d\xi \qquad [by\ (4)]$$

$$= \frac{\lambda_1}{\lambda_2} \int_a^b y_1(x)\, y_2(x) dx$$

$$\Rightarrow \qquad \left(1 - \frac{\lambda_1}{\lambda_2}\right) \int_a^b y_1(x)\, y_2(x) dx = 0$$

$$\Rightarrow \qquad \int_a^b y_1(x)\, y_2(x) dx = 0 \qquad (\lambda_1 \neq \lambda_2)$$

(ii) If possible, we assume on the contrary that there is an eigen value λ_0 (say) which is not real. So $\lambda_0 = \alpha_0 + i\beta_0,\ \beta_0 \neq 0$ (5)

Where α_0 and β_0 are real.

Let $y_0(x) \neq 0$ be the corresponding eigen function. Then

$$y_0(x) = \lambda_0 \int_a^b K(x,\xi)\, y_0(\xi) d\xi \qquad (6)$$

We claim that the eigen function $y_0(x)$ corresponding to a non real eigen value λ_0 is not real values. If $y_0(x)$ is real valued, then separating the real and imaginary parts in (6), we get

$$y_0(x) = \alpha_0 \int_a^b K(x,\xi)\, y_0(\xi) d\xi \qquad (7)$$

And

$$0 = \beta_0 \int_a^b K(x,\xi)\, y_0(\xi) d\xi \qquad (8)$$

$$\Rightarrow \qquad \int_a^b K(x,\xi)\, y_0(\xi) d\xi = 0 \quad (\beta_0 \neq 0)$$

Hence from (7), we get $y_0(x) = 0$, a contradiction. Thus $y_0(x)$ cannot be a real valued function.

Let us consider

$$y_0(x) = \alpha(x) + i\beta_0(x), \qquad \beta(x) \neq 0 \qquad (9)$$

Changing i to $-i$ in (6), we obtain

$$\overline{y_0(x)} = \overline{\lambda_0} \int_a^b K(x,\xi)\, \overline{y_0(\xi)} d\xi \qquad (10)$$

This shows that $\overline{\lambda_0}$ is an eigen value with corresponding eigen function $\overline{y_0(x)}$. Since λ_0 is non-real by assumption. So λ_0 and $\overline{\lambda_0}$ are two different eigen values. Thus by part (i), we have

$$\int_a^b y_0(x)\, \overline{y_0(\xi)} d\xi = 0$$

$$\Rightarrow \qquad \int_a^b |y_0(x)|^2 dx = 0$$

$$\Rightarrow \qquad \int_a^b |\alpha(x) + i\beta(x)|^2 dx = 0$$

$$\Rightarrow \qquad \int_a^b ([\alpha(x)]^2 + [\beta(x)]^2)\, dx = 0$$

$$\Rightarrow \qquad \alpha(x) = (x) = 0$$

$$\Rightarrow \qquad y_0(x) = 0$$

A contradiction because eigen functions are non-zero. This contradiction shows that our assumption that λ_0 is not real is wrong. Hence λ_0 must be real.

Remark.

1. After finding the resolvent kernel $R(x,\xi:\lambda)$ the solution of given integral equation is given by

$$u(x) = f(x) + \lambda \int_a^b R(x,\xi:\lambda) f(\xi) d\xi$$

2. This method cannot be used when $\lambda = 1$

Examples

Example.8. Using the Fredholm determinant, find the resolvent kernel of

$$K(x,\xi) = 2x - \xi, \qquad 0 \le x \le 1, 0 \le \xi \le 1$$

Solution: Here the kernel is

$$K(x,\xi) = 2x - \xi \qquad (1)$$

The resolvent kernel $R(x,\xi:\lambda)$ is given by

$$R(x,\xi:\lambda) = \frac{D(x,\xi:\lambda)}{D(\lambda)}, \qquad D(\lambda) \neq 0 \qquad (2)$$

Where

$$D(x,\xi:\lambda) = K(x,\xi) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \lambda^n B_n(x,\xi)$$

And

$$D(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \lambda^n c_n \qquad (3)$$

Where $B_n = \int_a^b \int_a^b \dots \dots \int_a^b \begin{vmatrix} K(x,\xi) & K(x,t_1) & \dots & K(x,t_n) \\ K(t,\xi) & K(t_1,t_1) & \dots & K(t_1,t_n) \\ \vdots & \vdots & \vdots & \vdots \\ K(t_n,\xi) & K(t_n,t_1) & \dots & K(t_n,t_n) \end{vmatrix} dt_1 dt_2 \dots dt_n$

and $\int_a^b \int_a^b \dots \dots \int_a^b \begin{vmatrix} K(t_1,t_1) & K(t_1,t_2) & \dots & K(t_1,t_n) \\ K(t_2,t_1) & K(t_2,t_2) & \dots & K(t_2,t_n) \\ \vdots & \vdots & \vdots & \vdots \\ K(t_n,t_1) & K(t_n,t_2) & \dots & K(t_n,t_n) \end{vmatrix} dt_1 dt_2 \dots dt_n$

therefore,

$$B_1(x,\xi) = \int_0^1 \begin{vmatrix} 2x-\xi & 2x-t_1 \\ 2t_1-\xi & 2t_1-t_1 \end{vmatrix} dt_1$$

$$= \int_0^1 (2xt_1 - \xi t_1 - 4xt_1 + 2t_1^2 + 2x\xi - \xi t_1) dt_1$$

$$= \int_0^1 (-2xt_1 - 2\xi t_1 + 2t_1^2 + 2x\xi) dt_1$$

$$B_1(x,\xi) = -x - \xi + \frac{2}{3} + 2x\xi$$

$$B_2(x,\xi) = \int_0^1 \int_0^1 \begin{vmatrix} 2x-\xi & 2x-t_1 & 2x-t_2 \\ 2t_1-\xi & 2t_1-t_1 & 2t_1-t_2 \\ 2t_2-\xi & 2t_2-t_1 & 2t_2-t_2 \end{vmatrix} dt_1 dt_2$$

Which on solving gives, $B_2(x,\xi) = 0$

In general $B_n(x,\xi) = 0$ for all n ≥ 2

Now, $c_1 = \int_0^1 (2t_1 - t_1)\, dt_1 = \frac{1}{2}$

$$c_2 = \int_0^1 \int_0^1 \begin{vmatrix} 2t_1-t_1 & 2t_1-t_2 \\ 2t_2-t_1 & 2t_2-t_2 \end{vmatrix} dt_1 dt_2 = \frac{1}{3}$$

Now, since $B_n = 0$ for all n ≥ 2

⇒ $c_n = 0$ for all n ≥ 3

Thus, from (3) we get

$$D(x,\xi:\lambda) = (2x-\xi) + (-1)\lambda\left(2\xi x - x - \xi + \frac{2}{3}\right)$$

$$= 2x - \xi + \lambda\left(x + \xi - 2x\xi - \frac{2}{3}\right)$$

$$D(\lambda) = 1 + (-1)^1\lambda c_1 + \frac{(-2)^2}{2!}\lambda c_2 = 1 - \frac{\lambda}{2} + \frac{\lambda^2}{6}$$

Hence the resolvent kernel is given by:

$$R(x,\xi:\lambda) = \frac{(2x-\xi) + \lambda\left(x + \xi - 2x\xi - \frac{2}{3}\right)}{1 - \frac{\lambda}{2} + \frac{\lambda^2}{6}}.$$

13.11 SUMMARY

A Fredholm Integral Equation is of the type

$$h(x)u(x) = f(x) + \int_a^b K(x,\xi)u(\xi)d\xi \ \ for\ all\ x \in [a,b]$$

(i) If h(x) = 0, the above equation reduces to:

$$-f(x) = \int_a^b K(x,\xi)u(\xi)d\xi$$

This equation is called Fredholm integral equation of first kind.

(ii) If h(x) = 1, the above equation becomes:

$$u(x) = f(x) + \int_a^b K(x,\xi)u(\xi)d\xi$$

This equation is called Fredholm integral equation of second kind.

(iii) If h(x) = 1, f(x) = 0 the above equation becomes:

$$u(x) = \int_a^b K(x,\xi)u(\xi)d\xi$$

This equation is called Homogeneous Fredholm integral equation of second kind.

If the Fredholm integral equation

$$u(x) = f(x) + \lambda\int_a^b K(x,\xi)u(\xi)d\xi \qquad (1)$$

is such that: (a) $K(x,\xi)$ is a non-zero real continuous function in the rectangle $R = I \times I$, where I = [a, b]

and $|K(x, \xi)| < M$ in R.

(b) $f(x)$ is an non-zero real valued continuous function on I.

(c) λ is a constant satisfying the inequality, $|\lambda| < \frac{1}{M(b-a)}$.

Then equation (1) has one and only one continuous solution in the interval I and this solution is given by the absolutely and uniformly convergent series $u(x) = f(x) + \lambda k[f(x)] + \lambda^2 k^2[f(x)] + \cdots to\ \infty$.

A kernel $K(x, \xi)$ of an integral equation is called separable if it can be expressed in the form

$$K(x, \xi) = \sum_{i=1}^{n} a_i(x) b_i(\xi) = a_1(x) b_1(\xi) + a_2(x) b_2(\xi) + \cdots . + a_n(x) b_n(\xi)$$

The values of λ for which $\Delta(\lambda) = 0$ are known as eigen values (or characteristic number) of Fredholm integral equation. The non-trivial solution corresponding to Eigen values are known as Eigen functions (or characteristic functions).

The kernel $K(x, \xi)$ of an integral equation is said to be symmetric if

$K(x, \xi) = K(\xi, x)\ for\ all\ x\ and\ \xi.$

Two function $\phi_1(x)$ and $\phi_2(x)$ continuous on an interval (a, b) are said to be orthogonal if

$$\int_a^b \phi_1(x)\, \phi_2(x) dx = 0.$$

13.12 TERMINAL QUESTIONS

Q.1 Explain the solution procedure of a Fredholm integral equation.

Q.2 Reduce the following boundary value problems to Fredholm integral equation.

a. $y'' - \lambda\, y = 0, a < x < b, y(a) = 0 = y(b)$

b. $y'' + \lambda y = 0, y(0) = 0, y(1) = 0$

c. $y'' + \lambda y = x;\ y(0) = 0, y(1) = 0$

d. $y'' + \lambda y = 2x + 1, y(0) = y'(1), y'(0) = y(1)$

e. $y'' + \lambda y = e^x\ y(0) = y'(0), y(1) = y'(1).$

2. Determine the Resolvent Kernel associated with $K(x, \xi) = x\xi$ in the interval (0, 1) in the form of a power series in λ

Answer. $R(x, \xi : \lambda) = \frac{3}{3-\lambda} x\xi,\ |\lambda| < 3$

3. Solve the following integral equations by finding the resolvent kernel:

a. $u(x) = f(x) + \lambda \int_0^1 e^{(x-\xi)} u(\xi) d\xi$

b. $u(x) = 1 + \lambda \int\limits_0^1 xe^{(\xi)} u(\xi) d\xi$

c. $u(x) = x + \lambda \int\limits_0^1 xe^{(\xi)} u(\xi) d\xi$

d. $u(x) = x + \lambda \int\limits_0^1 x\xi\, u(\xi) d\xi$

4. Solve the integral equations by the method of degenerate kernel:

a. $u(x) = x + \lambda \int\limits_0^1 (xt^2 + x^2 t) u(t) dt$

b. $u(x) = e^x + \lambda \int\limits_0^1 2e^x e^t u(t) dt$

5. Using Fredholm determinant, find the resolvent kernel of $K(x, \xi) = 1 + 3x\xi$.

6. Solve the following integral equations by finding the resolvent kernel:

$$u(x) = f(x) + \lambda \int\limits_0^1 e^{a(x^2 - \xi^2)} u(\xi) d\xi$$

7. Solve the integral equations by the method of degenerate kernel:

$$u(x) = x + \lambda \int\limits_0^1 (1 + x + t) u(t)\, dt$$

Answer

Q.2 (a) $y(x) = \lambda \int_a^b K(x,\xi) y(\xi) d\xi$where $K(x,\xi) = \begin{cases} \frac{(x-b)(\xi-a)}{b-a} & if \quad a \le \xi \le x \\ \frac{(x-a)(\xi-b)}{b-a} & if \quad x \le \xi \le b \end{cases}$

(b) $y(x) = \lambda \int_0^l K(x,\xi) y(\xi) d\xi$where $K(x,\xi) = \begin{cases} \frac{\xi(l-x)}{l} & if \quad 0 \le \xi \le x \\ \frac{x(l-\xi)}{l} & if \quad x \le \xi \le l \end{cases}$

(c) $y(x) = \frac{1}{6}(x^3 - 3x) + \lambda \int_0^l K(x,\xi) y(\xi) d\xi$where $K(x,\xi) = \begin{cases} x & , \quad x > \xi \\ \xi & , \quad x < \xi \end{cases}$

(d) $y(x) = f(x) + \lambda \int_0^1 K(x,\xi) y(\xi) d\xi$ where $f(x) = \frac{1}{6}[2x^3 + 3x^2 - 17x - 5]$ and

$K(x,\xi) = \begin{cases} 1 + x(1-\xi) & \xi < x \\ (1-\xi) + (2-\xi)x & \xi > x \end{cases}$

(e) $y(x) = e^x + \lambda \int_0^1 K(x,\xi) y(\xi) d\xi$ where $K(x,\xi) = \begin{bmatrix} -x(1+\xi) & ;\ x > \xi \\ -(1+xs)\xi & ;\ x < \xi \end{bmatrix}$

Q.3 (a) $u(x) = f(x) + \frac{\lambda}{1-\lambda} \int_0^1 e^{(x-\xi)} f(\xi) d\xi$

(b) $u(x) = 1 + \frac{\lambda x}{1-\lambda}(e-1)$

(c) $u(x) = x + \frac{\lambda x}{1-x}$

(d) $u(x) = x + \frac{\lambda x}{1-\lambda}$

Q.4 (a) $u(x) = \frac{(240-60\lambda)x + 80\lambda x^2}{240-120\lambda-\lambda^2}$

(b) $u(x) = \frac{e^x}{1-\lambda(e^2-1)}$

Q.5 $R(x,\xi:\lambda) = \frac{(1+3x\xi) - \lambda\left(1-3\xi x - \frac{3\xi}{2} - \frac{3x}{2}\right)}{1-2\lambda+\frac{\lambda^2}{4}}$

Q.6 $u(x) = f(x) + \frac{\lambda}{1-\lambda} \int_0^1 e^{a(x^2-\xi^2)} f(\xi) d\xi$

Q.7 $u(x) = x + \frac{\lambda}{12-24\lambda-\lambda^2}[10 + (6+\lambda)x]$

UNIT-14 VOLTERRA INTEGRAL EQUATIONS

Structure

14.1 Introduction
14.2 Objectives
14.3 Integral Equation
14.4 Volterra Integral Equation
14.5 Homogeneous Integral Equation
14.6 Solution of Volterra Integral Equation
14.7 Laplace transform method to solve an integral equation
14.8 Solution of Volterra Integral Equation of first kind
14.9 Method of Iterated kernel/Resolvent kernel to solve the Volterra integral equation
14.10 Summary
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14.1 INTRODUCTION

Volterra integral equations, named after mathematician Vito Volterra, describe dynamic processes and systems with memory effects. Unlike Fredholm equations, the unknown function only appears inside the integral sign. These equations are essential for modeling systems in physics, biology, economics, and engineering, where present states depend on past inputs. Particularly valuable for nonlinear systems analysis, they offer insights into complex dynamics like viscoelastic materials and biochemical reactions. Their applications extend to control theory, mathematical biology, and ecology, facilitating the study of population dynamics and predator-prey interactions. Additionally, they provide a framework for understanding fractional calculus, relevant in physics, engineering, and finance.

Hence the Volterra integral equations serve as indispensable tools for comprehending and analyzing real-world phenomena across diverse scientific and engineering domains. Volterra integral equations are closely related to fractional calculus, which deals with derivatives and integrals of non-integer order. Volterra integral equations are particularly useful for studying nonlinear systems, where the output does not vary linearly with the input.

14.2 OBJECTIVES

After reading this unit the learner should be able to understand about:

- Initial value problem reduced to Volterra integral equations.
- Method of successive substitution to solve Volterra integral equation of second kind.

- Method of successive approximation to solve Volterra integral equation of second kind.
- Resolved kernel as a series.
- Laplace transform method for a difference kernel

14.3 INTEGRAL EQUATION

An integral equation is one in which function to be determined appears under the integral sign. The most general form of a linear integral equation is

$$h(x)u(x) = f(x) + \int_{a}^{b(x)} K(x,\xi)\, \mathrm{u}(\xi)\mathrm{d}\xi \quad \text{for all } x \in [a,b]$$

In which u(x) is the function to be determined and $K(x,\xi)$ is called the Kernel of integral equation.

14.4 VOLTERRA INTEGRAL EQUATION

A Volterra integral equation is of the type:

$$h(x)u(x) = f(x) + \int_{a}^{x} K(x,\xi)\, \mathrm{u}(\xi)\mathrm{d}\xi \quad \text{for all } x \in [a,b]$$

that is, in Volterra equation h(x) = x

(i) If h(x) = 0 the above equation reduces to

$$-f(x) = \int_{a}^{x} K(x,\xi)\, \mathrm{u}(\xi)\mathrm{d}\xi$$

This equation is called Volterra integral equation of first kind.

(ii) If h(x) = 1 the above equation reduces to

$$u(x) = f(x) + \int_{a}^{x} K(x,\xi)\, \mathrm{u}(\xi)\mathrm{d}\xi$$

This equation is called Volterra integral equation of second kind.

14.5 HOMOGENEOUS INTEGRAL EQUATION

If f(x) = 0 for all $x \in [a,b]$, then the reduced equation

$$h(x)u(x) = \int_{a}^{b(x)} K(x,\xi)\, \mathrm{u}(\xi)\mathrm{d}\xi$$

is called homogeneous integral equation. Otherwise, it is called non-homogeneous integral equation.

Leibnitz Rule: The Leibnitz rule for differentiation under integral sign:

$$\frac{d}{dx}\left[\int_{\alpha(x)}^{\beta(x)} F(x,\xi)d\xi\right] = \int_{\alpha(x)}^{\beta(x)} \frac{\partial F}{\partial x}d\xi + F\big(x,\beta(x)\big)\frac{d\beta(x)}{dx} - F(x,\alpha(x))\frac{d\alpha(x)}{dx}$$

In particular, we have

$$\frac{d}{dx}\left[\int_a^x K(x,\xi)\,\mathrm{u}(\xi)\mathrm{d}\xi\right] = \int_a^x \frac{\partial K}{\partial x}\mathrm{u}(\xi)\mathrm{d}\xi + \mathrm{K}(\mathrm{x},\mathrm{x})\mathrm{u}(\mathrm{x})$$

Lemma: If n is a positive integer, then

$$\int_a^x\int_a^{x_1} \dots\int_a^{x_{n-2}}\int_a^{x_{n-1}} F(x_n)\,dx_n dx_{n-1}\dots dx_1 = \frac{1}{1-n!}\int_a^x (x-\xi)^{n-1}f(\xi)d\xi$$

Proof: If $I_n(x) = \int_a^x (x-\xi)^{n-1}f(\xi)d\xi$, then $I_n(a) = 0$ and for $n = 1, I_1(x) = \int_a^x f(\xi)d\xi$

Using Leibnitz rule, we get $\frac{dI_1}{dx} = f(x)$.

Now, differentiating $I_n(x)$ w.r.t. x and using Leibnitz rule, we get

$$\frac{dI_1}{dx} = \frac{d}{dx} = \int_a^x \frac{\partial}{\partial x}[(x-\xi)^{n-1}]\,f(\xi)d\xi = (n-1)\int_a^x (x-\xi)^{n-2}f(\xi)d\xi$$

or

$$\frac{dI_n(x)}{dx} = (n-1)I_{n-1}(x)\ for\ n > 1$$

Taking successive derivatives, we get

$$\frac{d^{n-1}}{dx^{n-1}}I_n(x) = (n-1)(n-2)\dots.2.1\ I_1(x)$$

Again, differentiating,

$$\frac{d^n}{dx^n}I_n(x) = n-1!\frac{d}{dx}I_{1(x)} = n-1!\,f(x) \tag{1}$$

We observe that,

$$I_n^{(m)}(a) = 0\ for\ m = 0, 1, 2, \dots\dots, n-1 \tag{2}$$

Integrating (1) over the interval [a, x] and using (2) for m = n – 1, we obtain

$$I_n^{(n-1)}(x) = (n-1)!\int_a^x f(x_1)dx_1$$

Again integrating it and using (2) for m = n – 2, we get

$$\frac{d^{n-2}}{dx^{n-2}} I_n(x) = I_n^{(n-2)}(x) = n-1! \int_a^x \int_a^{x_1} f(x_2) dx_2 \, dx_1$$

Continuing like this, n times, we obtain

$$I_n(x) = (n-1)! \int_a^x \int_a^{x_1} \dots \dots \int_a^{x_{n-1}} f(x_n) dx_n dx_{n-1} \dots \dots . . dx_1$$

Which provides the required result

Examples

Example.1. Transform the initial value equation $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0; y(0) = 1, \; y'(0) = 0$ to Volterra integral equation.

Solution: Consider, $\frac{d^2y}{dx^2} = \phi(x)$ ….(1)

Then $\frac{dy}{dx} = \int_0^x \phi(\xi)d\xi + C_1$

Using the condition $y'(0) = 0$, we get, $C_1 = 0$

$$\frac{dy}{dx} = \int_0^x \phi(\xi)d\xi \qquad \dots (2)$$

Again integrating from 0 to x and using the above lemma, we get

$$y = \int_0^x (x-\xi)\phi(\xi)d\xi + C_2$$

Using the condition $y(0) = 1$, we get $C_2 = 1$ so

$$y = \int_0^x (x-\xi)\phi(\xi)d\xi + 1 \qquad \dots (3)$$

From the relations (1), (2) and (3) the given differential equation reduces to:

$$\phi(x) + x\int_0^x \phi(\xi)d\xi + \int_0^x (x-\xi)\phi(\xi)d\xi + 1 = 0$$

or

$$\phi(x) = -1 - \int_0^x (2x-\xi)^{n-1}\phi(\xi)d\xi$$

Which represents a Volterra integral equation of second kind.

14.6 SOLUTION OF VOLTERRA INTEGRAL EQUATION

Weierstrass M-Test- Suppose $\sum f_n(z)$ is an infinite series of single valued functions defined in a bounded closed domain D. Let $\sum M_n$ be a series of positive constant (independent of z) such that

(i) $|f_n(z)| \le M_n$ for all n and for all $z \in D$

(ii) $\sum M_n$ is convergent.

Then the series $\sum f_n$ is uniformly and absolutely convergent in D.

Theorem 1. Let $(x) = f(x) + \lambda \int_a^x K(x,\xi)u(\xi)d\xi$ be a non-homogeneous Volterra integral equation of second kind with constant a and λ. f(x) is a non-zero real values continuous function in the interval $I = [a,b], K(x,\xi)$ is a non-zero real values continuous function defined in the rectangle $R = I \times I = \{(x,\xi): a \le x, \xi \le b\}$ and $|K(x,\xi)| \le M \ in \ R$.

Then the given equation has one and only one continuous solution u(x) in I and this solution is given by the absolutely and uniformly convergent series.

$$u(x) = f(x) + \lambda \int_a^x K(x,t)f(t)\,dt + \lambda^2 \int_a^x \int_a^t K(x,t)\,K(t,t_1)f(t_1)dt_1 dt + \cdots \ldots$$

Proof: This theorem can be proved by applying either of the following two methods:

(i) Method of successive substitution.

(ii) Method of successive approximation.

Let us apply these methods one by one

(i) **Method of successive substitution:** The given integral equation is

$$u(x) = f(x) + \lambda \int_a^x K(x,t)u(t)\,dt \qquad (1)$$

Substituting value of u(t) from (1) into itself, we get

$$u(x) = f(x) + \lambda \int_a^x K(x,t)\left[f(t) + \lambda \int_a^t K(t,t_1)t(t_1)dt_1\right]dt$$

$$= f(x)\ \lambda \int_a^x K(x,t)f(t)\,dt + \lambda^2 \int_a^x \int_a^t K(t,t_1)K(t_1)dt_1\,dt \qquad (2)$$

Again substituting the value of $u(t_1)$ from (1) into (2), we get

$$u(x) = f(x) + \ \lambda \int_a^x K(x,t)f(t)\,dt + \lambda^2 \int_a^x \int_a^t K(x,t)K(t,t_1)f(t_1)dt_1\,dt$$

$$+\lambda^3 \int_a^x \int_a^t \int_a^{t_1} K(x,t)K(t,t_1)K(t_1,t_2)u(t_2)dt_2 dt_1 dt$$

Proceeding in the same way, we get after n steps

$u(x)$

$$= f(x) + \lambda \int_a^x K(x,t) f(t)\, dt$$

$$+ \cdots .. + \lambda^n \int_a^x \int_a^t \dots \dots \int_a^{t_{n-2}} K(x,t)K(t,t_1) \dots \dots K(t_{n-2},t_{n-1}) f(t_{n-1}) dt_{n-1} dt_{n-2} \dots . dt_1 dt + R_{n+1}(x)$$

where

$$R_{n+1}(x) = \lambda^{n+1} \int_a^x \int_a^t \dots \dots \int_a^{t_{n-1}} K(x,t)K(t,t_1) \dots \dots K(t_{n-1},t_n) u(t_n) dt_{n-1} dt_n \dots . dt_1 dt$$

Consider the infinite series

$$f(x) + \lambda \int_a^x K(x,t) f(t)\, dt + \lambda^2 \int_a^x \int_a^t K(x,t)K(t,t_1) f(t_1) dt_1\, dt \qquad (5)$$

Neglecting the first term, let $v_n(x)$ denotes the nth term of infinite series in (5). Since f(x) is continuous over I, so it is bounded.

Let $|f(x)| \le N$ in I. also, it is given that $|K(x,t)| \le M \; in \; R$. Therefore,

$$|v_n(x)| \le |\lambda|^{\mathrm{n}} \int_a^x \int_a^t \dots \dots . \int_a^{t_{n-2}} M^n N \, dt_{n-1} dt_1 dt$$

Thus we have

$$|v_n(x)| \le |\lambda|^{\mathrm{n}} M^n N \frac{(x-a)^n}{n!} \le |\lambda|^{\mathrm{n}} M^n N \frac{(b-a)^n}{n!} \qquad (6)$$

The series whose nth term is $|\lambda|^{\mathrm{n}} M^n N \frac{(b-a)^n}{n!}$ is a series of positive terms and is convergent by ratio test for all values of a, b, $|\lambda|$, M and N.

Thus, by Weierstrass M-test, the series $\sum v_n(x)$ is absolutely and uniformly convergent in I.

If u(x) given by (2) is continuous in I, then is bounded in I, that is,

u(x) ≤ U for all x in I. then

$$|R_{n+1}(x)| \le |\lambda|^{\mathrm{n+1}} M^{n+1} u \frac{(x-a)^{n+1}}{(n+1)!} \le |\lambda|^{\mathrm{n+1}} M^{n+1} u \frac{(b-a)^{n+1}}{(n+1)!} \to 0 \; as \; n \to 0$$

$$\Rightarrow \qquad \lim_{n \to \infty} R_{n+1}(x) = 0 \qquad (8)$$

From equations (3), (4) and (8), we obtain

$$u(x) = f(x) + \lambda \int_a^x K(x,t) f(t)\, dt + \lambda^2 \int_a^x \int_a^t K(x,t)K(t,t_1) f(t_1) dt_1\, dt + \cdots \dots .. to\ \infty$$

which is the required series.

Now, we verify that this series is actually a solution of the given Volterra integral (1). Substituting the series for u(x) in the R.H.S. of the given equation, we get

$$R.H.S. = f(x) + \lambda \int_a^x K(x,\xi)\left[f(\xi) + \lambda \int_a^x K(\xi,t)f(t)\,dt + \lambda^2 \int_a^x \int_a^t K(\xi,t)K(t,t_1)f(t_1)dt_1\,dt + \cdots to\ \infty\right]$$

$$f(x) + \lambda \int_a^x K(x,\xi)f(\xi)\,d\xi + \lambda^2 \int_a^x \int_a^\xi K(x,\xi)K(\xi,t)f(t)dt\,d\xi + \cdots to\ \infty = u(x) = L.H.S.$$

(ii) Method of Successive Approximation: In this method, we select any real values function, say $u_0(x)$, continuous on I = [a, b] as the zeroth approximation. Substituting this zeroth approximation in the given Volterra integral equation.

$$u(x) = f(x) + \lambda \int_a^x K(x,t)u(t)\,dt \qquad (1)$$

We obtain the first approximation, say $u_1(x)$, given by

$$u_1(x) = f(x) + \lambda \int_a^x K(x,t)u_0(t)\,dt \qquad (2)$$

The value of $u_1(x)$, is given substituted for u(x) in (1) to obtain the second approximation $u_2(x)$ where

$$u_2(x) = f(x) + \lambda \int_a^x K(x,t)u_1(t)\,dt \qquad (3)$$

This process is continued to obtain nth approximation

$$u_n(x) = f(x) + \lambda \int_a^x K(x,t)u_{n-1}(t)\,dt\ for\ n = 1, 2, 3, \ldots \qquad (4)$$

This relation is known as recurrence relation.

Now, we can write

$$u_n(x) = f(x) + \lambda \int_a^x K(x,t)\left[f(t) + \lambda \int_a^t K(t,t_1)u_{n-2}(t_1)dt_1\right]dt$$

$$= f(x) + \lambda \int_a^x K(x,t)\,f(t)dt + \lambda^2 \int_a^x \int_a^t K(x,t)K(t,t_1)\left[f(t_1) + \lambda \int_a^{t_1} K(t_1,t_2)u_{n-3}\,(t_2)dt_2\right]dt_1 dt$$

or

$$u_n(x) = f(x) + \lambda \int_a^x K(x,t)\, f(t)dt + \lambda^2 \int_a^x \int_a^t K(x,t)K(t,t_1)\, f(t_1)dt_1 dt$$
$$+ \lambda^3 \int_a^x \int_a^t \int_a^{t_1} K(x,t)\, K(t,t_1)K(t_1,t_2)dt_2 dt_1 dt \qquad (5)$$

Continuing in this fashion, we get

$$u_n(x) = f(x) + \lambda \int_a^x K(x,t)\, f(t)dt$$
$$+ \lambda^{n-1} \int_a^x \int_a^t \ldots.. \int_a^{t_{n-3}} K(x,t)K(t,t_1) \ldots \ldots K(t_{n-3},t_{n-2})\, f(t_{n-2})dt_{n-2} \ldots dt_1 dt$$
$$+ R_n(x)F \qquad (6)$$

Where

$$R_n(x) = \lambda^n \int_a^x \int_a^t \ldots.. \int_a^{t_{n-2}} K(x,t)K(t,t_1) \ldots \ldots K(t_{n-2},t_{n-1})\, u_0(t_{n-1})dt_{n-1} \ldots dt_1 dt \qquad (7)$$

Since $u_0(x)$ is continuous on I so it is bounded. Let

$$|u_0(x)| \leq u \; in \; I \qquad (8)$$

Thus

$$|R_n(x)| \leq |\lambda|^n M^n u \frac{(x-a)^n}{n!} \leq |\lambda|^n M^n u \frac{(b-a)^n}{n!} \to 0 \; as \; n \to \infty$$

So

$$\lim_{n\to\infty} R_n(x) = 0 \qquad (9)$$

Thus, as n increases, the sequence $< u_n(x) >$ approaches to a limit. We denote this limit by u(x) that is,

$$u(x) = \lim_{n\to\infty} u_n(x)$$

So,

$$u(x) = f(x) + \lambda \int_a^x K(x,t)\, f(t)dt + \lambda^2 \int_a^x \int_a^t K(x,t)K(t,t_1)\, f(t_1)dt_1 dt + \cdots ..to\infty \qquad (10)$$

As in the method of successive substitution, we can prove that the series (10) is absolutely and uniformly convergent and hence the series on R.H.S. of (10) is the desid solution of given Volterra integral equation.

Uniqueness- Let, if possible, the given Volterra integral equation has another solution v(x). we make, by our choice, the zeroth approximation $u_0(x) = v(x)$, then all approximations $u_1(x) \ldots \ldots, u_n(x)$ will be identical with v(x) that is,

$$u_n(x) = v(x) for \; all \; n$$

$$\lim_{n\to\infty} u_n(x) = v(x)$$

$$u(x) = v(x)$$

This proves uniqueness of solution, with this, the proof of the theorem is completed.

Examples

Example.2. Using the method of successive approximation solve the integral equation,

$$u(x) = x - \int_0^x (x-\xi)u(\xi)d\xi \qquad (1)$$

Solution: Let the zeroth approximation be $u_0(x) = 0$

Then the first approximation $u_1(x)$ is given by:

$$u_1(x) = x - \int_0^x 0.\,d\xi = x \qquad (2)$$

Thus we have

$$u_2(x) = x - \int_0^x (x,\xi)u_1(\xi)d\xi = x - \int_0^x (x-\xi)\xi d\xi$$

$$= x - \left[\frac{x\xi^2}{2}\right]_0^x + \left[\frac{\xi^3}{3}\right]_0^x$$

$$= x - \frac{x^3}{2} + \frac{x^3}{3}$$

$$= x - \frac{x^3}{6}$$

$$= x - \frac{x^3}{3!} \qquad (3)$$

Now we have

$$u_3(x) = x - \int_0^x (x-\xi)u_2(\xi)d\xi$$

$$= x - \int_0^x (x-\xi)\left(\xi - \frac{\xi^3}{6}\right)d\xi$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} \qquad (4)$$

From equations (2), (3) and (4), we conclude that the nth approximation, $u_n(x)$ will be

$$u_n(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \ldots . + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} \qquad (5)$$

Which is obviously the nth partial sum of Maclaurin's series of sinx. Hence by the method of successive approximation, solution of given integral equation is

$$u(x) = \lim_{n\to\infty} u_n(x) = \sin x$$

Hence the solution.

14.7 LAPLACE TRANSFORM METHOD TO SOLVE AN INTEGRAL EQUATION

The Laplace transform of a function f(x) defined on interval (0, ∞) is given by

$$L[f(x)] = f(s) = \int_0^\infty f(x)\, e^{-5x} dx \qquad \ldots.(1)$$

Here s is called Laplace variable or Laplace parameter. Also . $f(x) = L^{-1}[f(s)]$ is called inverse Laplace transform.

Some important results:

(1) $L(\sin x) = \frac{1}{s^2+1}$ (2) $L[\cos x] = \frac{s}{s^2+1}$ (3) $L[e^{ax}] = \frac{1}{s-a}$

(4) $L[x^n] = \frac{n!}{s^n+1}, n \geq 0$ (5) $L[f'(x)] = sf(s) - f(0)$ (6) $L[1] = \frac{1}{s}$

Convolution: The convolution of two functions $f_1(x)$ and $f_2(x)$ is denoted by $(f_1 * f_2)\ (x)$ and is defined as

$$(f_1 * f_2)\ (x) = \int_0^x f_1(x-\xi) f_2(\xi) d\xi$$

Convolution theorem. (without proof): Laplace transform of convolution of two functions is equal to the product their respective Laplace transforms, that is

$$[(f_1 * f_2)\ (x)] = L[f_1(x)].L[f_2(x)]$$

Difference integral or convolution integral: Consider the integral equation

$$u(x) = f(x) + \int_a^{b(x)} K(x,\xi) u(\xi) d\xi$$

Let the kernel $K(x,\xi)$ be a function of $x - \xi$, say $g(x-\xi)$ then the integral equation becomes

$$u(x) = f(x) + \int_a^{b(x)} g(x-\xi) u(\xi) d\xi$$

In this case, the kernel $K(x,\xi) = g(x-\xi)$ is called difference kernel and the corresponding integral is called difference integral or convolution integral.

Working Procedure: Consider the integral equation

$$u(x) = f(x) + \lambda \int_a^x K(x,\xi)u(\xi)d\xi$$

Where $K(x,\xi)$ is difference kernel of the type $g(x-\xi)$ then,

$$u(x) = f(x) + \lambda \int_a^x g(x-\xi)u(\xi)d\xi$$

$$u(x) = f(x) + \lambda[g(x) * u(x)]$$

Applying Laplace transform on both sides, we get

$$U(s) = F(s) + \lambda G(s)U(s)$$

Where U(s), F(s) and G(s) represent the Laplace Transform of u(x), f(x) and g(x) respectively. Then

$$U(s) = \frac{F(s)}{1-\lambda G(s)}$$

Applying inverse Laplace Transform

$$u(x) = L^{-1}\left[\frac{F(s)}{1-\lambda G(s)}\right]$$

Note. Method of Laplace Transform is applicable to those integral equations only where the kernel is difference Kernel.

Examples

Example.3. Use the method of Laplace Transform to solve the integral equation.

$$u(x) = x - \int_0^x (x-\xi)u(\xi)d\xi \qquad (1)$$

Solution: Here $K(x,\xi) = x - \xi = g(x-\xi) \Rightarrow g(x) = x$

Thus (1) can be written as u(x) = x – g(x) * u(x)

Applying Laplace Transform on both sides

$$U(s) = L[x] - L[x]U(s)$$

$$= \frac{1}{s^2} - \frac{1}{s^2}U(s)$$

$$\Rightarrow \; U(s) = \frac{\frac{1}{s^2}}{1+\frac{1}{s^2}} = \frac{1}{s^2+1}$$

$$So, \qquad u(x) = L^{-1}\left[\frac{1}{s^2+1}\right] = \sin x.$$

14.8 SOLUTION OF VOLTERRA INTEGRAL EQUATION OF FIRST KIND

Consider the non-homogeneous Volterra integral equation of first kind

$$u(x) = \lambda \int_0^x K(x,\xi)u(\xi)d\xi \; ...(1)$$

Where the kernel $K(x,\xi)$ is the difference Kernel of the type

$$K(x,\xi) = g(x-\xi)$$

Then (1) can be written as

$$f(x) = \lambda g(x) * u(x)$$

Applying Laplace Transform on both sides:

$$F(s) = \lambda\, G(s)U(s)$$

$$U(s) = \frac{1}{\lambda}\frac{F(s)}{G(s)}$$

Applying inverse Laplace Transform on both sides:

$$u(x) = \frac{1}{\lambda}L^{-1}\left[\frac{F(s)}{G(s)}\right]$$

Examples

Example.4. Solve the integral equation $\sin x = \lambda \int_0^x e^{x-\xi}u(\xi)d\xi$ (1)

Solution: Here $K(x,\xi) = e^{x-\xi} = g(x-\xi)$

$$\Rightarrow \qquad g(x) = e^x$$

Thus (1) can be written as

$$\sin x = \lambda\, g(x) * u(x)$$

Applying Laplace Transform on both sides

$$L[\sin x] = \lambda\, L[e^x]L\,[u(x)]$$

$$\Rightarrow \qquad \frac{1}{s^2+1} = \frac{\lambda}{s-1}U(s)$$

$$\Rightarrow \qquad U(s) = \frac{1}{\lambda}\frac{s-1}{s^2+1} = \frac{1}{\lambda}\left[\frac{s}{s^2+1} - \frac{1}{s^2+1}\right]$$

$$So, \qquad u(x) = \frac{1}{\lambda}\, L^{-1}\left[\frac{s}{s^2+1} - \frac{1}{s^2+1}\right]$$

$$u(x) = \frac{1}{\lambda}\,(\cos x - \sin x)$$

Theorem 2. Prove that the Volterra integral equation of first kind

$f(x) = \lambda \int_0^x K(x,\xi)u(\xi)d\xi$ can be transformed to a Volterra integral equation of second kind, provided that $K(x, x) \neq 0$.

Proof: The given equation is

$$f(x) = \lambda \int_0^x K(x,\xi)u(\xi)d\xi \qquad (1)$$

Differentiating (1) w.r.t. x and using Leibnitz rule, we obtain

$$\frac{df}{dx} = \lambda \int_0^x \frac{\partial K}{\partial x} u(\xi)d\xi + \lambda K(x,x)u(x).1$$

$$-\lambda K(x,x)u(x) = \lambda \int_0^x \frac{\partial K}{\partial x} u(\xi)d\xi - \frac{df}{dx}$$

$$u(x) = \frac{1}{\lambda K(x,x)}.\frac{df}{dx} + \int_0^x -\frac{1}{K(x,x)}\frac{\partial K}{\partial x} u(\xi)d\xi$$

$$u(x) = g(x) + \int_0^x H(x,\xi)u(\xi)d\xi$$

Where $g(x) = \frac{1}{\lambda K(x,x)}.\frac{df}{dx}$ and $H(x,\xi) = -\frac{1}{K(x,x)}\frac{\partial K}{\partial x}$. Here represents the desired Volterra integral equation of second kind.

Example 5. Reduce the integral equation $\sin x = \lambda \int_0^x e^{x-\xi} u(\xi)d\xi$ to the second kind and hence solve it.

Solution: The given equation is

$$\sin x = \lambda \int_0^x e^{x-\xi} u(\xi)d\xi \ (1)$$

Differentiating equation (1) with respect to x, we get

$$\cos x = \lambda \int_0^x e^{x-\xi} u(\xi)d\xi + \lambda e^{x-x}u(x).1$$

$$\Rightarrow \cos x = \lambda \int_0^x e^{x-\xi} u(\xi)d\xi + \lambda u(x)$$

$$\Rightarrow u(x) = \frac{1}{\lambda}\cos x - \int_0^x e^{x-\xi} u(\xi)d\xi \ (2)$$

Which is Volterra integral equation of second kind and can be simply solved by the method of Laplace transform.

14.9 METHOD OF ITERATED KERNEL/RESOLVENT KERNEL TO SOLVE THE VOLTERRA INTEGRAL EQUATION

Let us Consider the volterra integral equation

$$u(x) = f(x) = \lambda \int_{a}^{x} K(x,\xi)u(\xi)d\xi \quad (1)$$

We take

$$K_1(x,\xi) = K(x,\xi) \quad (2)$$

and

$$K_{n+1}(x,\xi) = \int_{\xi}^{x} K(x,t)K_n(t,\xi)dt; n = 1, 2, 3, \dots \dots \dots. \quad (3)$$

From here, we get a sequence of new kernels and these kernels are called iterated kernels.

We know that (1) has one and only one series solution given by

$$u(x) = f(x) + \lambda \int_{a}^{x} K(x,t)\,f(t)dt + \lambda^2 \int_{a}^{x}\int_{a}^{t} K(x,t)K(t,t_1)\,f(t_1)dt_1dt + \cdots .. to\infty \quad (4)$$

We write this series solution in the form

$$u(x) = u_0(x) + \lambda u_1(x) + \lambda^2 u_2(x) + \cdots \dots . to\infty \quad (5)$$

Then comparing (4) and (5), we have

$$u_0(x) = f(x)$$

$$u_1(x) = \int_{a}^{x} K(x,t)\,f(t)dt = \int_{a}^{x} K_1(x,t)\,f(t)dt$$

$$and \qquad u_2(x) = \int_{a}^{x}\int_{a}^{t} K(x,t)K(t,t_1)\,f(t_1)dt_1dt$$

By interchanging the order of integration, we have

$$u_2(x) = \int_{a}^{x} f(t_1)\left[\int_{t_1}^{x} K(x,t)K_1(t,t_1)dt\right]dt_1$$

$$= \int_{a}^{x} f(t_1)\,K_2(x,t_1)dt_1 = \int_{a}^{x} f(t)\,K_2(x,t)dt$$

similarly $$u_n(x) = \int_{a}^{x} f(t)\,K_n(x,t)dt$$

Thus (5) becomes

$$u(x) = f(x) + \lambda \int_a^x K_1(x,t) f(t) dt + \lambda^2 \int_a^x K_2(x,t) f(t) dt + \cdots \ldots to\ \infty$$

$$u(x) = f(x) + \lambda \int_a^x [K_1(x,t) + \lambda K_2(x,t) + \lambda^2 K_3(x,t) + \cdots \ldots \ldots \infty]\ f(t) dt$$

$$= f(x) + \lambda \int_A^X R(x,t:\lambda) f(t) dt \qquad \ldots.(6)$$

Where $R(x,t:\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} K_n(x,t)$

Thus (6) is the solution of given integral (1).

Neumann Series

The series $K_1 + \lambda K_2 + \lambda^2 K_3 + \cdots \ldots \ldots .\ to\ \infty$ is known as the Neumann series.

Resolvent Kernel

The sum of Neumann Series $R(x,t:\lambda)$ is known as the Resolvent Kernel.

Examples

Example.6. With the aid of Resolvent Kernel find the solution of the integral equation.

$$\phi(x) = x + \int_0^x (\xi - x)\phi(\xi) d\xi$$

Solution: Here $\quad K_1(x,\xi) = K(x,\xi) = \xi - x \quad$ …(1)

and $\quad K_{n+1}(x,\xi) = \int_0^x K(x,t) K_n(t,\xi) dt \quad$ …. (2)

Putting n = 1, 2, 3, ….in the equation (2), we have,

$$K_2(x,\xi) = \int_\xi^x K(x,t) K_1(t,\xi) dt = \int_\xi^x (t-x)(\xi - t) dt = -\frac{1}{3!}(\xi - x)^3$$

And

$$K_3(x,\xi) = \int_\xi^x K(x,t) K_2(t,\xi) dt = \int_\xi^x (t-x)(\xi - t) dt = -\frac{1}{3!}(\xi - x)^5$$

The Resolvent Kernel is defined as

$$R(x,\xi:\lambda) = \sum_{n=1}^{\infty} \lambda^n K_n\,(x,\xi) = \frac{\xi - x}{1!} - \frac{(\xi - x)^3}{3!} + \frac{(\xi - x)^5}{5!} = -\frac{1}{3!}(\lambda = 1)$$

$$= \sin(\xi - x)$$

The solution of the integral equation is given by

$$\phi(x) = f(x) + \lambda \int_0^x R(x,\xi:\lambda) f(\xi) d\xi$$

$$= x + \int_0^x \xi \sin(\xi - x) d\xi$$

$$= x + \sin x - x \;\; [Integrating\ by\ parts]$$

$$= \sin x.$$

14.10 SUMMARY

An integral equation is one in which function to be determined appears under the integral sign. The most general form of a linear integral equation is

$$h(x)u(x) = f(x) + \int_a^{b(x)} K(x,\xi)\, \mathrm{u}(\xi)\mathrm{d}\xi \;\; \text{for all } \mathrm{x} \in [\mathrm{a}, \mathrm{b}]$$

In which u(x) is the function to be determined and $K(x,\xi)$ is called the Kernel of integral equation.

A Volterra integral equation is of the type:

$$h(x)u(x) = f(x) + \int_a^x K(x,\xi)\, \mathrm{u}(\xi)\mathrm{d}\xi \;\; \text{for all } \mathrm{x} \in [\mathrm{a}, \mathrm{b}]$$

that is, in Volterra equation h(x) = x

(iii) If h(x) = 0 the above equation reduces to

$$-f(x) = \int_a^x K(x,\xi)\, \mathrm{u}(\xi)\mathrm{d}\xi$$

This equation is called Volterra integral equation of first kind.

(iv) If h(x) = 1 the above equation reduces to

$$u(x) = f(x) + \int_a^x K(x,\xi)\, \mathrm{u}(\xi)\mathrm{d}\xi$$

This equation is called Volterra integral equation of second kind.

If f(x) = 0 for all x ∈ [a, b], then the reduced equation

$$h(x)u(x) = \int_a^{b(x)} K(x,\xi)\, \mathrm{u}(\xi)\mathrm{d}\xi$$

is called homogeneous integral equation. Otherwise, it is called non-homogeneous integral equation.

If n is a positive integer, then

$$\int_a^x \int_a^{x_1} \dots \int_a^{x_{n-2}} \int_a^{x_{n-1}} F(x_n)\, dx_n dx_{n-1} \dots . dx_1 = \frac{1}{1-n!} \int_a^x (x-\xi)^{n-1} f(\xi) d\xi$$

The Laplace transform of a function f(x) defined on interval (0, ∞) is given by

$$L[f(x)] = f(s) = \int_0^\infty f(x)\, e^{-5x} dx \dots . (1)$$

Here s is called Laplace variable or Laplace parameter. Also. $f(x) = L^{-1}[f(s)]$ is called inverse Laplace transform.

14.11 TERMINAL QUESTIONS

Q.1 Write the solution procedure of Volterra Integral equation.

Q.2 Reduce following initial value problem into Volterra integral equations:

a. $y'' + xy = 1,\ y'(0) = 0 = y(0)$

b. $\frac{d^2y}{dx^2} + A(x)\frac{dy}{dx} + B(x)y = g(x),\ y(a) = c_1\ and\ y'(a) = c_2$

c. $y'' + \lambda y = 0, y(0) = 1, y'(0) = 0$

d. $y'' - 5\,y' + 6y = 0,\ y(0) = 0, y'(0) = -1$

Q.3 Using the method of successive approximation, solve the integral equation,

$$y(x) = e^x + \int_0^x e^{x-t} y(t) dt.$$

Using the method of successive approximation, solve the integral equation:

Q.4 $u(x) = 1 + \int_0^x (x-\xi) u(\xi) d\xi.$

Q.5 $u(x) = 1 + \int_0^x (\xi - x) u(\xi) d\xi$

Q.6 $u(x) = 1 + \int_0^x u(\xi) d\xi$

Q.7 $u(x) = e^{x^2} + \int_0^x e^{x^2-t^2} u(t) dt$

Q.8 $u(x) = (1+x) + \int_0^x (x-\xi) u(\xi) d\xi\ with\ u_0(x) = 1$

Q.9 Use the method of Laplace Transform to solve the following integral equations.

$$(a)\quad u(x) = 1 + \int_0^x (x - \xi)u(\xi)d\xi$$

$$(b)\quad u(x) = 1 + \int_0^x (\xi - x)u(\xi)d\xi$$

$$(c)\quad u(x) = 1 + \int_0^x u(\xi)d\xi$$

Q.10 Solve the integral equation $x = \int_0^x \cos(x - \xi)u(\xi)d\xi$

Q.11 Obtaining the Resolvent Kernel, solve the following Volterra integral equation of second kind:

$$(a)\ u(x) = f(x) + \lambda \int_0^x e^{x-\xi}u(\xi)d\xi$$

$$(b)\ \phi(x) = 1 + \int_0^x (\xi - x)\phi(\xi)d\xi$$

$$(c)\ \phi(x) = e^{x^2} + \int_0^x e^{x^2-\xi^2}\phi(\xi)d\xi$$

Q.12 Solve the following problems.

a. Reduce following initial value problem into Volterra integral equations:

$$y'' - 2xy' - 3y = 0; \qquad y(0) = 0, y'(0) = 0$$

b. Using the method of successive approximation, solve the integral equation,

$$u(x) = (1 + x) - \int_0^x u(\xi)d\xi \ \ with\, u_0(x) = 1$$

c. Use the method of Laplace Transform to solve the following integral equations.

$$u(x) = e^{-x} + \int_0^x \sin(x - \xi)\, u(\xi)d\xi$$

Answer

Q.2 **(a)** $y(x) = \frac{x^2}{2} - \int_0^x (x-\xi)\phi(\xi)d\xi$

(b) $f(x) = c_1 + c_2(x-a) + \int_a^x (x-\xi)g(\xi)d\xi + A(a)c_1(x-a)$,

where $K(x,\xi) = (x-\xi)[A'(\xi) - B(\xi)] - A(\xi)$.

(c) $y(x) = 1 - \lambda \int_a^x (x-\xi)y(\xi)d\xi$.

(d) $y(x) = (6x-5) + \int_a^x (5 - 6x + 6\xi)\phi(\xi)d\xi$

Q.3 $y(x) = \lim_{n\to\infty} e^x \left[1 + x + \frac{x^2}{2!} + \cdots \dots \dots \dots . + \frac{x^n}{n!}\right] = e^x . e^x = e^{2x}$

Q.4 Cosh x

Q.5 Cos x

Q.6 e^x

Q.7 $e^{x(x+1)}$

Q.8 e^x

Q.9 **(a)** cosh x

(b) cos x.

(c) e^x

Q.10 $1 + \frac{x^2}{2}$.

Q.11 **(a)** $u(x) = f(x) + \lambda e^{(1+\lambda)x} \int_0^x e^{-(1+\lambda)\xi} f(\xi)d\xi$.

(b) cos x.

(c) $e^{x(x+1)}$

Q.12 **(a)** $y(x) = \int_0^x (x+\xi)y(\xi)d\xi$ **(b)** 1 **(c)** $2e^{-x} - 1 + x$.